A functional Hilbert space approach to the theory of wavelets

A Functional Hilbert Space Approach to the Theory of Wavelets

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Abstract

We approach the theory of wavelets from the theory of functional Hilbert spaces. Starting with a Hilbert space $\mathcal{H}$, we consider a subset V of $\mathcal{H}$, for which the span is dense in $\mathcal{H}$. We define a function of positive type on the index set $\mathbb{I}$ which labels the elements of V. This function of positive type induces uniquely a functional Hilbert space, which is a subspace of $\mathbb{C}^{\mathbb{I}}$ and there exists a unitary mapping from $\mathcal{H}$ onto this functional Hilbert space. Such functional Hilbert spaces, however, are not easily characterized.
Next we consider a group G for the index set $\mathbb{I}$ and create the set V using a representation $\mathcal{R}$ of the group on $\mathcal{H}$. The unitary mapping between $\mathcal{H}$ and the functional Hilbert space is easily recognized as the wavelet transform. We do not insist the representation to be irreducible and derive a generalization of the wavelet theorem as formulated by Grossmann, Morlet and Paul. The functional Hilbert space can in general not be identified with a closed subspace of $\mathbb{L}_2(G)$, in contrast to the case of unitary, irreducible and square integrable representations.
Secondly, we take for G a semi-direct product of two locally compact groups $S \rtimes T$, where S is abelian. In this case we give a more tangible description for the functional Hilbert space, which is easier to grasp.
Finally, we provide an example where we take $\mathcal{H} = \mathbb{L}_2(\mathbb{R}^2)$ and the Euclidean motion group for G. This example is inspired by an application of biomedical imaging, namely orientation bundle theory, which was the motivation for this report.

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1 Introduction

In the last twenty years a lot has been written about the theory of wavelets. In 1985 Grossmann, Morlet and Paul published an article [GMP] which can be seen as the fundament of the theory of wavelet transformations based on group representations. Their main result was that the wavelet transformation

$$\big(W_\psi f\big)(g) = (\mathcal{U}_g\psi, f)_{\mathcal{H}} \tag{1.1}$$

defines a unitary mapping from a Hilbert space $\mathcal{H}$ onto $\mathbb{L}_2(G)$ for a suitable vector $\psi \in \mathcal{H}$, where G is a locally compact group with a unitary, irreducible and square integrable representation $\mathcal{U}$ of G in $\mathcal{H}$. A square integrable representation is a representation for which a ψ exist such that

$$C_\psi = \frac{1}{\|\psi\|_{\mathcal{H}}^2} \int_G (\mathcal{U}_g\psi, \psi)_{\mathcal{H}}\, d\mu_G(g) < \infty, \tag{1.2}$$

where μ_G is a left invariant Haar measure.

The irreducibility condition is very strong. A lot of interesting representations are not irreducible at all.Therefore it is often suggested, to replace the condition of irreducibility by the condition that the representation is cyclic, i.e. it has a cyclic vector, i.e. a vector for which the span of the orbit under $\mathcal{U}$ is dense in the Hilbert space. But no really successful unitarity results were obtained. For a nice overview of some posed suggestions, see [FM].

This report is mainly focused on the questions when and in what way the above wavelet transform defines a unitary mapping. In our opinion we succeeded answering these questions in the most general way by using the theory of functional Hilbert spaces (often also named the theory of reproducing kernels). The idea of working with these kind of spaces is inspired by the identity

$$\big|\big(W_\psi f\big)(g)\big| \leq \|\mathcal{U}_g\psi\|_{\mathcal{H}} \|f\|_{\mathcal{H}}. \tag{1.3}$$

This identity states that if the wavelet transformation defines a unitary mapping from (a subspace of) $\mathcal{H}$ onto another space $\mathbb{C}_K^G$, then point evaluation on elements of this space is a continuous continuous linear functional. So the latter space is a functional Hilbert space and it admits a reproducing kernel K. Furthermore, this space will consist of complex-valued functions on G. This explains the notation $\mathbb{C}_K^G$.

First we derive a unitarity result using functional Hilbert spaces where no group representations are involved at all. Starting from a Hilbert space $\mathcal{H}$ and a subset $V \subset \mathcal{H}$ labelled with index set $\mathbb{I}$, we construct a functional Hilbert space $\mathbb{C}_K^{\mathbb{I}}$ and a unitary mapping between $\mathcal{H}$ and $\mathbb{C}_K^{\mathbb{I}}$. This construction is the most fundamental result of this report.

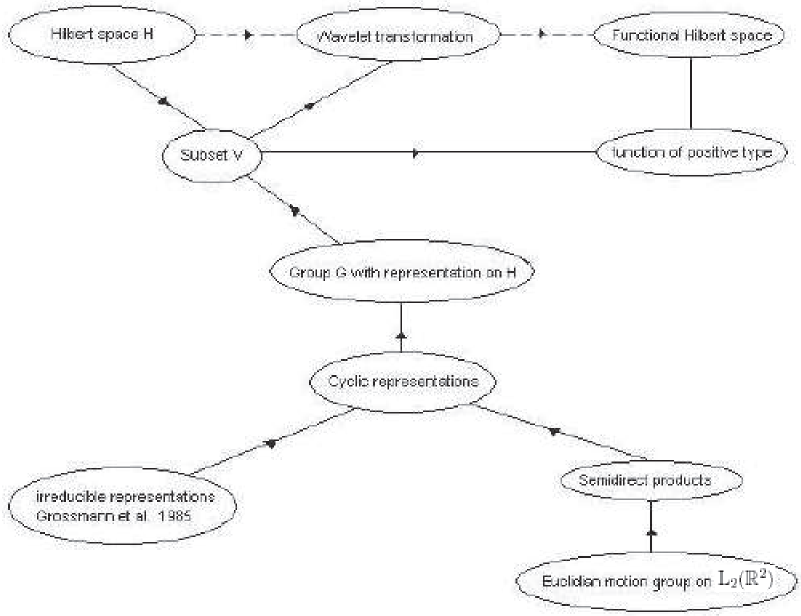

Figure 1: Overview of the contents

Later, we take a group G for $\mathbb{I}$ and construct V by means of a representation $\mathcal{R}$ of G on $\mathcal{H}$ and use this unitarity result, which leads straightforwardly to the wavelet transformation. The conditions we impose on the representation are quite simple: none! For every representation and $\psi \in \mathcal{H}$ we define a unitary mapping from a closed subspace of $\mathcal{H}$ to a functional Hilbert space $\mathbb{C}_K^G$. This closed subspace is equal to $\mathcal{H}$ if and only if ψ is a cyclic vector.

Although the above solves the unitarity questions, the functional Hilbert space is not easily characterized. Therefore we are challenged to find an alternative description of it. As mentioned before, this has already be done for irreducible representations but we also managed to give an easy to grasp description of the functional Hilbert space in the case $\mathcal{H} = \mathbb{L}_2(S)$ and $G = S \rtimes T$ for an abelian group S and an arbitrary group T. As an example we work out the case $S = \mathbb{R}^2$ and $T = \mathbb{T}$ so G is the Euclidean motion group. The idea to consider semi-direct products is not new, several articles have been written on this subject. See for example [FK], [FM].

2 Constructing unitary maps from a Hilbert space to a functional Hilbert space

2.1 Introduction

In this section our aim is to construct unitary maps from a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathbb{C}_K^{\mathbb{I}}$ which is a vector subspace of $\mathbb{C}^{\mathbb{I}}$, where $\mathbb{I}$ is a set. Here $\mathbb{C}^{\mathbb{I}}$ stands for the space of all complex-valued functions on $\mathbb{I}$.

We say that a Hilbert space $\mathcal{H}$ consisting of functions on a set $\mathbb{I}$ is a **functional Hilbert space**, if the point evaluation is continuous. Then, by the Riesz-representation theorem, there exists a set $\{K_m \mid m \in \mathbb{I}\}$ with

$$(K_m, f)_{\mathcal{H}} = f(m), \tag{2.4}$$

for all $m \in \mathbb{I}$ and $f \in \mathcal{H}$. It follows that the span of the set $\{K_m \mid m \in \mathbb{I}\}$ is dense in $\mathbb{C}_K^{\mathbb{I}}$. Indeed, if $f \in \mathcal{H}$ is orthogonal to all K_m then $f = 0$ on $\mathbb{I}$.

Then define $K(m, m') = K_{m'}(m) = (K_m, K_{m'})_{\mathcal{H}}$, for $m, m' \in \mathbb{I}$. K is called the **reproducing kernel**. It is obvious that K is a **function of positive type** on $\mathbb{I}$, i.e.,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} K(m_i, m_j)\overline{c_i} c_j \geq 0, \tag{2.5}$$

for all $n \in \mathbb{N},\ c_1, ..., c_n \in \mathbb{C},\ m_1, ..., m_n \in \mathbb{I}$.

So to every functional Hilbert space there belongs a reproducing kernel, which is a function of positive type. Conversely, as Aronszajn pointed out in his paper [A], a function K of positive type on a set $\mathbb{I}$, induces uniquely a functional Hilbert space consisting of functions on $\mathbb{I}$ with reproducing kernel K. We will denote this space with $\mathbb{C}_K^{\mathbb{I}}$. Without giving a detailed proof we mention that $\mathbb{C}_K^{\mathbb{I}}$ can be constructed as follows; start with $K : \mathbb{I} \times \mathbb{I} \to \mathbb{C}$, a function of positive type and define $K_m = K(\cdot, m)$. Now take the span $\langle \{K_m \mid m \in \mathbb{I}\} \rangle$ and define the inner product on this span as

$$\left(\sum_{i=1}^{l} \alpha_i K_{m_i}, \sum_{j=1}^{n} \beta_j K_{m_j} \right)_{\mathbb{C}_K^{\mathbb{I}}} = \sum_{i=1}^{l} \sum_{j=1}^{n} \overline{\alpha_i} \beta_j K(m_i, m_j). \tag{2.6}$$

This is a pre-Hilbert space. After taking the completion we arrive at the functional Hilbert space $\mathbb{C}_K^{\mathbb{I}}$.

There exists a useful characterization of the elements of $\mathbb{C}_K^{\mathbb{I}}$.

Lemma 2.1 *Let K be a function of positive type on $\mathbb{I}$ and F a complex-valued function on $\mathbb{I}$. Then the function F belongs to $\mathbb{C}_K^{\mathbb{I}}$ if and only if there exists a constant $\gamma > 0$ such that*

$$\left| \sum_{j=1}^{l} \alpha_j \overline{F(m_j)} \right|^2 \leq \gamma \sum_{k,j=1}^{l} \overline{\alpha_k} \alpha_j K(m_k, m_j), \tag{2.7}$$

for all $l \in \mathbb{N}$ and $\alpha_j \in \mathbb{C},\ m_j \in \mathbb{I},\ 1 \leq j \leq l$.

Proof: See [Ma, Lemma 1.7, pp.31] or [An, Th. II.1.1].

This lemma enables us to give an expression for the norm of an arbitrary element in $\mathbb{C}_K^{\mathbb{I}}$.

Lemma 2.2 *Let $F \in \mathbb{C}_K^{\mathbb{I}}$. Then*

$$\|F\|^2_{\mathbb{C}_K^{\mathbb{I}}} = \sup \left\{ \left| \sum_{j=1}^{l} \alpha_j \overline{F(m_j)} \right|^2 \left(\sum_{k,j=1}^{l} \overline{\alpha_k} \alpha_j K(m_k, m_j) \right)^{-1} \right.$$

$$\left. \left| \ l \in \mathbb{N},\ \alpha_j \in \mathbb{C},\ m_j \in \mathbb{I},\ \left\| \sum_{k=1}^{l} \alpha_k K_{m_k} \right\|_{\mathbb{C}_K^{\mathbb{I}}} \neq 0 \right. \right\}. \tag{2.8}$$

Proof: The statement is equivalent to

$$\|F\|^2_{\mathbb{C}_K^{\mathbb{I}}} = \sup \left\{ \frac{\left|(F, G)_{\mathbb{C}_K^{\mathbb{I}}}\right|^2}{\|G\|^2_{\mathbb{C}_K^{\mathbb{I}}}} \mid G \in \langle \{K_m \mid m \in \mathbb{I}\} \rangle \right\}.$$

Since $\langle \{K_m \mid m \in \mathbb{I}\} \rangle$ is dense in $\mathbb{C}_K^{\mathbb{I}}$, the statement follows. □

For a detailed discussion of functional Hilbert spaces see [A], [An] or [Ma].

2.2 Construction of a unitary map

Starting with some labelled subset V of $\mathcal{H}$, we will construct a functional Hilbert space by means of a function of positive type on the index set, using the construction as described in the introduction. Moreover, there exists a natural unitary mapping from $\overline{\langle V \rangle}$ to this functional Hilbert space.

Let $\mathcal{H}$ be a Hilbert space. Let $\mathbb{I}$ be an index set and

$$V := \{\phi_m \mid m \in \mathbb{I}\}, \tag{2.9}$$

be a subset of $\mathcal{H}$. We can easily build a function $K : \mathbb{I} \times \mathbb{I} \to \mathbb{C}$ of positive type on $\mathbb{I}$ by

$$K(m, m') = (\phi_m, \phi_{m'})_{\mathcal{H}}. \tag{2.10}$$

From this function of positive type the space $\mathbb{C}_K^{\mathbb{I}}$ can be constructed.

The following theorem is the starting point of this report.

Theorem 2.3 (Abstract Wavelet Theorem) *Define* $W : \overline{\langle V \rangle} \to \mathbb{C}_K^{\mathbb{I}}$ *by*

$$\big(Wf\big)(m) = (\phi_m, f)_{\mathcal{H}}. \tag{2.11}$$

Then W *is a unitary mapping.*

Proof:
Here $\overline{\langle V \rangle}$ inherits the inner product from $\mathcal{H}$. First we show that $Wf \in \mathbb{C}_K^{\mathbb{I}}$ for any element $f \in \overline{\langle V \rangle}$ and that W is bounded (and therefore continuous). If $f \in \overline{\langle V \rangle}$ then

$$\left|\sum_{j=1}^{l} \alpha_j \overline{\big(Wf\big)(m_j)}\right|^2 = \left|\sum_{j=1}^{l} \alpha_j \overline{(\phi_{m_j}, f)_{\mathcal{H}}}\right|^2 = \left|\left(\sum_{j=1}^{l} \alpha_j \phi_{m_j}, f\right)_{\mathcal{H}}\right|^2$$
$$\leq \left\|\sum_{j=1}^{l} \alpha_j \phi_{m_j}\right\|_{\mathcal{H}}^2 \|f\|_{\mathcal{H}}^2 = \left(\sum_{k,j=1}^{l} \overline{\alpha_k} \alpha_j K(m_k, m_j)\right) \|f\|_{\mathcal{H}}^2,$$

for all $l \in \mathbb{N}$, $\alpha_1, ..., \alpha_l \in \mathbb{C}$, and $m_1, ..., m_n \in \mathbb{I}$. So $Wf \in \mathbb{C}_K^{\mathbb{I}}$ by Lemma 2.1 and $\|Wf\|_{\mathbb{C}_K^{\mathbb{I}}}^2 \leq \|f\|_{\mathcal{H}}^2$, by Lemma 2.2. Next we prove that W is an isometry. Because $\big(\phi_{m'}\big)(m) = K(m, m')$, W maps a linear combination $\sum_i \alpha_i \phi_{m_i}$ onto the linear combination $\sum_i \alpha_i K(\cdot, m_i)$. So $W(\langle V \rangle) = \langle \{K(\cdot, m) | m \in M\} \rangle$. Moreover, it maps $\langle V \rangle$ isometrically onto $\langle \{K(\cdot, m) | m \in \mathbb{I}\} \rangle$, because

$$\left(W\big(\sum_i \alpha_i \phi_{m_i}\big), W\big(\sum_j \beta_j \phi_{m'_j}\big)\right)_{\mathbb{C}_K^{\mathbb{I}}} = \left(\sum_i \alpha_i K(\cdot, m_i), \sum_j \beta_j K(\cdot, m'_j)\right)_{\mathbb{C}_K^{\mathbb{I}}}$$
$$= \sum_{i,j} \overline{\alpha}_i \beta_j K(m_i, m'_j) = \sum_{i,j} \overline{\alpha}_i \beta_j (\phi_{m_i}, \phi_{m'_j})_{\mathcal{H}}.$$

Since $\langle V \rangle$ is dense in $\overline{\langle V \rangle}$ and W is bounded on $\overline{\langle V \rangle}$ it follows that W is an isometry, . Furthermore, $W[\langle V \rangle]$ is dense in $\mathbb{C}_K^{\mathbb{I}}$. So W is also surjective and therefore unitary. □

In most cases we are mainly interested in the case $\overline{\langle V \rangle} = \mathcal{H}$, i.e. V is total in $\mathcal{H}$. To get a feeling for what is happening we now deal with two illustrating examples.

2.2.1 The special case $\mathbb{I} = \mathbb{N}$

Let $\mathcal{H}$ be a separable Hilbert space consisting of functions on the set $\mathbb{I} = \mathbb{N}$. Now let $V = \{\phi_m \mid m \in \mathbb{N}\}$ consist of an orthonormal basis, so $\overline{\langle V \rangle} = \mathcal{H}$. Then,

$$K(m, m') = (\phi_m, \phi_{m'})_{\mathcal{H}} = \delta_{mm'}, \tag{2.12}$$

for all $m, m' \in \mathbb{N}$. This means that we just get $\mathbb{C}_K^{\mathbb{N}} = l_2(\mathbb{N})$. The unitary map W gives us the sequence of expansion coefficients c_m of a vector $f \in \mathcal{H}$ with respect to the orthonormal basis. This is the most trivial example of a **frame**.

2.2.2 The special case $\mathbb{I} = \mathcal{H}$

Now let $\mathbb{I} = \mathcal{H}$ and $V = \{m | m \in \mathcal{H}\} = \mathcal{H}$. The function of positive type is just the inner product

$$K(m, m') = (m, m')_{\mathcal{H}}. \tag{2.13}$$

This means that $\mathbb{C}_K^{\mathbb{I}} = \mathbb{C}_{(\cdot,\cdot)_{\mathcal{H}}}^{\mathcal{H}}$. This is the functional Hilbert representation of an arbitrary Hilbert space. It is equal to the topological dual space $\mathcal{H}'$, the space of all continuous linear functions on $\mathcal{H}$.

2.2.3 The functional Hilbert space

The functional Hilbert space $\mathbb{C}_K^{\mathbb{I}}$ is an abstract construction. We are challenged to find alternative characterizations of these functional Hilbert spaces.

In the literature two major classes of functional Hilbert spaces appear; Hilbert spaces of Bargmann-type and of Sobolev-type. The first type consists of a nullspace of unbounded operators on $\mathbb{L}_2(\mathbb{I}, \mu)$ and the second of the domain of unbounded operators on $\mathbb{L}_2(\mathbb{I}, \mu)$. For Bargmann-type spaces see [B]. For Sobolev-type, see [EG1] and [EG2].

3 Functional Hilbert spaces on groups

3.1 Construction of V using group representations

From now on we will assume $\mathbb{I}$ to be a group G. Furthermore, we assume the group to have a **representation** on $\mathcal{H}$, i.e. a map $\mathcal{R}$ from G onto $\mathcal{B}(\mathcal{H})$, the space of all bounded operators on $\mathcal{H}$, which satisfies

$$\mathcal{R}_g\mathcal{R}_h = \mathcal{R}_{gh} \quad \forall_{g,h\in G}, \tag{3.1}$$

$$\mathcal{R}_e = I \tag{3.2}$$

where e is the identity element of G. Here and in the sequel we denote the representation with $\mathcal{R} : g \mapsto \mathcal{R}_g$. Given a vector $\psi \in \mathcal{H}$ we can construct the set V in (2.9) as follows

$$V_\psi = \{\mathcal{R}_g\psi \mid g \in G\}. \tag{3.3}$$

We will call ψ a **generating wavelet** or just a **wavelet**. Starting with such a set V_ψ we can construct a functional Hilbert subspace $\mathbb{C}_K^G$ and a unitary mapping W_ψ between $\overline{\langle V_\psi\rangle}$ and this functional Hilbert space, as described in section 2. The unitary map W_ψ will be called the **wavelet transformation**.

We state the following consequence of Theorem 2.3.

Theorem 3.1 (Wavelet Theorem for group representations) *Let $\mathcal{R}$ be a representation of a group G in a Hilbert space $\mathcal{H}$. Let $\psi \in \mathcal{H}$. Define the function $K : G \times G \to \mathbb{C}$ of positive type by*

$$K(g, g') = (\mathcal{R}_g\psi, \mathcal{R}_{g'}\psi)_\mathcal{H}. \tag{3.4}$$

Define the set V_ψ by

$$V_\psi = \{\mathcal{R}_g\psi \mid g \in G\}. \tag{3.5}$$

Then the wavelet transformation $W_\psi : \overline{\langle V_\psi\rangle} \to \mathbb{C}_K^G$ defined by

$$\big(W_\psi f\big)(g) = (\mathcal{R}_g\psi, f)_\mathcal{H}, \tag{3.6}$$

is a unitary mapping.

Of course, the wavelet transformation W_ψ could be defined on the entire space $\mathcal{H}$, but then the unitarity is lost in the case $\overline{\langle V_\psi\rangle} \neq \mathcal{H}$. For a vector $f \perp V_\psi$ we then get $W_\psi f = 0$.

Usually we are interested in the case $\overline{\langle V \rangle} = \mathcal{H}$. If $\overline{\langle V_\psi \rangle} = \mathcal{H}$ for some $\psi \in \mathcal{H}$, we call ψ a **cyclic vector** or a**cyclic wavelet** and the representation is called a **cyclic representation** if a cyclic wavelet exists.

Theorem 3.2 (Wavelet Theorem for Cyclic Representations) *Let $\mathcal{R}$ be a representation of a group G in a Hilbert space $\mathcal{H}$. Let ψ be a cyclic wavelet. Define a function $K : G \times G \to \mathbb{C}$ of positive type by*

$$K(g, g') = (\mathcal{R}_g \psi, \mathcal{R}_{g'} \psi)_{\mathcal{H}}. \tag{3.7}$$

The wavelet transformation $W_\psi : \mathcal{H} \to \mathbb{C}_K^G$ defined by

$$\big(W_\psi f\big)(g) = (\mathcal{R}_g \psi, f)_{\mathcal{H}}, \tag{3.8}$$

is a unitary mapping.

It is obvious that W_ψ can be defined as a unitary mapping on the entire space $\mathcal{H}$ if and only if $\mathcal{R}$ is cyclic and ψ is a cyclic wavelet.

Note that up till now there are no restrictions have been imposed on the Hilbert space $\mathcal{H}$, the group G or the representation $\mathcal{R}$. In particular, there are no topological conditions on G and $\mathcal{R}$.

3.2 Unitary representations

The kind of representations, which have our special interest, are **unitary representations**, i.e. representations $\mathcal{U}$ for which the $\mathcal{U}_g$ are unitary for all $g \in G$. These kind of representations have some nice properties. We will use the symbol $\mathcal{U}$ instead of $\mathcal{R}$ to denote a representation that is unitary.

The function of positive type, from which the functional Hilbert space can be constructed, is given by $K(g, h) = (\mathcal{U}_g \psi, \mathcal{U}_h \psi)_{\mathcal{H}}$. Because the representation is unitary this simplifies to

$$K(g, h) = (\mathcal{U}_g \psi, \mathcal{U}_h \psi)_{\mathcal{H}} = (\mathcal{U}_{h^{-1}g} \psi, \psi)_{\mathcal{H}} =: F(h^{-1}g). \tag{3.9}$$

In abstract harmonic analysis, the function $F : G \to \mathbb{C}$ is said to be of **positive type** if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} F(g_i^{-1} g_j) c_j \overline{c_i} \geq 0, \tag{3.10}$$

for all $n \in \mathbb{N}$, $c_1, ..., c_n \in \mathbb{C}$ and $g_1, ..., g_n \in G$. Remark that this definition is stronger then the definition as formulated in (2.5), since $K(hg_1, hg_2) = K(g_1, g_2)$ for all $h, g_1, g_2 \in G$.

Define $\mathcal{U}_L : G \to \mathcal{B}(\mathbb{C}_K^G)$ by

$$\big(\mathcal{U}_{Lg} f\big)(h) = f(g^{-1}h), \tag{3.11}$$

for all $f \in \mathbb{C}_K^G$ and $g, h \in G$.

Theorem 3.3 *Let G be a group and $F : G \to \mathbb{C}$ be a function of positive type. Define $K(g,h) = F(h^{-1}g)$ for all $g, h \in G$. Then $\mathcal{U}_L$ is a unitary representation of G in $\mathbb{C}_K^G$.*

Proof:
Now,

$$\begin{aligned} \big(\mathcal{U}_{Lh} K_{g_1}\big)(g_2) &= K_{g_1}(h^{-1}g_2) = (K_{h^{-1}g_2}, K_{g_1}) = K(h^{-1}g_2, g_1) = F(g_1^{-1}h^{-1}g_2) \\ &= F((hg_1)^{-1}g_2) = K(g_2, hg_1) = (K_{g_2}, K_{hg_1}) = K_{hg_1}(g_2), \end{aligned}$$

for all $h, g_1, g_2 \in G$. So $\mathcal{U}_{Lh} K_g = K_{hg}$ for all $g, h \in G$. Furthermore,

$$(\mathcal{U}_{Lh} K_{g_1}, \mathcal{U}_{Lh} K_{g_2})_{\mathbb{C}_K^G} = (K_{hg_1}, K_{hg_2})_{\mathbb{C}_K^G}$$

$$= K(hg_1, hg_2) = F(g_2^{-1}g_1) = K(g_1, g_2) = (K_{g_1}, K_{g_2})_{\mathbb{C}_K^G},$$

for all $h, g_1, g_2 \in G$. Hence, $\mathcal{U}_{Lh}$ is unitary on $\langle\{K_g \mid g \in G\}\rangle$ for all $h \in G$. Therefore it follows by denseness of $\langle\{K_g \mid g \in G\}\rangle$, that $\mathcal{U}_L$ is a unitary representation. □

The representation $\mathcal{U}_L$ is called the **left regular representation**. Remark the intertwining relation $\mathcal{U}_{Lg} W_\psi = W_\psi \mathcal{U}_g$.

The following theorems give us a guarantee that all functional Hilbert spaces, which are subspaces of $\mathbb{C}^G$ and induced by a function of positive type in the sense of (3.10), can also be constructed by some unitary (not necessarily cyclic) representation of G in $\mathcal{H}$ and a wavelet $\psi \in \mathcal{H}$, where $\mathcal{H}$ is unitarily equivalent to $\mathbb{C}_K^G$.

Theorem 3.4 *Let G be a group and $F : G \to \mathbb{C}$ be a function of positive type. Define $K(g,h) = F(h^{-1}g)$ for all $g, h \in G$. Then there exist a $\psi \in \mathbb{C}_K^G$ and a unitary representation $\mathcal{U}$ of G in $\mathbb{C}_K^G$ such that*

$$F(g) = (\mathcal{U}_g \psi, \psi)_{\mathbb{C}_K^G}, \tag{3.12}$$

for all $g \in G$.

Proof:
The representation $\mathcal{U}_R$ is unitary. Moreover

$$(\mathcal{U}_{Lg}F, F)_{\mathbb{C}_K^G} = (\mathcal{U}_{Lg}K_e, K_e)_{\mathbb{C}_K^G} = (K_g, K_e)_{\mathbb{C}_K^G} = K(g, e) = F(g),$$

for all $g \in G$. $\Box$

Corollary 3.5 *Let G be a group and $F: G \to \mathbb{C}$ a function of positive type. Define $K(g,h) = F(h^{-1}g)$ for all $g, h \in G$. Let $\mathcal{H}$ be a Hilbert space, which is unitarily equivalent to $\mathbb{C}_K^G$. Then there exist a $\psi \in \mathcal{H}$ and a unitary representation $\mathcal{U}$ of G in $\mathcal{H}$ such that*

$$F(g) = (\mathcal{U}_g\psi, \psi)_{\mathcal{H}}, \tag{3.13}$$

for all $g \in G$.

Proof:
By assumption, there exist a unitary mapping $\mathcal{T}$ from $\mathcal{H}$ to $\mathbb{C}_K^G$. Now, the element $\psi = \mathcal{T}^{-1}F$ and the unitary representation defined by $\mathcal{U}_g = \mathcal{T}^{-1}\mathcal{U}_{Lg}\mathcal{T}$ for all $g \in G$ do the trick. $\Box$

We recall that all separable Hilbert spaces of infinite dimension are unitarily equivalent. So are all finite dimensional Hilbert spaces of equal dimension.

3.3 Topological conditions

Some elementary topological conditions which can be posed on the representation $\mathcal{R}$, are straightforwardly transferred to the wavelet transformation.

Let $\mathcal{R}$ be a **bounded representation**, i.e. a representation for which the mapping $g \mapsto \|\mathcal{R}_g\|$ is bounded. Define $\|\mathcal{R}\| = \sup_{g\in G} \|\mathcal{R}_g\|$. Let $f \in \overline{\langle V_\psi \rangle}$. Then,

$$\big|\big(W_\psi f\big)(g)\big| = |(\mathcal{R}_g\psi, f)_{\mathcal{H}}| = \|\mathcal{R}_g\psi\|_{\mathcal{H}}\|f\|_{\mathcal{H}} \leq \|\mathcal{R}\|\|\psi\|_{\mathcal{H}}\|f\|_{\mathcal{H}}, \tag{3.14}$$

for all $g \in G$. Hence, the wavelet transform $W_\psi f$ for an arbitrary $f \in \mathcal{H}$ is bounded on G. Also the reproducing kernel is bounded on $G \times G$. A unitary representation is an example of a bounded representation.

Assume G is a **topological group**, i.e. a group on which a topology is defined, such that the group operations, multiplication and inversion, are continuous. Let $\mathcal{R}$ be a **continuous**

representation, i.e. a representation for which $\mathcal{R}_g f \to \mathcal{R}_h f$ whenever $g \to h$, for all $h \in G$ and $f \in \mathcal{H}$. Let $f \in \mathcal{H}$. Then $W_\psi[f]$ is a continuous function on G. Indeed, if $g \to h$ then

$$|\big(W_\psi f\big)(g) - \big(W_\psi f\big)(h)| = |((\mathcal{R}_g - \mathcal{R}_h)\psi, f)_{\mathcal{H}}| \leq \|(\mathcal{R}_g - \mathcal{R}_h)\psi\|_{\mathcal{H}} \|f\|_{\mathcal{H}} \to 0. \quad (3.15)$$

Also the reproducing kernel is a continuous function on $G \times G$.

4 Cyclic representations

Because of Theorem 3.2 the cyclic representations have our special attention. But it is not often straightforward to see whether a representation is cyclic or not. And even if so, one still has to find a cyclic vector. In this section we pose an idea to find candidates for cyclic vectors. Moreover, we work out an example which deals with diffusion on a sphere. For this case we managed, to find an interesting cyclic vector, with the aid of Theorem 4.1.

4.1 A fundamental theorem

Let $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ be a sequence of Hilbert spaces. Then define the **orthogonal direct sum** of the sequence as the Hilbert space

$$\bigoplus_{n=1}^{\infty} \mathcal{H}_n = \left\{ a \in \prod_{n=1}^{\infty} \mathcal{H}_n \mid \sum_{n=1}^{\infty} \|a_n\|^2_{\mathcal{H}_n} < \infty \right\}, \quad (4.1)$$

with the inner product

$$(a, b)_{\oplus} = \sum_{n=1}^{\infty} (a_n, b_n)_{\mathcal{H}_n}. \quad (4.2)$$

The following Theorem is inspired by Theorem B.3 in Appendix B.

Theorem 4.1 *Let $\mathbb{I}$ be a set. Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of functions of positive type on $\mathbb{I}$ such that a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ exists satisfying the following conditions*

1. $\forall\ n \in \mathbb{N}: \quad \lambda_n > 0$
2. $\sup_n \lambda_n < \infty$.

Then $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{\lambda_n K_n}$ is dense in $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$. If in addition the sequences satisfy the conditions

3. $\forall\, x \in \mathbb{I}: \ \ \sum_{n=1}^{\infty} \lambda_n K_n(x,x) < \infty$

4. $\forall\, n \in \mathbb{N}: \ \ \mathbb{C}^{\mathbb{I}}_{\lambda_n K_n} \cap \mathbb{C}^{\mathbb{I}}_{\sum_{m=1,m\neq n}^{\infty} \lambda_m K_m} = \{0\}.$

Then

$$\psi_x = (\lambda_1 K_{1;x}, \lambda_2 K_{2;x}, \ldots) \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n} \tag{4.3}$$

for all $x \in \mathbb{I}$. *Furthermore*

$$\overline{\langle \{\psi_x \mid x \in \mathbb{I}\}\rangle} = \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}. \tag{4.4}$$

Proof:
Assume the first two conditions are satisfied.

First, we remark that from the definition it straightforwardly follows that $\mathbb{C}^{\mathbb{I}}_{K_n} = \mathbb{C}^{\mathbb{I}}_{\lambda_n K_n}$ as a set and $(f,g)_{\mathbb{C}^{\mathbb{I}}_{K_n}} = \lambda_n (f,g)_{\mathbb{C}^{\mathbb{I}}_{\lambda_n K_n}}$ for all $n \in \mathbb{N}$ and $f,g \in \mathbb{C}^{\mathbb{I}}_{K_n}$.

Secondly, write $\|\cdot\|_{\lambda\oplus}$ for the norm of $\bigoplus_{n=1}^{\infty} \mathbb{C}^{G}_{\lambda_n K_n}$. Let $f = (f_1, f_2, \ldots) \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{\lambda_n K_n}$. Then, it follows by

$$\begin{aligned} \|f\|_{\oplus} &= \sum_{n=1}^{\infty} (f_n, f_n)_{\mathbb{C}^{\mathbb{I}}_{K_n}} = \sum_{n=1}^{\infty} \lambda_n (f_n, f_n)_{\mathbb{C}^{\mathbb{I}}_{\lambda_n K_n}} \\ &\leq \sup_n \lambda_n \sum_{n=1}^{\infty} \|f_n\|^2_{\mathbb{C}^{\mathbb{I}}_{\lambda K_n}} = \sup_n \lambda_n \|f\|_{\lambda\oplus}, \end{aligned}$$

that $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{\lambda_n K_n} \subset \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$.

Finally, the set

$$\{f \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n} \mid \exists N \in \mathbb{N}\ \forall n > N\ [f_n = 0]\}$$

is dense in $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$ and contained in $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{\lambda_n K_n}$. Hence $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{\lambda_n K_n}$ is dense in $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$.

Now, assume in addition that the last two condition are satisfied.

Because $\psi_x \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{\lambda_n K_n}$ we have in particular $\psi_x \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$ for all $x \in \mathbb{I}$. Then by Theorem B.3, Theorem B.4 and Theorem B.5, the set $\langle \{\psi_x \mid x \in \mathbb{I}\}$ is dense in $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{\lambda_n K_n}$. Moreover, because $\|\cdot\|_{\oplus} \leq \sup_n \lambda_n \|\cdot\|_{\lambda\oplus}$ and $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{\lambda_n K_n}$ is dense in $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$, it follows that $\langle \{\psi_x \mid x \in \mathbb{I}\}\rangle$ is dense in $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$ □

It is easy to see that

$$(f, \psi_x)_{\mathcal{H}} = \sum_{n=1}^{\infty} \lambda_n f_n(x), \tag{4.5}$$

for all $x \in \mathbb{I}$, which will turn out to be a useful identity.

4.2 An example: diffusion on a sphere

We now deal with an example concerning the problem of diffusion on a sphere. For a detailed discussion about some statements which we do not prove, see for example [Mu, Ch. 3].

Let S^{q-1} for $q \geq 3$ be the unit sphere in $\mathbb{R}^q$ and $G = SO(q)$ the special orthogonal matrix group. Let the group $SO(q)$ act on S^{q-1} in the usual way, $(A, x) = Ax$. The group acts **transitively** on S^{q-1}, i.e. for all $x, y \in S^{q-1}$ there exists an $A \in SO(q)$ such that $x = Ay$.

Let $\mathcal{H}$ be the Hilbert space $\mathbb{L}_2(S^{q-1})$. Define the representation $\mathcal{R} : G \to \mathcal{B}(\mathbb{L}_2(S^{q-1}))$ by

$$\big(\mathcal{U}_A f\big)(x) = f(A^{-1}x), \tag{4.6}$$

for all $A \in SO(q)$, $f \in \mathbb{L}_2(S^{q-1})$ and almost all $x \in S^{q-1}$.

First, it is well-known that the space $\mathbb{L}_2(S^{q-1})$ decomposes in $\mathbb{L}_2(S^{q-1}) \cong \oplus_{n=1}^{\infty} \mathbb{C}_{K_n}^{S^{q-1}}$ where $\mathbb{C}_{K_n}^{S^{q-1}}$ is the space of all spherical harmonic polynomials of order n. For all $n \in \mathbb{N}$ the space $\mathbb{C}_{K_n}^{S^{q-1}}$ is finite dimensional, therefore a functional Hilbert space. The reproducing kernel is given by

$$K_{n;x} = \frac{q + 2n - 2}{q - 2} C_N^{q/2-1}((\cdot, x)_2), \tag{4.7}$$

for all $x \in S^{q-1}$, where $C_N^{q/2-1}$ are the Gegenbauer polynomials. Since

$$\|K_{n;x}\|^2_{\mathbb{L}_2(S^q)} = \frac{q + 2n - 2}{q - 2} C_N^{q/2-1}((x, x)_2) = \frac{q + 2n - 2}{q - 2} C_N^{q/2-1}(1) = \frac{q + 2n - 2}{q - 2}, \tag{4.8}$$

for all $x \in S^{q-1}$, it is straightforward to see that this orthogonal sum satisfies the condition of Theorem 4.1 for some sequence $\{\lambda_n\}_{n \in \mathbb{N}}$.

Secondly, we have to choose a sequence $\{\lambda_n\}_{\,n \in \mathbb{N}}$. Let $t > 0$. Then it is obvious that $\lambda_n = \mathrm{e}^{-tn(n+q-2)}$ defines a sequence that satisfies the conditions in Theorem 4.1. Now define for all $x \in S^{q-1}$

$$\psi_x = \sum_{n=1}^{\infty} \mathrm{e}^{-tn(n+q-2)} K_{n;x}. \tag{4.9}$$

Then $\psi_x \in \mathbb{L}_2(S^{q-1})$ by Theorem 4.1. Fix $y \in S^{q-1}$. Then,

$$\mathcal{U}_A \psi_y = \psi_{Ay}, \tag{4.10}$$

for all $A \in SO(q)$ by (4.7). Finally, by the transitivity of the action of the group we get by Theorem 4.1

$$\overline{\langle \{ U_A \psi_y \mid A \in SO(q) \} \rangle} = \overline{\langle \{ \psi_x \mid x \in S^{q-1} \} \rangle} = \mathbb{L}_2(S^{q-1}). \tag{4.11}$$

Hence ψ_y is a cyclic vector for all $y \in S^{q-1}$ and $\mathcal{U}$ is a cyclic representation.

We summarize.

Theorem 4.2 *Let $q \geq 3$, $\mathcal{H} = \mathbb{L}_2(S^{q-1})$ and $G = SO(q)$. Let $y \in S^{q-1}$ and $t > 0$. Define $\psi_y \in \mathbb{L}_2(S^{q-1})$ by*

$$\psi_y = \sum_{n=1}^{\infty} \mathrm{e}^{-tn(n+q-2)} \frac{q+2n-2}{q-2} C_N^{q/2-1}((\cdot, x)_2). \tag{4.12}$$

Define a function $K : SO(q) \times SO(q) \to \mathbb{C}$ of positive type by

$$K(A, A') = (\mathcal{U}_A \psi_y, \mathcal{U}_{A'} \psi_y)_{\mathcal{H}}. \tag{4.13}$$

Then the wavelet transformation $W_{\psi_y} : \mathbb{L}_2(S^{q-1}) \to \mathbb{C}_K^{SO(q)}$ defined by

$$\big(W_\psi f\big)(A) = (\mathcal{U}_A \psi_y, f)_{\mathbb{L}_2(S^{q-1})} = (\psi_{Ay}, f)_{\mathbb{L}_2(S^{q-1})}, \tag{4.14}$$

is a unitary mapping.

The choice $\lambda_n = \mathrm{e}^{-tn(n+q-2)}$ was not without reason. The spherical harmonic polynomials of order n are the eigenvectors of the Laplace-Beltrami operator $\triangle_S$ with eigenvalue $n(n+q-2)$. Therefore the functions of the form $(t, x) \mapsto \mathrm{e}^{-tn(n+q-2)} p_n(x)$ with p_n a spherical harmonic polynomial of order n are solutions of the evolution equation

$$u_t = -\triangle_S u. \tag{4.15}$$

Let $f \in \mathbb{L}_2(S)$. With (4.5) it is easy to see that

$$\big(W_{\psi_y} f\big)(A) = (\psi_{Ay}, f)_{\mathbb{L}_2(S^{q-1})} = \sum_{n=1}^{\infty} \mathrm{e}^{-tn(n+q-2)} \big(P_n f\big)(Ay), \tag{4.16}$$

where P_n stands for the projection operator corresponding to the space of all spherical harmonic polynomials of order n. So we could interpret the above wavelet transformation as the solution at time t and point Ay of the evolution equation (4.15) with initial condition $u(0, \cdot) = f(\cdot)$. Therefore the cyclic vector ψ_y is the fundamental solution for the evolution equation.

The choice $\lambda_n = \mathrm{e}^{-tn}$ would correspond with scaling of the harmonics.

5 Examples of wavelet transformations based on cyclic representations

We now work out two examples based on cyclic representations. In the first, the representation is irreducible and therefore cyclic in a trivial way. The second example concerns $\mathcal{H} = \mathbb{L}_2(S)$ with S a locally compact abelian group and $G = S \rtimes T$ the semi-direct product of S with some other (not necessarily abelian) group T.

From now on we do pose a topological condition on G. Recall that a **topological group** is a group on which a topology is defined, such that the group operations, multiplication and inversion, are continuous. We always assume the topology to be Hausdorff. Moreover, we always assume the group G to be a **locally compact group**, i.e. a topological group, in which every group element has a compact neighbourhood.

It is well-known that every locally compact group G has a **left invariant Haar measure**, which we denote by μ_G. A left invariant Haar measure on G is a Radon measure on G such that $\mu_G(gE) = \mu_G(E)$ for all $g \in G$ and Borel sets E.

5.1 Irreducible unitary representations

We call a representation $\mathcal{R}$ of a group G in a Hilbert space $\mathcal{H}$ **irreducible** if the only closed subspaces of $\mathcal{H}$ which are invariant under all $\mathcal{R}_g$ for all $g \in G$ are $\mathcal{H}$ and $\{0\}$. An irreducible representation is in particular cyclic and every nonzero vector is cyclic. Indeed, for every nonzero $\psi \in \mathcal{H}$ the set $\overline{\langle V_\psi \rangle}$ is a subspace which is invariant under all $\mathcal{R}_g$ with $g \in G$ and it is not empty, so $\overline{\langle V_\psi \rangle} = \mathcal{H}$.

The representation $\mathcal{R}$ is called **square integrable** if there exist a $\psi \in \mathcal{H}$ with $\psi \neq 0$ and

$$C_\psi := \frac{1}{(\psi, \psi)_\mathcal{H}} \int_G \left|(\mathcal{R}_g \psi, \psi)_\mathcal{H}\right|^2 \mathrm{d}\mu_G(g) < \infty. \tag{5.1}$$

If the group representation is unitary, irreducible and square integrable, then the functional Hilbert space will always be a closed subspace of $\mathbb{L}_2(G)$, whenever the wavelet $\psi \in \mathcal{H}$ satisfies (5.1). This was first shown by Grossman, Morlet and Paul [GMP] in 1985. In this report we will give a new proof of this theorem. For our proof we need an extension of the Schur's lemma, which is presented in Appendix A. Moreover, we need a lemma which is valid for all bounded representations. Hence, let $\mathcal{R}$ be a bounded representation of a group G in a Hilbert space $\mathcal{H}$. Let $\psi \in \mathcal{H}$. First define the linear mapping $\mathcal{W}_\psi$ as

$$\mathcal{W}_\psi = W_\psi|_\mathcal{D}\,, \tag{5.2}$$

where $\mathcal{D} = \{f \in \mathcal{H} \mid W_\psi f \in \mathbb{L}_2(G)\}$.

Lemma 5.1 *Let $\psi \in \mathcal{H}$. The wavelet transform $\mathcal{W}_\psi : \mathcal{D} \to \mathbb{L}_2(G)$ is a closed operator.*

Proof:
Let $f_n \to f$ in $\mathcal{H}$ and $\mathcal{W}_\psi f_n \to \Phi$, for some $\Phi \in \mathbb{L}_2(G)$. Then we have to show that $f \in \mathcal{D}$ and $\mathcal{W}_\psi f = \Phi$. The group G is locally compact, therefore it is sufficient to show that for *any* compact $\Omega \subset G$

$$\int_\Omega |W_\psi f - \Phi|^2 \, \mathrm{d}\mu_{\mathrm{G}} = 0 \ ,$$

to conclude that $W_\psi f = \Phi$.
Note that by boundedness of the representation

$$\big|\big(W_\psi f\big)(g) - \big(W_\psi f_n\big)(g)\big| = |(\mathcal{R}_g\psi, f - f_n)_\mathcal{H}| \leq \|\mathcal{R}\|\|\psi\|_\mathcal{H}\|f - f_n\|_\mathcal{H},$$

for all $g \in G$ and $n \in \mathbb{N}$.
Now the statement follows from

$$\begin{aligned}
&\int_\Omega |W_\psi f - \Phi|^2 \ \mathrm{d}\mu_G(g) \\
&\leq 2\int_\Omega |W_\psi f - \mathcal{W}_\psi f_n|^2 \, \mathrm{d}\mu_G + 2\int_\Omega |\Phi - \mathcal{W}_\psi f_n|^2 \, \mathrm{d}\mu_G \\
&\leq 2\mu(\Omega) \sup_{g \in G} \big|\big(W_\psi f\big)(g) - \big(\mathcal{W}_\psi f_n\big)(g)\big|^2 + 2\int_\Omega |\Phi - \mathcal{W}_\psi f_n|^2 \, \mathrm{d}\mu_G \\
&\leq 2\mu(\Omega)\|\mathcal{R}\|^2\|\psi\|^2_\mathcal{H}\|f_n - f\|^2_\mathcal{H} \ + 2\int_\Omega |\Phi - \mathcal{W}_\psi f_n|^2 \, \mathrm{d}\mu_G
\end{aligned}$$

for all $n \in \mathbb{N}$. As $f_n \to f$ we find $W_\psi f = \Phi$ on Ω. Therefore $f \in \mathcal{D}$ and $\mathcal{W}_\psi f = \Phi$. □

The left regular representation $\mathcal{L}$ of G on $\mathbb{L}_2(G)$ is defined by

$$\mathcal{L}_h f(g) = f(h^{-1}g), \tag{5.3}$$

for all $h \in G$, $f \in \mathbb{L}_2(G)$ and almost every $g \in G$.

We now prove a Theorem by Morlet, Grossmann and Paul.

Theorem 5.2 (The Wavelet Reconstruction Theorem) *Let $\mathcal{U}$ be an irreducible, unitary and square integrable representation of a locally compact group G on a Hilbert space $\mathcal{H}$. Let $\psi \in \mathcal{H}$ such that (5.1) holds. Then the wavelet transform is a linear isometry (up to a constant) from the Hilbert space $\mathcal{H}$ onto a closed subspace $\mathbb{C}_K^G$ of $\mathbb{L}_2(G, \mathrm{d}\mu)$:*

$$\|\mathcal{W}_\psi f\|^2_{\mathbb{L}_2(G)} = C_\psi \|f\|^2_{\mathcal{H}}. \tag{5.4}$$

Here, the space $\mathbb{C}_K^G$ is the functional Hilbert space with reproducing kernel

$$K_\psi(g, g') = \frac{1}{C_\psi}(\mathcal{U}_g\psi, \mathcal{U}_{g'}\psi). \tag{5.5}$$

Proof:
The domain $\mathcal{D}$ of operator $\mathcal{W}_\psi : \mathcal{D} \to \mathbb{L}_2(G)$ is by definition the set of all $f \in \mathcal{H}$ for which $W_\psi f \in \mathbb{L}_2(G)$. By assumption $\psi \in \mathcal{D}$. Moreover, it follows by the left-invariance of $\mathrm{d}\mu_G$ that the span $\mathcal{S}_\psi = \langle \{\mathcal{U}_g\psi \mid g \in G\}\rangle$ of the orbit of ψ, is a subspace of $\mathcal{D}$, since for any $\eta = \mathcal{U}_h\psi$, we have

$$\begin{aligned}
\int_G \left|\left(W_\psi \eta\right)(g)\right|^2 \,\mathrm{d}\mu_G(g) &= \int_G |(\mathcal{U}_g\psi, \mathcal{U}_h\psi)|^2 \,\mathrm{d}\mu_G(g) \\
&= \int_G |(\mathcal{U}_{h^{-1}g}\psi, \psi)|^2 \;\mathrm{d}\mu_G(g) \\
&= \int_G |(\mathcal{U}_g\psi, \psi)|^2 \;\mathrm{d}\mu_G(g) \\
&= C_\psi |(\psi, \psi)|^2 = C_\psi |(\mathcal{U}_h\psi, \mathcal{U}_h\psi)|^2 < \infty.
\end{aligned}$$

Obviously $\mathcal{S}_\psi$ is invariant under $\mathcal{U}$ and since $\mathcal{U}$ was assumed to be irreducible, this space is dense in $\mathcal{H}$. By Lemma 5.1 operator $\mathcal{W}_\psi$ is closed, since a unitary representation is bounded. So, $\mathcal{W}_\psi$ is a closed densely defined operator and therefore operator $\mathcal{W}_\psi^*\mathcal{W}_\psi$ is self-adjoint, by a theorem of J. von Neumann (see [Y, Theorem VII.3.2]).

It is easy to see that

$$\left(\mathcal{W}_\psi \mathcal{U}_h f\right)(g) = (\mathcal{U}_g\psi, \mathcal{U}_h f)_{\mathcal{H}} = (\mathcal{U}_{h^{-1}}\mathcal{U}_g\psi, f)_{\mathcal{H}} = (\mathcal{U}_{h^{-1}g}\psi, f)_{\mathcal{H}},$$

for all $g, h \in G$ and $f \in \mathcal{H}$. Therefore, if $f \in \mathcal{D}$ then $\mathcal{U}_h f \in \mathcal{D}$ and $\mathcal{W}_\psi\mathcal{U}_h f = \mathcal{L}_h\mathcal{W}_\psi f$. Hence $\mathcal{W}_\psi\mathcal{U}_h = \mathcal{L}_h\mathcal{W}_\psi$. For the adjoint operator the same is true. If $\Phi \in \mathcal{D}(\mathcal{W}_\psi^*)$, $f \in \mathcal{D}(\mathcal{W}_\psi)$ and $h \in G$

$$\begin{aligned}
(\mathcal{L}_h\Phi, \mathcal{W}_\psi f)_{\mathbb{L}_2(G)} &= (\Phi, \mathcal{L}_{h^{-1}}\mathcal{W}_\psi f)_{\mathbb{L}_2(G)} = (\Phi, \mathcal{W}_\psi\mathcal{U}_{h^{-1}} f)_{\mathbb{L}_2(G)} \\
&= (\mathcal{W}_\psi^*\Phi, \mathcal{U}_{h^{-1}} f)_{\mathcal{H}} = (\mathcal{U}_h\mathcal{W}_\psi^*\Phi, f)_{\mathcal{H}}.
\end{aligned}$$

So for all $\Phi \in \mathcal{D}(\mathcal{W}_\psi^*)$ we have $\mathcal{L}_h\Phi \in \mathcal{D}(\mathcal{W}_\psi^*)$ and furthermore $\mathcal{W}_\psi^*\mathcal{L}_h = \mathcal{U}_h\mathcal{W}_\psi^*$. In particular $\mathcal{W}_\psi^*\mathcal{W}_\psi\mathcal{U}_g = \mathcal{U}_g\mathcal{W}_\psi^*\mathcal{W}_\psi$ for all $g \in G$ and $\mathcal{D}(\mathcal{W}_\psi^*\mathcal{W}_\psi)$ is invariant under $\mathcal{U}$.
By the topological version of Schur's lemma, Theorem A.1, it now follows that there is a $c \in \mathbb{C}$ such that $\mathcal{W}_\psi^*\mathcal{W}_\psi = cI$ on $\mathcal{D}(\mathcal{W}_\psi^*\mathcal{W}_\psi)$. But because $\mathcal{W}_\psi^*\mathcal{W}_\psi$ is closed and bounded on $\mathcal{D}(\mathcal{W}_\psi^*\mathcal{W}_\psi)$ we can conclude from the closed graph theorem that $\mathcal{W}_\psi^*\mathcal{W}_\psi = c\,I$ on the entire Hilbert space $\mathcal{H}$. From $\|\mathcal{W}_\psi\psi\|^2 = C_\psi\|\psi\|^2$ it follows that $c = C_\psi$. □

5.2 Semi-direct products

In this section we will work out the wavelet construction for the special case $\mathcal{H} = \mathbb{L}_2(S, \mu_S)$ with S some locally compact abelian group. Here μ_S is a left invariant Haar measure. Given a locally compact group T we will define a natural unitary representation (not necessarily irreducible) of the semi-direct product $S \rtimes T$ on $\mathbb{L}_2(S)$. From this unitary representation a wavelet transformation and a corresponding functional Hilbert space can be constructed for a suitable choice of $\psi \in \mathbb{L}_2(S)$.

5.2.1 Introduction

We first recall the notion of the semi-direct product of two groups. We also mention some elementary topics from harmonic analysis.

Definition 5.3 *Let S and T be groups and let $\tau : T \to Aut(S)$ be a group homomorphism. The **semi-direct product** $S \rtimes_\tau T$ is defined to be the group with underlying set $S \times T$ and group operation*

$$(s,t)(s',t') = (s\tau(t)s', tt'), \tag{5.6}$$

for all $(s,t),(s',t') \in S \times T$

From now on we only consider a group G which is a semidirect product $G = (S,+) \rtimes (T,\cdot)$ for some locally compact group T and a group homomorphism $\tau : T \to \mathrm{Aut}(S)$ such that

$$(s,t) \mapsto \tau(t)s \tag{5.7}$$

is a continuous mapping from $S \rtimes T$ onto S. Since S and T are locally compact, G is also locally compact. Note that $\tilde{S} = \{(s,e_2) \in G \mid s \in S\}$ and $\tilde{T} = \{(e_1,t) \in G \mid t \in T\}$ are closed subgroups of G.

A locally compact group has a (left invariant) Haar measure. If we talk about a measure on the group or integration over the group, this is always with respect to a (left invariant) Haar measure. Let μ_T, μ_S, μ_G be Haar measures of resp. T, S, G. There exists a relation which relates these Haar measures. To this end, we need the notion of **modular function**.

Definition 5.4 *Let H be a locally compact group and μ a Haar measure on H. Then for each $h \in H$*

$$\mu_h(E) = \mu(Eh), \qquad E \in \text{ Bor } (H), \tag{5.8}$$

defines a Haar measure, where Bor (H) is the set of Borel sets. Because all Haar measures are equal up to a constant, there exists for all $h \in H$ a $\Delta_H(h) > 0$ such that

$$\mu_h = \Delta_H(h)\mu. \tag{5.9}$$

*The function $\Delta_H : h \mapsto \Delta_H(h)$ on H is called the **modular function**. The modular function is a continuous homomorphism from H into $(\mathbb{R}^+, \cdot)$.*

Now $\tilde{T} = \{(e, t) \in G \mid t \in T\}$ is subgroup of G and it has a Haar measure $\mu_{\tilde{T}}$ corresponding to μ_T. Starting from μ_S, μ_T, the Haar measure μ_G can be chosen such that

$$\int_G f(g) \, \mathrm{d}\mu_G(g) = \int_S \left\{ \int_T f(s, t)\rho^{-1}(t) \, \mathrm{d}\mu_T(t) \right\} \mathrm{d}\mu_S(s), \tag{5.10}$$

for all $f \in \mathbb{L}_1(G)$. Furthermore,

$$\int_S f(\tau(t)^{-1}s) \, \mathrm{d}\mu_S(s) = \rho(t) \int_S f(s) \, \mathrm{d}\mu_S(s), \tag{5.11}$$

for all $t \in T$ and $f \in \mathbb{L}_1(S)$. Here $\rho(t) = \frac{\Delta_{\tilde{T}}(e,t)}{\Delta_G(e,t)}$. It follows that ρ is continuous and strictly positive. For further details, we refer to [R, (8.1.12) and (8.1.10)] .

In the case $S = \mathbb{R}^n$ we simply get $\rho(t) = |\det \tau(t)|$, which can easily be proved by the transformation of variables formula.

We define in a natural way a representation of the semi-direct product $S \rtimes T$ in $\mathbb{L}_2(S)$. Define $\mathcal{U} : G \to \mathcal{B}(\mathbb{L}_2(S))$ as follows

$$\mathcal{U}_{(s,t)} f = \mathcal{T}_s \mathcal{P}_t f, \tag{5.12}$$

where

$$\big(\mathcal{T}_{s_1} f\big)(s_2) = f(s_2 - s_1), \tag{5.13}$$

$$\big(\mathcal{P}_t f\big)(s) = \rho^{-\frac{1}{2}}(t) f(\tau(t)^{-1}s), \tag{5.14}$$

for all $s_1 \in S$ and $t \in T$ and almost all $s_2 \in S$. Note that $\mathcal{P}_t f \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$ for all $t \in T$, if $f \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$. It is easily verified that $\mathcal{U}$ is a unitary representation. Moreover, we will prove that it is cyclic.

5.2.2 The wavelet transformation

We recall that, with the use of the unitary representation $\mathcal{U}$, for any $\psi \in \mathcal{H}$ we now can define the unitary map $W_\psi : \overline{\langle V_\psi \rangle} \to \mathbb{C}_K^G$ as formulated in Theorem 3.1. In this section we set the wavelet transformation in a useful different form, making use of Fourier transformation for abelian groups. Let $f \in \mathbb{L}_2(S)$ and $\hat{S}$ be the dual group. Then $\hat{S}$ exists of all continuous homomorphisms of S onto the circle group. Then define the Fourier transform as

$$\big(\mathcal{F}f\big)(\gamma) = \int_S f(s)\overline{\langle s, \gamma\rangle}\mathrm{d}\mu_S(s), \tag{5.15}$$

for all $\gamma \in \hat{S}$ and $f \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$, where $\langle \cdot, \cdot \rangle$ stands for the dual pairing, $\langle s, \gamma \rangle = \gamma(s)$ for all $s \in S$ and $\gamma \in \hat{S}$. This defines, after extension, a unitary mapping from $\mathbb{L}_2(S)$ onto $\mathbb{L}_2(\hat{S}, \mathrm{d}\mu_{\hat{S}}(\gamma))$ where the left Haar measure $\mu_{\hat{S}}(\gamma)$ related to μ_S. The inversion is given by

$$\big(\mathcal{F}^{-1}F\big)(s) = \int_{\hat{S}} F(\gamma)\langle s, \gamma\rangle \mathrm{d}\mu_{\hat{S}}(\gamma), \tag{5.16}$$

for all $F \in \mathbb{L}_1(\hat{S}) \cap \mathbb{L}_2(\hat{S})$. For a detailed discussion of the Fourier transformation on locally compact abelian groups, see for example [Fo].

Lemma 5.5 *Let $\psi \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$. Then, $\big(W_\psi f\big)(\cdot, t) \in \mathbb{L}_2(S)$ for all $f \in \mathbb{L}_2(S)$ and $t \in T$.*

Proof:
Let $f \in \mathbb{L}_2(S)$ and $t \in T$. Then

$$(\mathcal{T}_s \mathcal{P}_t \psi, f)_{\mathbb{L}_2(S)} = \int_S \overline{(\mathcal{P}_t\psi)(s'-s)} f(s') d\mu_S(s'),$$

for all s. So we arrive at a convolution. A convolution of a $\mathbb{L}_1$ function with a $\mathbb{L}_2$ function is again a $\mathbb{L}_2$ function. See [Fo, Proposition 2.39]). $\square$

This means that for all elements Φ of our functional Hilbert space $\mathbb{C}_K^G$, the function $\Phi(\cdot, t)$ will be in $\mathbb{L}_2(S)$ for fixed $t \in T$. Hence, the Fourier transform of $\Phi(\cdot, t)$ is well-defined.

Now use Fourier transformation and Plancherel to get a different presentation of the wavelet transform of an arbitrary function $f \in \mathbb{L}_2(S)$

$$\begin{aligned} \big(W_\psi f\big)(s,t) &= (\mathcal{T}_s\mathcal{P}_t\psi, f)_{\mathbb{L}_2(S)} = (\mathcal{F}\mathcal{T}_s\mathcal{P}_t\psi, \mathcal{F}f)_{\mathbb{L}_2(\hat{S})} \\ &= (\overline{\langle s, \cdot\rangle}\mathcal{F}\mathcal{P}_t\psi, \mathcal{F}f)_{\mathbb{L}_2(\hat{S})} = \big(\mathcal{F}^{-1}\left[\overline{\mathcal{F}\mathcal{P}_t\psi}\mathcal{F}f\right]\big)(s), \end{aligned} \tag{5.17}$$

for all $s \in S$ and $t \in T$. We notice that $\mathcal{F}f \in \mathbb{L}_2(\hat{S})$ and $\mathcal{F}\mathcal{P}_t\psi \in \mathbb{L}_\infty(\hat{S})$ for all $t \in T$. Hence $\overline{\mathcal{F}\mathcal{P}_t\psi}\mathcal{F}f \in \mathbb{L}_2(\hat{S})$ and $\big(W_\psi f\big)(\cdot, t) \in \mathbb{L}_2(S)$ for all $t \in T$. Moreover, since $\overline{\mathcal{F}\mathcal{P}_t\psi}\mathcal{F}f \in \mathbb{L}_1(\hat{S})$ we get $\big(W_\psi f\big)(\cdot, t) \in \mathcal{C}_0(S)$ for all $t \in T$ and $f \in \mathbb{L}_2(S)$. With $\mathcal{C}_0(S)$ we denote the space of continuous functions on S which vanish at infinity.

Lemma 5.6 *Let $\psi \in \mathbb{L}_2(S)$. Suppose*

$$\mu_{\hat{S}}\Big(\{\gamma \in \hat{S} \mid \forall t \in T\big[\big(\mathcal{F}\mathcal{P}_t\psi\big)(\gamma) = 0\big]\}\Big) = 0.$$

Then ψ is a cyclic vector.

Proof:
Remark that the measure does not depend on the representant. Let $f \in V_\psi^\perp$. Then $W_\psi f = 0$ by the remark after Theorem 3.1. Hence, $\overline{\mathcal{F}\mathcal{P}_t\psi}\mathcal{F}f = 0$ for all $t \in T$, by (5.17). Therefore, $\mathcal{F}f = 0$ by the assumption. □

Corollary 5.7 *Let $\psi \in \mathbb{L}_2(S)$. If $\mathcal{F}\psi \neq 0$ a.e., then ψ is a cyclic vector. Moreover, if S is metrizable then the representation $\mathcal{U}$ is cyclic.*

Proof:
The first statement follows immediately from Lemma 5.6.
If the group S is metrizable, then $\hat{S}$ is σ-compact by [R, Thm. 4.2.7]. Therefore, there exists a $\psi \in \mathbb{L}_2(S)$ such that $\mathcal{F}\psi > 0$, by the σ-compactness of $\hat{S}$. The conclusion now follows from the first statement. □

5.2.3 Alternative description of $\mathbb{C}_K^{S \rtimes T}$ for non-vanishing wavelets

In this subsection we will derive an alternative description of the functional Hilbert space for some special wavelets.

Since W_ψ is unitary,

$$(\Phi, \Psi)_{\mathbb{C}_K^{S \rtimes T}} = (W_\psi^{-1}\Phi, W_\psi^{-1}\Psi)_{\mathbb{L}_2(S)}, \tag{5.18}$$

for all $\Phi, \Psi \in \mathbb{C}_K^{\mathbb{R}^2 \rtimes T}$. Hence, if we are able to find an explicit expression for W_ψ^{-1} we can derive an alternative description of the inner product on the functional Hilbert space. To this end, equation (5.17) appears to be very useful.

In this subsection we will assume that the wavelet $\psi \in \mathbb{L}_2(S)$ satisfies the condition

$$\mu_{\hat{S}}\big(\{\gamma \in \hat{S} \mid \big(\mathcal{F}\mathcal{P}_t\psi\big)(\gamma) = 0\}\big) = 0, \tag{5.19}$$

for all $t \in T$. Remark that the measure of the set does not depend on the choice of the representant. Such wavelets will be called **non-vanishing wavelets**. By Theorem 3.2 and Lemma 5.6, the wavelet is cyclic and the wavelet transformation W_ψ defines a unitary mapping from $\mathbb{L}_2(S)$ onto $\mathbb{C}_K^{S \rtimes T}$.

For non-vanishing wavelets there exists a simple inversion formula.

Lemma 5.8 *Let $f \in \mathbb{L}_2(S)$ and $\Phi = W_\psi f$. Then,*

$$f = \mathcal{F}^{-1}\left[\overline{\left(\mathcal{F}\mathcal{P}_t\psi\right)^{-1}}\mathcal{F}[\Phi(\cdot,t)]\right] \tag{5.20}$$

for all $t \in T$.

Proof:
From (5.19) it follows that $\left(\mathcal{F}\mathcal{P}_t\psi\right)^{-1}$ exists almost everywhere on $S \rtimes T$. The lemma now straightforwardly follows from (5.17). □

Although (5.20) leads to an expression for W_ψ^{-1}, it appears to be desirable to write W_ψ^{-1} as

$$W_\psi^{-1}\Phi = \int_T \mathcal{F}^{-1}\left[\overline{\left(\mathcal{F}\mathcal{P}_t\psi\right)^{-1}}\mathcal{F}[\Phi(\cdot,t)]\right]\ A(t)\rho^{-1}(t)\ \mathrm{d}\mu_T(t), \tag{5.21}$$

for all $\Phi \in \mathbb{C}_K^{S \rtimes T}$, where $A : T \to \mathbb{R}^+ \cup \{0\}$ is a function such that

$$\int_T A(t)\rho^{-1}(t)\ \mathrm{d}\mu_T(t) = 1. \tag{5.22}$$

The main advantage is that (5.21) takes in account all $t \in T$, whilst (5.20) forces us to choose a $t \in T$. Especially when we imbed $\mathbb{C}_K^{S \rtimes T}$ as a closed subspace of a larger space, expression (5.21) turns out to be more useful.

Theorem 5.9 *Let $\Phi, \Psi \in \mathbb{C}_K^{S \rtimes T}$. Then,*

$$(\Phi,\Psi)_{\mathbb{C}_K^{S \rtimes T}} = \int_{\hat{S}}\int_T \overline{\mathcal{F}[\Phi(\cdot,t)](\gamma)}\mathcal{F}[\Psi(\cdot,t)](\gamma)\ \frac{A(t)}{\left|\left(\mathcal{F}\mathcal{P}_t\psi\right)(\gamma)\right|^2\rho(t)}\ \mathrm{d}\mu_T(t)\mathrm{d}\mu_{\hat{S}}(\gamma). \tag{5.23}$$

Proof:

$$\begin{aligned}
(\Phi,\Psi)_{\mathbb{C}_K^{S \rtimes T}} &= (W_\psi^{-1}\Phi, W_\psi^{-1}\Psi)_{\mathbb{L}_2(S)} = (f, W_\psi^{-1}\Psi)_{\mathbb{L}_2(S)} = (\mathcal{F}f, \mathcal{F}W_\psi^{-1}\Psi)_{\mathbb{L}_2(\hat{S})} \\
&= \int_{\hat{S}}\int_T \overline{\left(\mathcal{F}f\right)(\gamma)}\mathcal{F}[\Psi(\cdot,t)](\gamma)\ \frac{A(t)}{\overline{\left(\mathcal{F}\mathcal{P}_t\psi\right)(\gamma)}\rho(t)}\ \mathrm{d}\mu_T(t)\mathrm{d}\mu_{\hat{S}}(\gamma) \\
&= \int_{\hat{S}}\int_T \left(\mathcal{F}\mathcal{P}_t\psi\right)(\gamma)\overline{\left(\mathcal{F}f\right)(\gamma)}\mathcal{F}[\Psi(\cdot,t)](\gamma)\ \frac{A(t)}{|\left(\mathcal{F}\mathcal{P}_t\psi\right)(\gamma)|^2\rho(t)}\ \mathrm{d}\mu_T(t)\mathrm{d}\mu_{\hat{S}}(\gamma) \\
&= \int_{\hat{S}}\int_T \overline{\mathcal{F}[\Phi(\cdot,t)](\gamma)}\mathcal{F}[\Psi(\cdot,t)](\gamma)\ \frac{A(t)}{\left|\left(\mathcal{F}\mathcal{P}_t\psi\right)(\gamma)\right|^2\rho(t)}\ \mathrm{d}\mu_T(t)\mathrm{d}\mu_{\hat{S}}(\gamma).
\end{aligned}$$

$\square$

We summarize the previous in the following theorem.

Theorem 5.10 *Let $\psi \in \mathbb{L}_2(S)$ be a wavelet which satisfies condition (5.19) Then the wavelet transformation W_ψ defined by*

$$\big(W_\psi f\big)(s,t) = (\mathcal{T}_s \mathcal{P}_t \psi, f)_{\mathbb{L}_2(S)}, \quad f \in \mathbb{L}_2(S), \ \ (s,t) \in S \rtimes T, \tag{5.24}$$

is a unitary mapping from $\mathbb{L}_2(S)$ to $\mathbb{C}_K^{S\rtimes T}$. Here, $\mathbb{C}_K^{S\rtimes T}$ is the functional Hilbert space with reproducing kernel

$$K(g,h) = (\mathcal{U}_g \psi, \mathcal{U}_h \psi)_{\mathbb{L}_2(S)} = (\mathcal{U}_{h^{-1}g}\psi, \psi)_{\mathbb{L}_2(S)}, \tag{5.25}$$

for all $g, h \in S \rtimes T$. Let $A : T \to \mathbb{R}^+ \cup \{0\}$ be a function such that

$$\int_T A(t)\rho^{-1}(t) \ \mathrm{d}\mu_T(t) = 1. \tag{5.26}$$

The inner product on $\mathbb{C}_K^{S\rtimes T}$ can be written as

$$(\Phi, \Psi)_{\mathbb{C}_K^{S\rtimes T}} = \int_{\hat{S}} \int_T \overline{\mathcal{F}[\Phi(\cdot,t)](\gamma)} \mathcal{F}[\Psi(\cdot,t)](\gamma) \ \frac{A(t)}{\left|\big(\mathcal{F}\mathcal{P}_t\psi\big)(\gamma)\right|^2 \rho(t)} \ \mathrm{d}\mu_T \mathrm{d}(t) \mu_{\hat{S}}(\gamma), \tag{5.27}$$

for all $\Phi, \Psi \in \mathbb{C}_K^{S\rtimes T}$.

Remark that A is not unique.

We can imbed $\mathbb{C}_K^{S\rtimes T}$ in a larger space $\mathbb{H}_{\psi,A}$, such that $\mathbb{C}_K^{S\rtimes T}$ is a closed subspace of $\mathbb{H}_{\psi,A}$. Define the space $\mathbb{H}_{\psi,A}$ as

$$\begin{aligned}\mathbb{H}_{\psi,A} = \Big\{ \Phi \in \mathbb{C}^{S\rtimes T} \ \ \Big| \ \ & \Phi(\cdot,t) \in \mathbb{L}_2(S) \text{ for almost all } t \in T, \\ & \int_{\hat{S}} \int_T \left|\mathcal{F}[\Phi(\cdot,t)](\gamma)\right|^2 \frac{A(t)}{\left|\big(\mathcal{F}\mathcal{P}_t\psi\big)(\gamma)\right|^2 \rho(t)} \ \mathrm{d}\mu_T(t) \mathrm{d}\mu_{\hat{S}}(\gamma) < \infty \Big\},\end{aligned} \tag{5.28}$$

with the inner product

$$(\Phi, \Psi)_{\mathbb{H}_{\psi,A}} = \int_{\hat{S}} \int_T \overline{\mathcal{F}[\Phi(\cdot,t)](\gamma)} \mathcal{F}[\Psi(\cdot,t)](\gamma) \ \frac{A(t)}{\left|\big(\mathcal{F}\mathcal{P}_t\psi\big)(\gamma)\right|^2 \rho(t)} \ \mathrm{d}\mu_T(t) \mathrm{d}\mu_{\hat{S}}(\gamma), \tag{5.29}$$

for all $\Phi, \Psi \in \mathbb{H}_{\psi,A}$.

Theorem 5.11 *$\mathbb{C}_K^{S\rtimes T}$ is a closed subspace of $\mathbb{H}_{\psi,A}$. The operator $\Phi \mapsto [g \mapsto (K(\cdot,g),\Phi)_{\mathbb{H}_{\psi,A}}]$ is the projection operator from $\mathbb{H}_{\psi,A}$ onto $\mathbb{C}_K^{S\rtimes T}$.*

Proof:
It is obvious that $\mathbb{C}_K^{S\rtimes T}$ is a closed subspace of $\mathbb{H}_{\psi,A}$. Let $\Phi \in \mathbb{H}_{\psi,A}$. Then it can be written as

$$\Phi = \Phi_1 + \Phi_2,$$

with $\Phi_1 \in \mathbb{C}_K^{S\rtimes T}$ and $\Phi_2 \in (\mathbb{C}_K^{S\rtimes T})^\perp$. Then for all $g \in S \rtimes T$

$$(K(\cdot,g),\Phi)_{\mathbb{H}_{\psi,A}} = (K(\cdot,g),\Phi_1)_{\mathbb{H}_{\psi,A}} + (K(\cdot,g),\Phi_2)_{\mathbb{H}_{\psi,A}} = \Phi_1(g).$$

Therefore Φ is mapped to Φ_1. □

5.2.4 Alternative description of $\mathbb{C}_K^{S\rtimes T}$ for admissible wavelets

In this subsection we will replace condition (5.19) by another condition. Also in this case an alternative description can be given, using (5.17) and (5.18).

Definition 5.12 *Let $\psi \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$. Define $M_\psi : \hat{S} \to [0,\infty) \cup \{\infty\}$ as*

$$M_\psi(\gamma) = \int_T \frac{\left|\left(\mathcal{F}\mathcal{P}_t\psi\right)(\gamma)\right|^2}{\rho(t)}\, d\mu_T(t). \tag{5.30}$$

The function M_ψ is a substitute for the constant C_ψ given by (5.1) in the irreducible case. We note that $\mathcal{F}\mathcal{P}_t\psi \in \mathcal{C}_0(\hat{S})$ for all $t \in T$, so M_ψ can be defined pointwise.

Definition 5.13 *We call $\psi \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$ an **admissible wavelet** iff*

$$0 < M_\psi < \infty \text{ a.e.}$$

In this section we will assume that $\psi \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$ an admissible wavelet. All the admissible wavelets are cyclic, so lead to a unitary mapping from the entire space $\mathbb{L}_2(S)$ onto $\mathbb{C}_K^{S\rtimes T}$ by Theorem 3.2. This is shown in the following lemma.

Lemma 5.14 *Every admissible wavelet is a cyclic wavelet, i.e. $\overline{\langle V_\psi \rangle} = \mathbb{L}_2(S)$.*

Proof:
If $f \in \mathbb{L}_2(S)$, then with (5.17) we get

$$f \in \langle V_\psi \rangle^\perp \quad \Leftrightarrow \Big(\forall t \in T\Big[|\overline{\mathcal{F}\mathcal{P}_t\psi}\mathcal{F}f|^2 = 0 \text{ a.e. on } S\Big]\Big). \tag{5.31}$$

Let $f \in \langle V_\psi \rangle^\perp$. Then

$$M_\psi |\mathcal{F}f|^2 = \int_T \frac{\left|\overline{\mathcal{F}\mathcal{P}_t\psi}\mathcal{F}f\right|^2}{\rho(t)} \mathrm{d}\mu_T(t) = 0 \quad \text{a.e. on } \hat{S}.$$

Because ψ is an admissible wavelet, the function $M_\psi > 0$ a.e.. Hence $|\mathcal{F}f|^2 = 0$ a.e. and therefore $f = 0$. □

Using the function M_ψ we can also give an expression for W_ψ^{-1}.

Lemma 5.15 *Let $\psi \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$ be an admissible wavelet. Let $f \in \mathbb{L}_2(S)$. Then*

$$f = W_\psi^{-1}\Phi = \mathcal{F}^{-1}\Big(\int_T \mathcal{F}[\Phi(\cdot,t)]\mathcal{F}\mathcal{P}_t\psi \; M_\psi^{-1}\rho^{-1}(t) \; \mathrm{d}\mu_T(t)\Big), \tag{5.32}$$

where $\Phi = W_\psi f \in \mathbb{C}_K^{S \rtimes T}$

Proof:
We recall that $0 < M_\psi < \infty$ a.e. on $\hat{S}$, hence also $0 < M_\psi^{-\frac{1}{2}} < \infty$ a.e. on $\hat{S}$. The lemma now easily follows from (5.17) since

$$\begin{aligned}
&\mathcal{F}^{-1}\Big(\int_T \mathcal{F}[\Phi(\cdot,t)]\mathcal{F}\mathcal{P}_t\psi \; M_\psi^{-1}\rho^{-1}(t) \; \mathrm{d}\mu_T(t)\Big) \\
&\qquad = \mathcal{F}^{-1}\Big(M_\psi^{-1} \int_T \mathcal{F}f \; |\mathcal{F}\mathcal{P}_t\psi|^2 \rho^{-1}(t) \; \mathrm{d}\mu_T(t)\Big) \\
&\qquad = \mathcal{F}^{-1}\big(M_\psi^{-1} M_\psi \mathcal{F}f\big) = f.
\end{aligned}$$

□

We are now able to give an alternative description of the norm of $\mathbb{C}_K^{S \rtimes T}$ using (5.17) and (5.18) and the previous lemma.

Theorem 5.16 *If $\Phi \in \mathbb{C}_K^{S \rtimes T}$ then $M_\psi^{-\frac{1}{2}}\mathcal{F}[\Phi(\cdot,t)] \in \mathbb{L}_2(\hat{S})$ for almost every $t \in T$. Moreover,*

$$\|\Phi\|^2_{\mathbb{C}_K^{S \rtimes T}} = \int_{\hat{S}} \int_T \left|\mathcal{F}[\Phi(\cdot,t)](\gamma)\right|^2 \; M_\psi^{-1}(\gamma)\rho^{-1}(t) \; d\mu_T(t)\mathrm{d}\mu_{\hat{S}}(\gamma). \tag{5.33}$$

Proof:
Let $\Phi \in \mathbb{C}_K^{S\rtimes T}$. Then there exists a function $f \in \mathbb{L}_2(S)$ such that $W_\psi f = \Phi$.

$$\begin{aligned}
(\Phi,\Phi)^2_{\mathbb{C}_K^{S\rtimes T}} &= (f, W_\psi^{-1}\Phi)_{\mathbb{L}_2(S)} = (\mathcal{F}f, \mathcal{F}W_\psi^{-1}\Phi)_{\mathbb{L}_2(S)} \\
&= \int_{\hat{S}} \overline{\mathcal{F}f}(\gamma) \int_T \mathcal{F}[\Phi(\cdot,t)](\gamma)\big(\mathcal{F}\mathcal{P}_t\psi\big)(\gamma)\ M_\psi^{-1}(\gamma)\rho^{-1}(t)\ \mathrm{d}\mu_T(t)\mathrm{d}\mu_{\hat{S}}(\gamma) \\
&= \int_{\hat{S}} \int_T \mathcal{F}[\Phi(\cdot,t)](\gamma)\overline{\mathcal{F}f(\gamma)}\big(\mathcal{F}\mathcal{P}_t\psi\big)(\gamma)\ M_\psi^{-1}(\gamma)\rho^{-1}(t)\ \mathrm{d}\mu_T(t)\mathrm{d}\mu_{\hat{S}}(\gamma) \\
&= \int_{\hat{S}} \int_T \mathcal{F}[\Phi(\cdot,t)](\gamma)\overline{\mathcal{F}[\Phi(\cdot,t)](\gamma)}\ M_\psi^{-1}(\gamma)\rho^{-1}(t)\ \mathrm{d}\mu_T(t)\mathrm{d}\mu_{\hat{S}}(\gamma)
\end{aligned}$$

Therefore,

$$\int_{\hat{S}} \int_T \big|\mathcal{F}[\Phi(\cdot,t)](\gamma)\big|^2\ M_\psi^{-1}\rho^{-1}(t)\ \mathrm{d}\mu_T(t)\mathrm{d}\mu_{\hat{S}}(\gamma) = \|\Phi\|^2_{\mathbb{C}_K^{S\rtimes T}}.$$

The integrand is positive, so by a theorem of Fubini we are allowed to change integrals and in particular

$$\int_{\hat{S}} |M_\psi \mathcal{F}[\Phi(\cdot,t)]|^2\ \mathrm{d}\mu_{\hat{S}}(\gamma) < \infty,$$

for almost all $t \in T$. Therefore $M_\psi^{-\frac{1}{2}}\mathcal{F}[\Phi(\cdot,t)] \in \mathbb{L}_2(\hat{S})$ for almost all $t \in T$. $\square$

Because of Lemma 5.16 and (5.10) we can define the linear operator $T_{M_\psi} : \mathbb{C}_K^{S\rtimes T} \to \mathbb{L}_2(S \rtimes T)$ by

$$\left(T_{M_\psi}\Phi\right)(s,t) = \left(\mathcal{F}^{-1}[M_\psi^{-\frac{1}{2}}\ \mathcal{F}[\Phi(\cdot,t)]]\right)(s), \tag{5.34}$$

for almost all $(s,t) \in S \rtimes T$.

We summarize the previous in the following theorem.

Theorem 5.17 *Let $\psi \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$ be an admissible wavelet. Then the wavelet transformation W_ψ defined by*

$$\big(W_\psi f\big)(s,t) = (\mathcal{T}_s\mathcal{P}_t\psi, f)_{\mathbb{L}_2(S)}, \quad f \in \mathbb{L}_2(S),\ \ (s,t) \in S \rtimes T, \tag{5.35}$$

is a unitary mapping from $\mathbb{L}_2(S)$ onto $\mathbb{C}_K^{S\rtimes T}$. Here, $\mathbb{C}_K^{S\rtimes T}$ is the functional Hilbert space with reproducing kernel

$$K(g,h) = (\mathcal{U}_g\psi, \mathcal{U}_h\psi)_{\mathbb{L}_2(S)} = (\mathcal{U}_{h^{-1}g}\psi, \psi)_{\mathbb{L}_2(S)}, \tag{5.36}$$

for all $g, h \in S \rtimes T$. The inner product on $\mathbb{C}_K^{S \rtimes T}$ can be written as

$$(\Phi, \Psi)_{\mathbb{C}_K^{S \rtimes T}} = (T_{M_\psi}\Phi, T_{M_\psi}\Psi)_{\mathbb{L}_2(S \rtimes T)}, \tag{5.37}$$

for all $\Phi, \Psi \in \mathbb{C}_K^{S \rtimes T}$.

Corollary 5.18 *If $M_\psi = 1$ on $\hat{S}$, then $\mathbb{C}_K^{S \rtimes T}$ is a closed subspace of $\mathbb{L}_2(S \rtimes T)$.*

By Lemma 5.16, our functional Hilbert space is a closed subspace of

$$\mathbb{H}(S, \mu_S) \otimes \mathbb{L}_2(T, \rho^{-1}\mu_T), \tag{5.38}$$

where

$$\mathbb{H}(S, \mu_S) = \{f \in \mathbb{L}_2(S, \mu_S) \mid M_\psi^{-\frac{1}{2}}\mathcal{F}f \in \mathbb{L}_2(\hat{S})\}. \tag{5.39}$$

The inner product on $\mathbb{H}(S, \mu_S)$ is defined by

$$(f, g)_{\mathbb{H}(S, \mu_s)} = (M_\psi^{-\frac{1}{2}}\mathcal{F}f, M_\psi^{-\frac{1}{2}}\mathcal{F}g)_{\mathbb{L}_2(\hat{S})} \tag{5.40}$$

We recall that $\mathbb{H}(S, \mu_S)$ is a vector subspace of $\mathbb{L}_2(S)$, because of Lemma 5.5. Hence we always arrive at a kind of Sobolev space on S. Now denote the inner product on $\mathbb{H}(S, \mu_S) \otimes \mathbb{L}_2(T, \rho^{-1}\mu_T)$ by $(\cdot, \cdot)_\otimes$. It follows from Lemma 5.16 that $(\cdot, \cdot)_\otimes|_{\mathbb{C}_K^{S \rtimes T}} = (\cdot, \cdot)_{\mathbb{C}_K^{S \rtimes T}}$.

Theorem 5.19 *$\mathbb{C}_K^{S \rtimes T}$ is a closed subspace of $\mathbb{H}(S, \mu_S) \otimes \mathbb{L}_2(T, \rho^{-1}\mu_T)$. The operator $\Phi \mapsto [g \mapsto (K(\cdot, g), \Phi)_\otimes]$ is the projection operator from $\mathbb{H}(S, \mu_S) \otimes \mathbb{L}_2(T, \rho^{-1}\mu_T)$ onto $\mathbb{C}_K^{S \rtimes T}$.*

Proof:
It is obvious that $\mathbb{C}_K^{S \rtimes T}$ is a closed subspace of $\mathbb{H}(S, \mu_S) \otimes \mathbb{L}_2(T, \rho^{-1}\mu_T)$. Let $\Phi \in \mathbb{H}(S, \mu_S) \otimes \mathbb{L}_2(T, \rho^{-1}\mu_T)$. Then it can be written as

$$\Phi = \Phi_1 + \Phi_2,$$

with $\Phi_1 \in \mathbb{C}_K^{S \rtimes T}$ and $\Phi_2 \in (\mathbb{C}_K^{S \rtimes T})^\perp$. Then for all $g \in S \rtimes T$

$$(K(\cdot, g), \Phi)_\otimes = (K(\cdot, g), \Phi_1)_\otimes + (K(\cdot, g), \Phi_2)_\otimes = \Phi_1(g)$$

Therefore Φ is mapped to Φ_1. □

5.2.5 Compact groups

If the group T is compact, which we assume throughout this section, then we can simplify some expressions of the previous two sections.

Lemma 5.20 *$\rho(t) = 1$ for all $t \in T$.*

Proof:
Since T is compact the groups $\{\triangle_{\tilde{T}}(e,t) \mid t \in T\}$ and $\{\triangle_{S \rtimes T}(e,t) \mid t \in T\}$ are compact subgroups of $(\mathbb{R}^+, \cdot)$. The only compact subgroup of $(\mathbb{R}^+, \cdot)$ is $\{1\}$. So $\triangle_{\tilde{T}}(e,t) = 1$ and $\triangle_{S \rtimes T}(e,t) = 1$ for all $t \in T$. Hence $\rho(t) = 1$ for all $t \in T$. $\square$

For non-vanishing wavelets, a function $A : T \to \mathbb{R}^+ \cup \{0\}$ is to be chosen. In case compactness we can simply take $A(t) = |T|^{-1}$ for all $t \in T$. Then obviously

$$\int_T A(t)\rho^{-1}(t)\ \mathrm{d}\mu_{\mathrm{T}}(\mathrm{t}) = 1. \tag{5.41}$$

Theorem 5.21 *Let $\psi \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$. Then $M_\psi \in \mathbb{L}_1(\hat{S})$.*

Proof:
For all $t \in T$ the operator $\mathcal{F}\mathcal{P}_t$ is unitary from $\mathbb{L}_2(S)$ onto $\mathbb{L}_2(\hat{S})$ we get

$$\int_{\hat{S}} \frac{|\mathcal{F}\mathcal{P}_t\psi|^2}{\rho(t)}(\gamma)\mathrm{d}\mu_{\hat{S}}(\gamma) = \int_{\hat{S}} |\mathcal{F}\mathcal{P}_t\psi|^2(\gamma)\mathrm{d}\mu_{\hat{S}}(\gamma) = \|\psi\|^2_{\mathbb{L}_2(S)},$$

for all $t \in T$. Hence,

$$\int_{\hat{S}}\int_T \frac{|\mathcal{F}\mathcal{P}_t\psi|^2}{\rho(t)}(\gamma)\ \mathrm{d}\mu_T(t)\mathrm{d}\mu_{\hat{S}}(\gamma) \quad = \quad \int_T \|\psi\|^2_{\mathbb{L}_2(S)}\mathrm{d}\mu_T(t) = |T|\|\psi\|^2_{\mathbb{L}_2(S)},$$

by Fubini's theorem. $\square$

Corollary 5.22 *Let $\psi \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$ be a non-vanishing wavelet. Then ψ is admissible.*

6 $\mathbb{L}_2(\mathbb{R}^2)$ and the Euclidean motion group

6.1 The wavelet transformation

We now will work the previous section out in detail for a more explicit example. The circle group $\mathbb{T}$ is defined by the set

$$\mathbb{T} = \{z \mid z \in \mathbb{C}\ |z| = 1\}, \tag{6.1}$$

with complex multiplication. The group $\mathbb{T}$ has the following group homomorphism $\tau : \mathbb{T} \to \mathrm{Aut}(\mathbb{R}^2)$

$$\tau :\ z \mapsto R_z, \tag{6.2}$$

with

$$R_z = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta = \arg z. \tag{6.3}$$

Using this automorphism we can define the semi-direct product $\mathbb{R}^2 \rtimes \mathbb{T}$. The group product of $\mathbb{R}^2 \rtimes \mathbb{T}$ is given by

$$(x, z_1)(y, z_2) = (x + R_{z_1} y, z_1 z_2). \tag{6.4}$$

for all $(x, z_1), (y, z_2) \in \mathbb{R}^2 \rtimes \mathbb{T}$. The group $\mathbb{R}^2 \rtimes \mathbb{T}$ is called the **Euclidean motion group**. It has the following unitary representation on $\mathbb{L}_2(\mathbb{R}^2)$

$$\big(\mathcal{U}_{(b,z)} f\big)(x) = \big(\mathcal{T}_b \mathcal{P}_z f\big)(x) = f(R_z^{-1}(x - b))), \tag{6.5}$$

with

$$\big(\mathcal{T}_b f\big)(x) = f(x - b), \ \ \big(\mathcal{P}_z f\big)(x) = f(R_z^{-1} x), \tag{6.6}$$

for all $b \in \mathbb{R}^2$, $z \in \mathbb{T}$, $f \in \mathbb{L}_2(\mathbb{R}^2)$ and almost every $x \in \mathbb{R}^2$.

We consider the wavelet transformation using the representation $\mathcal{U}$, as above, of the group $G = \mathbb{R}^2 \rtimes \mathbb{T}$ in the Hilbert space $\mathbb{L}_2(\mathbb{R}^2)$. The wavelet transform $W_\psi : \mathbb{L}_2(\mathbb{R}^2) \to \mathbb{C}_K^{\mathbb{R}^2 \rtimes \mathbb{T}}$ for cyclic wavelets is defined by

$$\big(W_\psi f\big)(b, z) = \big(\mathcal{T}_b \mathcal{P}_z \psi, f\big)_{\mathbb{L}_2(\mathbb{R}^2)}, \tag{6.7}$$

for all $f \in \mathbb{L}_2(\mathbb{R}^2)$.

Since $\mathbb{T}$ is compact, $\rho(z) = 1$ for all $z \in \mathbb{T}$. We normalize the Haar measure on $\mathbb{T}$ such that $\mathbb{T}$ has total measure one. Then we choose the Haar measure of $\mathbb{R}^2 \rtimes \mathbb{T}$ as $\mu_{\mathbb{R}^2 \rtimes \mathbb{T}}$.

We end this introduction with the remark that every non-vanishing wavelet $\psi \in \mathbb{L}_1(\mathbb{R}^2) \cap \mathbb{L}_2(\mathbb{R}^2)$ is also admissible, by Corollary 5.22.

6.2 Admissible wavelets

First we mention the method involving admissible wavelets. We recall that $\psi \in \mathbb{L}_1(\mathbb{R}^2) \cap \mathbb{L}_2(\mathbb{R}^2)$ is called admissible if

$$0 < M_\psi < \infty \text{ a.e.}$$

where

$$M_\psi(\omega) = \int_{\mathbb{T}} \left|\big(\mathcal{F}\mathcal{P}_z\psi\big)(\omega)\right|^2 \, \mathrm{d}\mu_{\mathbb{T}}(z). \tag{6.8}$$

We can reformulate Theorem 5.17 as follows.

Theorem 6.1 *Let $\psi \in \mathbb{L}_1(\mathbb{R}^2) \cap \mathbb{L}_2(\mathbb{R}^2)$ be an admissible wavelet. Then W_ψ defined by*

$$\big(W_\psi f\big)(x,z) = (\mathcal{T}_x\mathcal{P}_z\psi, f)_{\mathbb{L}_2(\mathbb{R}^2)}, \quad f \in \mathbb{L}_2(\mathbb{R}^2),\ (x,z) \in \mathbb{R}^2 \rtimes \mathbb{T}, \tag{6.9}$$

is a unitary mapping from $\mathbb{L}_2(\mathbb{R}^2)$ onto $\mathbb{C}_K^{\mathbb{R}^2 \rtimes \mathbb{T}}$. Here, $\mathbb{C}_K^{\mathbb{R}^2 \rtimes \mathbb{T}}$ is the functional Hilbert space with reproducing kernel

$$K(g,h) = (\mathcal{U}_g\psi, \mathcal{U}_h\psi)_{\mathbb{L}_2(\mathbb{R}^2)}, \tag{6.10}$$

for all $g,h \in \mathbb{R}^2 \rtimes \mathbb{T}$. The inner product on $\mathbb{C}_K^{\mathbb{R}^2 \rtimes \mathbb{T}}$ can be written as

$$(\Phi, \Psi)_{\mathbb{C}_K^{\mathbb{R}^2 \rtimes \mathbb{T}}} = (T_{M_\psi}\Phi, T_{M_\psi}\Psi)_{\mathbb{L}_2(\mathbb{R}^2 \rtimes \mathbb{T})}, \tag{6.11}$$

for all $\Phi, \Psi \in \mathbb{C}_K^{\mathbb{R}^2 \rtimes \mathbb{T}}$.

We now analyse the function M_ψ, defined in (6.8) a little further. First we mention that $\mathbb{T}$ is a compact group, so $M_\psi \in \mathbb{L}_1(\mathbb{R}^2)$ by Theorem 5.21. Define for $m \in \mathbb{Z}$ the function $\eta_m : [0, 2\pi) \to \mathbb{C}$ by $\eta_m(\phi) = \mathrm{e}^{\mathrm{i}m\phi}$. Because $\mathbb{L}_2(\mathbb{R}^2) = \mathbb{L}_2(S^1) \otimes \mathbb{L}_2((0,\infty), r\ \mathrm{d}r)$, we can write all $\psi \in \mathbb{L}_2(\mathbb{R}^2)$ in the following way

$$\psi = \sum_{m=-\infty}^{\infty} \eta_m \otimes \chi_m, \tag{6.12}$$

where $\chi_m \in \mathbb{L}_2((0,\infty), r\ dr)$ for all $m \in \mathbb{Z}$. For the Fourier transform we can write in polar coordinates

$$\big(\mathcal{F}[\eta_m \otimes \chi_m]\big)(\rho, \phi_\omega) = \mathrm{i}^m \sqrt{2\pi}\mathrm{e}^{\mathrm{i}m\phi_\omega} \int_0^\infty r\chi_m(r) J_m(\rho r)\ dr \tag{6.13}$$

for all $\rho \in [0,\infty)$ and $\phi_\omega \in [0, 2\pi)$, where J_m is the m-th order Bessel function of the first kind. See for example [FH, pp. 24-25].

Now M_ψ is easily calculated.

$$\big(P_z\psi\big)(r,\phi) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi-\arg z)}\chi_m(r), \tag{6.14}$$

for all $r \in (0,\infty)$ and $\phi \in [0,2\pi)$. Hence,

$$\big(\mathcal{F}P_z\psi\big)(\rho,\varphi) = \sqrt{2\pi}\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\varphi-\arg z)}\mathrm{i}^m \int_0^\infty r\chi_m(r)J_m(\rho r)dr, \tag{6.15}$$

for all $\rho \in [0,\infty)$ and $\varphi \in [0,2\pi)$. Hence M_ψ is given by,

$$M_\psi(\omega) = 2\pi \sum_{m=-\infty}^{\infty} |\tilde{\chi}_m(\|\omega\|_2)|^2, \tag{6.16}$$

for all $\omega \in \mathbb{R}^2$, where $\tilde{\chi}_m$ defined by $\tilde{\chi}_m(\rho) = \int_0^\infty r\chi_m(r)J_m(\rho r)dr$ for all $\rho \in (0,\infty)$ and $m \in \mathbb{Z}$. Thus, the above sum completely determines the inner product. Furthermore, M_ψ only depends on the radius. By Lemma 5.21 we get the following relation between a chosen wavelet ψ and M_ψ

$$\int_{\mathbb{R}^2} M_\psi(\omega)\ \mathrm{d}\omega = \|\psi\|^2_{\mathbb{L}_2}. \tag{6.17}$$

This implies that M_ψ^{-1} is at least unbounded. Because M_ψ only depends on the radius, there exists a function $\tilde{M}_\psi : (0,\infty) \to (0,\infty)$ such that

$$M_\psi(\omega) = \tilde{M}_\psi(\|\omega\|_2), \tag{6.18}$$

for almost all $\omega \in \mathbb{R}^2$. Then $\tilde{M}_\psi \in \mathbb{L}_1((0,\infty), r\ dr)$.

We end this subsection with the remark, that $\psi \mapsto M_\psi$ is not injective; several different wavelets ψ can lead to the same M_ψ. If ψ_1 and ψ_2 are different admissible wavelets with the property $M_{\psi_1} = M_{\psi_2}$, then their corresponding functional Hilbert space are different closed subspaces of the same Hilbert space $\mathbb{H}(\mathbb{R}^2) \otimes \mathbb{L}_2(\mathbb{T})$ as defined in (5.38).

6.3 Wavelet $\psi_d : x \mapsto \frac{1}{d^2}K_0(d\|x\|_2)$

Take for $d > 0$ the vector $\psi_d \in \mathbb{L}_2(\mathbb{R}^2)$ defined by

$$\psi_d(x) = \frac{1}{d^2}K_0(d\|x\|_2), \tag{6.19}$$

for all $x \in \mathbb{R}^2$, where K_0 stands for the zeroth order modified Bessel function of the second kind. The Fourier transformation of this wavelet is given by

$$\big(\mathcal{F}\psi_d\big)(\omega) = (1 + d^2\|\omega\|_2^2)^{-1}, \tag{6.20}$$

for all $\omega \in \mathbb{R}^2$. See [AS, Expr. 11.4.44, pp 488].

The set V_{ψ_d} equals

$$V_{\psi_d} = \{T_b\mathcal{P}_z\psi_d \mid b \in \mathbb{R}^2, \;\; z \in \mathbb{T}\} = \{T_b\psi_d \mid b \in \mathbb{R}^2\}. \tag{6.21}$$

Moreover,

$$M_{\psi_d}(\omega) = |\mathcal{F}\psi_d|^2(\omega) = (1 + d^2\|\omega\|_2^2)^{-2}, \tag{6.22}$$

by a straightforward calculation. It is easily seen that ψ is an admissible wavelet. So by Theorem 6.1 the mapping W_{ψ_d} defined by

$$\big(W_{\psi_d}f\big)(b,z) = \int_{\mathbb{R}^2} T_b\mathcal{P}_z\psi_d f(x)\;\mathrm{d}x = \int_{\mathbb{R}^2} T_b\psi_d f(x)\;\mathrm{d}x, \tag{6.23}$$

is a unitary mapping from $\mathbb{L}_2(\mathbb{R}^2)$ to the space $\mathbb{C}_K^{\mathbb{R}^2\rtimes\mathbb{T}}$. But because $M_{\psi_d}^{-1}(\omega) = (1+d^2\|\omega\|_2^2)^2$ we get a subspace of a kind of a Sobolev-space

$$\begin{aligned}(\Phi,\Psi)_{\mathbb{C}_K^{\mathbb{R}^2\rtimes\mathbb{T}}} &= (M_{\psi_d}^{-\frac{1}{2}}\mathcal{F}[\Phi], M_{\psi_d}^{-\frac{1}{2}}\mathcal{F}[\Psi])_{\mathbb{L}_2(\mathbb{R}^2\rtimes\mathbb{T})}\\ &= ((1+d^2\triangle_2)\Phi, (1+d^2\triangle_2)\Psi)_{\mathbb{L}_2(\mathbb{R}^2\rtimes\mathbb{T})},\end{aligned} \tag{6.24}$$

for all $\Phi, \Psi \in \mathbb{C}_K^{\mathbb{R}^2\rtimes\mathbb{T}}$.

6.4 Wavelet $\psi_\alpha : x \mapsto \frac{\alpha}{4}\mathrm{e}^{-|x_1|-\alpha|x_2|}$

We illustrate the method for non-vanishing wavelets by means of the example $\psi_\alpha : x \mapsto \frac{\alpha}{4}\mathrm{e}^{-|x_1|-\alpha|x_2|}$. Then,

$$\big(\mathcal{F}\mathcal{P}_z\psi_\alpha\big)(\omega) = \frac{\alpha^2}{2\pi(1+(\omega_1\cos\theta+\omega_2\sin\theta)^2)(\alpha^2+(\omega_2\cos\theta-\omega_1\sin\theta)^2)}, \tag{6.25}$$

for all $\omega \in \mathbb{R}^2$ and with $\theta = \arg z$. Obviously, ψ_α is a non-vanishing wavelet. Hence, it is cyclic. As $\mathbb{T}$ is compact, $\rho(t) = 1$ for all $t \in T$. Moreover, define the function A by $A(t) = 1$ for all $t \in T$.

By (5.26), the inner product on $\mathbb{C}_K^G$ is given by

$$(\Phi,\Psi)_{\mathbb{C}_K^G} = \frac{1}{2\pi}\int_{\mathbb{R}^2\times[0,2\pi)} \overline{\mathcal{F}[\Phi(\cdot,\mathrm{e}^{\mathrm{i}\theta})](\omega)}\mathcal{F}[\Psi(\cdot,\mathrm{e}^{\mathrm{i}\theta})](\omega)[\mathcal{F}[\mathcal{P}_{\mathrm{e}^{\mathrm{i}\theta}}\psi_\alpha]]^{-2}(\omega)\ \mathrm{d}\omega\mathrm{d}\theta.$$

$$= \frac{2\pi}{\alpha^4}\int_{\mathbb{R}^2\times[0,2\pi)} \overline{\mathcal{F}[\Phi(\cdot,\mathrm{e}^{\mathrm{i}\theta})](\omega)}\mathcal{F}[\Psi(\cdot,\mathrm{e}^{\mathrm{i}\theta})](\omega)(1+\xi^2)^2(\alpha^2+\eta^2)^2\ \mathrm{d}\omega\mathrm{d}\theta, \tag{6.26}$$

where

$$\begin{pmatrix}\eta\\ \xi\end{pmatrix} = \begin{pmatrix}\omega_1\cos\theta - \omega_2\sin\theta\\ \omega_2\cos\theta+\omega_1\sin\theta\end{pmatrix}. \tag{6.27}$$

Using Plancherel we find

$$(\Phi,\Psi)_{\mathbb{C}_K^{\mathbb{R}^2\rtimes\mathbb{T}}} = 2\pi\int_{[0,2\pi)}\int_{\mathbb{R}^2} \overline{\big(D(\theta)\Phi\big)(x,\mathrm{e}^{\mathrm{i}\theta})}\ \big(D(\theta)\Psi\big)(x,\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}x\mathrm{d}\theta, \tag{6.28}$$

where

$$D(\theta) = \left(1-\frac{d^2}{d\xi^2}\right)\left(1-\frac{1}{\alpha^2}\frac{d^2}{d\eta^2}\right) \qquad \text{and} \qquad \begin{pmatrix}\frac{d}{d\eta}\\ \frac{d}{d\xi}\end{pmatrix} = \begin{pmatrix}\cos\theta\frac{d}{dx}-\sin\theta\frac{d}{dy}\\ \cos\theta\frac{d}{dy}+\sin\theta\frac{d}{dx}\end{pmatrix}, \tag{6.29}$$

for all $\theta\in[0,2\pi)$. So we arrive at some variant of a second order Sobolev space. For fixed θ it looks like a second order Sobolev space, but the derivatives rotate with θ.

By (5.21), the adjoint/inverse operator is given by

$$W_{\psi_\alpha}^*[\Phi] = 2\pi\mathcal{F}^{-1}\left[\int_0^{2\pi}\mathcal{F}(\Phi(\cdot,\mathrm{e}^{\mathrm{i}\theta}\ \left(\overline{\mathcal{F}\mathcal{P}_{\mathrm{e}^{\mathrm{i}\theta}}\psi_\alpha}\right)^{-1}\mathrm{d}\theta\right], \tag{6.30}$$

for all $\Phi\in\mathbb{C}_K^{\mathbb{R}^2\rtimes\mathbb{T}}$.

7 A topic from image analysis

7.1 Introduction

In many applications in medical images it is common use to construct a orientation-score of a grey-value image, a so-called orientation bundle function. Mostly, such an orientation bundle function is obtained by means of a convolution with some anisotropic rotated vector ψ. So, by the orientation bundle function of an image we mean the Wavelet transform of the image using the Euclidean motion group, with the representation as defined by (6.5).

In image analysis, the wavelet transformation is often regarded as an operator $\tilde{W}_\psi : \mathbb{L}_2(\mathbb{R}^2) \to \mathbb{L}_2(\mathbb{R}^2 \rtimes \mathbb{T})$. In this section we pay some attention to this point of view. Since every non-vanishing wavelet $\psi \in \mathbb{L}_1(\mathbb{R}^2) \cap \mathbb{L}_2(\mathbb{R}^2)$ is also straightforwardly admissible by Corollary 5.22, we only pay attention to the method of admissible wavelets. First, we state a theorem about the definition of $\tilde{W}_\psi$.

Theorem 7.1 *Let $\psi \in \mathbb{L}_1(\mathbb{R}^2) \cap \mathbb{L}_2(\mathbb{R}^2)$ be an admissible kernel. Define $\tilde{W}_\psi : \mathbb{L}_2(\mathbb{R}^2) \to \mathbb{L}_2(\mathbb{R}^2 \rtimes \mathbb{T})$ by $\tilde{W}_\psi f = W_\psi f$ for all $f \in \mathbb{L}_2(R^2)$. The adjoint operator $\tilde{W}_\psi^*$ is then given by*

$$\tilde{W}_\psi^* \Phi = \frac{1}{2\pi} \mathcal{F}^{-1} \left\{ \int_0^{2\pi} \mathcal{F}[\Phi(\cdot, \mathrm{e}^{\mathrm{i}\theta})] \mathcal{F} \mathcal{P}_{\mathrm{e}^{\mathrm{i}\theta}} \psi \, d\theta \right\}, \tag{7.1}$$

for all $\Phi \in \mathbb{L}_2(\mathbb{R}^2 \rtimes \mathbb{T})$. Moreover, $\tilde{W}_\psi$ is closed.

Proof:
Then the operator is well-defined from $\mathbb{L}_2(\mathbb{R}^2)$ onto $\mathbb{L}_2(\mathbb{R}^2 \rtimes \mathbb{T})$ with domain $\mathbb{C}_K^{\mathbb{R}^2 \rtimes \mathbb{T}}$, because $\mathbb{C}_K^{\mathbb{R}^2 \rtimes \mathbb{T}}$ is a subset of $\mathbb{L}_2(\mathbb{R}^2 \rtimes \mathbb{T})$ by the remark above Theorem 5.19. The adjoint is easily calculated by changing the order of integration and equation (5.17). Moreover, $\tilde{W}_\psi$ is closed by Lemma 5.1. □

Remark that in contrast to W_ψ^*, see (5.32) , the function M_ψ does not occur in $\tilde{W}_\psi^*$.

7.2 Sequences of wavelets

Let ψ be an admissible wavelet. Because of the identity

$$\|f\|_{\mathbb{L}_2(\mathbb{R}^2)} = \|\tilde{W}_\psi f\|_{\mathbb{C}_K^G} = \|M_\psi^{-\frac{1}{2}} \mathcal{F} \tilde{W}_\psi f\|_{\mathbb{L}_2(\mathbb{R}^2 \rtimes \mathbb{T})}, \tag{7.2}$$

for all $f \in \mathbb{L}_2(\mathbb{R}^2)$ and the fact that $M_\psi \in \mathbb{L}_1(\mathbb{R}^2)$, the operator $\tilde{W}_\psi^{-1}$ is unbounded.

Although for every admissible wavelet the operator $\tilde{W}_\psi^{-1}$ is unbounded, there exists sequences of admissible wavelet such that $\lim_{n\to\infty} \tilde{W}_{\psi_n}^* \tilde{W}_{\psi_n} f = f$ for all $f \in \mathbb{L}_2(\mathbb{R}^2)$ in $\mathbb{L}_2$-sense. The main idea is that there exists sequences for which $M_{\psi_n} \to 1$ uniformly on compact sets.

Lemma 7.2 *Let $\{e_n \mid n \in \mathbb{N}\}$ be a sequence in $\mathbb{L}_2(\mathbb{R}^2)$ which satisfies the following conditions*

1. $\lim_{n\to\infty}\big(\mathcal{F}e_n\big)(\omega)=1$, *uniformly on compact sets,*

2. $\sup_{n\in\mathbb{N}}\|\mathcal{F}e_n\|_{\mathbb{L}_\infty(\mathbb{R}^2)}<\infty$.

Then for all $f\in\mathbb{L}_2(\mathbb{R}^2)$

$$e_n * f \to f, \tag{7.3}$$

for $n\to\infty$ *in* $\mathbb{L}_2(\mathbb{R}^2)$*-sense.*

Proof:

$$\begin{aligned}\|e_n * f - f\|_{\mathbb{L}_2(\mathbb{R}^2)} &= \|(I-\mathcal{F}e_n)\mathcal{F}f\|_{\mathbb{L}_2(\mathbb{R}^2)}\\ &= \|(I-\mathcal{F}e_n)\mathcal{F}f\|_{\mathbb{L}_2(B_{0,R})}+\|(I-\mathcal{F}e_n)\mathcal{F}f\|_{\mathbb{L}_2(\mathbb{R}^2/B_{0,R})}\\ &\leq \|(I-\mathcal{F}e_n)\|_{\mathbb{L}_\infty(B_{0,R})}\|\mathcal{F}f\|_{\mathbb{L}_2(B_{0,R})}\\ &\qquad +(1+\sup_{n\in\mathbb{N}}\|\mathcal{F}e_n\|_{\mathbb{L}_\infty(\mathbb{R}^2)})\|\mathcal{F}f\|_{\mathbb{L}_2(\mathbb{R}^2/B_{0,R})},\end{aligned}$$

Now let first $n\to\infty$ and then $R\to\infty$. □

Theorem 7.3 *Let* $\psi_n\in\mathbb{L}_2(\mathbb{R}^2)$ *be an admissible wavelet for all* $n\in\mathbb{N}$*. Assume that* $M_{\psi_n}\in\mathbb{L}_2(\mathbb{R}^2)$ *for all* $n\in\mathbb{N}$ *and* $\{\mathcal{F}^{-1}M_{\psi_n}\mid n\in\mathbb{N}\}$ *satisfies the condition of Lemma 7.2. Then for all* $f\in\mathbb{L}_2(\mathbb{R}^2)$

$$\mathcal{F}^{-1}M_{\psi_n} * \big(W^*_{\psi_n}W_{\psi_n}f\big)\to f, \tag{7.4}$$

for $n\to\infty$ *in* $\mathbb{L}_2(\mathbb{R}^2)$ *sense.*

Proof:
$W^*_{\psi_n}W_{\psi_n}f=f$ and $\mathcal{F}^{-1}M_{\psi_n}$ satisfies the conditions of Lemma 7.2 by assumption. □

Remark that we can rewrite (7.4) as

$$\tilde{W}^*_{\psi_n}W_{\psi_n}f\to f, \tag{7.5}$$

for $n\to\infty$ in $\mathbb{L}_2(\mathbb{R}^2)$ sense, for all $f\in\mathbb{L}_2(\mathbb{R}^2)$.

So the corollary states that under the given assumptions we can reconstruct, in the limit, the image using the $\mathbb{L}_2$-adjoint.

An interesting idea, is not to model the image-space by $\mathbb{L}_2(\mathbb{R}^2)$, but by a subspace of $\mathbb{L}_2(\mathbb{R}^2)$. For example, closed subspace $\mathcal{H}_R$ of functions for which the support in Fourier domain is within a ball with radius R. Now we can choose an admissible wavelet such that M_ψ equals 1 on this set. Hence, in this case $\tilde{W}_\psi|_{\mathcal{H}_R}$ is an isometry from $\mathcal{H}_R$ onto $\mathbb{L}_2(S\rtimes T)$.

7.3 Example

In an article by Kalitzin, ter Haar Romeny and Viergever see [KHV], it is suggested to take the sequence of wavelets defined by

$$\psi_n(r,\phi) = \frac{1}{2} + \frac{1}{2}\sum_{m=-n}^{n} \mathrm{e}^{im\phi}\frac{r^{|m|}}{\sqrt{|m|!}}\mathrm{e}^{-r^2}, \tag{7.6}$$

for all $r \in (0,\infty)$, $\phi \in [0,2\pi)$ and $n \in \mathbb{N}$. The related function M_{ψ_n} is now easily calculated

$$M_{\psi_n}(\omega) = \sum_{m=0}^{n} \frac{\rho^{2m}}{m!}\mathrm{e}^{-\rho^2}, \tag{7.7}$$

where $\rho = \|\omega\|_2$. It is obvious that the sequence $\{M_{\psi_n} \mid n \in \mathbb{N}\}$ satisfies the conditions of Lemma 7.2. Hence, by Theorem 7.3, for all $f \in \mathbb{L}_2(\mathbb{R}^2)$

$$\tilde{W}^*_{\psi_n} W_{\psi_n} f \to f, \tag{7.8}$$

for $n \to \infty$ in $\mathbb{L}_2(\mathbb{R}^2)$ sense. For this sequence the limit $\lim_{n\to\infty}\psi_n$ does exist pointwise, but not of course in $\mathbb{L}_2(\mathbb{R}^2)$ sense.

A Schur's lemma

Schur's lemma is mostly known for the special case of irreducible representations $\mathcal{U}$ on a Hilbert space $\mathcal{H}$ of or compact group G. In these cases the proof is straightforward. The main idea is that if A has an eigen-value, then the eigen space is invariant under $\mathcal{U}_g$, which follows by the assumption $\mathcal{U}_g A = A\mathcal{U}_g$, and by irreducibility of $\mathcal{U}$ it then follows that $\overline{E_\lambda} = E_\lambda = \mathcal{H}$. Nevertheless, Schur's lemma has serious consequences such as the orthogonality relations by Weyl for compact groups. We will give a generalization of this theorem which is applied in the general wavelet theorem 5.2 and which is formulated as an exercise in [D, vol.V, pp.21]. This general Schur's lemma is very often used in literature with bad and incomplete references, therefore we include a proof.

Theorem A.1 (Schur's Lemma) *Let G be a locally compact group and let $g \mapsto \mathcal{U}_g$ be a unitary irreducible representation of G in a Hilbert space $\mathcal{H}$. If A is a (not necessarily bounded) closed densely defined operator on $\mathcal{H}$ such that the domain $\mathcal{D}(A)$ is invariant under the representation $\mathcal{U}$ such that*

$$\mathcal{U}_g A f = A\mathcal{U}_g f \qquad \text{for all } g \in G\ ,\, f \in \mathcal{D}(A)\ ,$$

then $A = cI$ for some $c \in \mathbb{C}$.

Proof:
First we will show the theorem for a self-adjoint bounded operator A with $\mathcal{D}(A) = \mathcal{H}$. It follows from the spectral theorem for self adjoint operators that A is in the norm closure of the linear span V of all orthogonal projections P commuting with all the bounded operators commuting with A. In particular $\mathcal{U}_g$ is a bounded operator commuting with A and therefore every $P \in V$ commutes with $\mathcal{U}_g$. Therefore the space on which P projects (which is closed since it equals the $\mathcal{N}(I - P)$) is invariant under $\mathcal{U}_g$. But $\mathcal{U}$ was supposed to be irreducible and therefore this space equals $\mathcal{H}$ or $\{0\}$, i.e. $P = 0$ or $P = I$. Since A is within the span of such P, we have that $A = cI$, for some constant $c \in \mathbb{R}$.

Every bounded operator can be decomposed $A = (1/2)(A + iA^*) + (1/2)(A - iA^*)$. Furthermore, by the unitarity of $\mathcal{U}_g$

$$A^*\mathcal{U}_g = A^*\mathcal{U}_{g^{-1}}^* = (\mathcal{U}_{g^{-1}}A)^* = (A\mathcal{U}_{g^{-1}})^* = \mathcal{U}_g A^*,$$

for all $g \in G$. Hence the result now also follows for any bounded operator A on a Hilbert space A.

Now we deal with the unbounded case: The domain $\mathcal{D}(A)$ is invariant under $\mathcal{H}$, therefore by the irreducibility of $\mathcal{U}$ it follows that $\mathcal{D}(A)$ is dense in H. Next we show that the domain $\mathcal{D}(A)$ is a Hilbert space (say $\mathcal{D}_A$) equipped with inner product $(f, g)_A = (g, h) + (Ag, Ah)$:
If $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{D}(A)$ with respect to $(\cdot, \cdot)_A$ then $(\{h_n, \mathcal{A}h_n\})$ is a Cauchy sequence in $\mathcal{H} \times \mathcal{H}$. Because $\mathcal{A}$ is closed the limit equals $(\{h, \mathcal{A}h\})$ with $h \in \mathcal{D}(A)$. This implies that $\|h_n - h\|_A$ converges to 0.
Obviously, the operator $\tilde{A} : \mathcal{D}_A \to \mathcal{H}$ given by $\tilde{A}f = Af$ is a bounded operator on a Hilbert space commuting with $\mathcal{U}_g$ for all $g \in G$ and as a result the operator $\tilde{A}^*\tilde{A} : \mathcal{D}_A \to \mathcal{D}_A$ is a bounded operator on the Hilbert space $\mathcal{D}_A$. As a result we have by the preceding that $\tilde{A}^*\tilde{A} = dI$, but then we have $(\tilde{A}f, \tilde{A}f) = d(f, f)_A$ and therefore

$$d(Af, Af) = (f, f) + (Af, Af) \Leftrightarrow (Af, Af) = |c|^2(f, f) \text{ , for all } f \in \mathcal{D}_A \text{ ,}$$

with $|c|^2 = 1/(d - 1)$. Now $\mathcal{A}$ is a closed operator and $\mathcal{D}(A)$ is dense and therefore

$$(Af, Af) = |c|^2(f, f) \text{ for all } f \in \mathcal{H} \text{ ,}$$

i.e. $B = (1/|c|)A$ is unitary. In particular A is bounded and therefore equal to cI by the previous part of the proof. □

See [T, Prop. 2.4.5] for a more general version of the Schur's Lemma.

B Orthogonal sums of functional Hilbert spaces

In this section we analyze the orthogonal direct sum of functional Hilbert spaces as defined in section 4. The sequel is based on a part of the article by Aronszajn [A, part I, 6]. Theorem B.1 and Corollary B.2 are proven by Aronszajn.

Theorem B.1 *Let K and L be two functions of positive type on a set $\mathbb{I}$. Then*

$$\mathbb{C}^{\mathbb{I}}_{K+L} = \{f_1 + f_2 \mid f_1 \in \mathbb{C}^{\mathbb{I}}_K,\ f_2 \in \mathbb{C}^{\mathbb{I}}_L\} = \mathbb{C}^{\mathbb{I}}_K + \mathbb{C}^{\mathbb{I}}_L. \tag{B.1}$$

Furthermore, if $\mathbb{C}^{\mathbb{I}}_K \cap \mathbb{C}^{\mathbb{I}}_L = \{0\}$ then

$$\|f_1 + f_2\|^2_{\mathbb{C}^{\mathbb{I}}_{K+L}} = \|f_1\|^2_{\mathbb{C}^{\mathbb{I}}_K} + \|f_2\|^2_{\mathbb{C}^{\mathbb{I}}_L}. \tag{B.2}$$

Hence it follows that $\mathbb{C}^{\mathbb{I}}_K \perp \mathbb{C}^{\mathbb{I}}_L$ in $\mathbb{C}^{\mathbb{I}}_{K+L}$.

Define the Hilbert space $\mathbb{C}^{\mathbb{I}}_K \oplus \mathbb{C}^{\mathbb{I}}_L$ as the Cartesian product $\mathbb{C}^{\mathbb{I}}_K \times \mathbb{C}^{\mathbb{I}}_L$. with the inner product defined by

$$((f_1, g_1), (f_2, g_2))_{\oplus} = (f_1, f_2)_{\mathbb{C}^{\mathbb{I}}_K} + (g_1, g_2)_{\mathbb{C}^{\mathbb{I}}_L}, \tag{B.3}$$

for all pairs $(f_1, g_1), (f_2, g_2) \in \mathbb{C}^{\mathbb{I}}_K \oplus \mathbb{C}^{\mathbb{I}}_L$. It is obvious that $\mathbb{C}^{\mathbb{I}}_K \oplus \mathbb{C}^{\mathbb{I}}_L$ with the above inner product is a Hilbert space.

The following theorem is a direct consequence of Theorem B.1.

Corollary B.2 *Assume $\mathbb{C}^{\mathbb{I}}_K \cap \mathbb{C}^{\mathbb{I}}_L = \{0\}$. Then the mapping defined by*

$$(f_1, f_2) \mapsto f_1 + f_2, \tag{B.4}$$

is a unitary mapping from $\mathbb{C}^{\mathbb{I}}_K \oplus \mathbb{C}^{\mathbb{I}}_L$ onto $\mathbb{C}^{\mathbb{I}}_{K+L}$.

This idea is easily generalized to an infinite sum of functions of positive type. Define the Hilbert space $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$ as in (4.1) and (4.2).

Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of functions of positive type on a set $\mathbb{I}$ such that

$$\sum_{n=1}^{\infty} K_n(x, x) < \infty, \tag{B.5}$$

for all $x \in \mathbb{I}$. Then by the estimate

$$\begin{aligned} |K_n(x,y)| &= |(K_{n;x}, K_{n;y})_{\mathbb{C}^{\mathbb{I}}_{K_n}}| \leq \|K_{n;x}\|_{\mathbb{C}^{\mathbb{I}}_{K_n}} \|K_{n;y}\|_{\mathbb{C}^{\mathbb{I}}_{K_n}} \\ &\leq \frac{1}{2}\|K_{n;x}\|^2_{\mathbb{C}^{\mathbb{I}}_{K_n}} + \frac{1}{2}\|K_{n;y}\|^2_{\mathbb{C}^{\mathbb{I}}_{K_n}} = \frac{1}{2}K_n(x,x) + \frac{1}{2}K_n(y,y) \end{aligned} \tag{B.6}$$

for all $x, y \in \mathbb{I}$ and $n \in \mathbb{N}$, the sum

$$K_\oplus(x,y) := \sum_{n=1}^{\infty} K_n(x,y), \tag{B.7}$$

converges absolutely on $\mathbb{I} \times \mathbb{I}$. As a result $K_\oplus$ is a function of positive type, since we may change the order of summation in the definition and use the property that K_n is a function of positive type for all $n \in \mathbb{N}$.

Furthermore, the sequence $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely for all $(f_1, f_2, \ldots) \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$ and $x \in \mathbb{I}$. Indeed, let $f = (f_1, f_2, \ldots) \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$ and $x \in \mathbb{I}$, then

$$\begin{aligned} \lim_{N\to\infty} \sum_{n=1}^{N} |f_n(x)| &= \lim_{N\to\infty} \sum_{n=1}^{N} |(K_{n;x}, f_n)_{\mathbb{C}^{\mathbb{I}}_K}| \leq \lim_{N\to\infty} \frac{1}{2} \sum_{n=1}^{N} \left\{ \|K_{n;x}\|^2_{\mathbb{C}^{\mathbb{I}}_K} + \|f_n\|^2_{\mathbb{C}^{\mathbb{I}}_K} \right\} \\ &= \lim_{N\to\infty} \frac{1}{2} \sum_{n=1}^{N} \left\{ K_n(x,x)_{\mathbb{C}^{\mathbb{I}}_K} + \|f_n\|^2_{\mathbb{C}^{\mathbb{I}}_K} \right\} < \infty. \end{aligned} \tag{B.8}$$

Hence $\sum_{n=1}^{\infty} |f_n(x)| < \infty$.

Now we are ready for the following theorem.

Theorem B.3 *Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of functions of positive type on a set $\mathbb{I}$ such that*

$$\sum_{n=0}^{\infty} K_n(x,x) < \infty, \tag{B.9}$$

for all $x \in \mathbb{I}$. Define $K_\oplus$ by

$$K_\oplus(x,y) = \sum_{n=1}^{\infty} K_n(x,y), \tag{B.10}$$

for all $x, y \in \mathbb{I}$. Then $K_\oplus$ is a function of positive type on $\mathbb{I}$. Moreover, define for $x \in \mathbb{I}$ the vector $\psi_x \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$ as

$$\psi_x = (K_{1;x}, K_{2;x}, \ldots). \tag{B.11}$$

Then the mapping $\Phi : \overline{\langle\{\psi_x \mid x \in \mathbb{I}\rangle} \to \mathbb{C}^{\mathbb{I}}_{\sum_{n=1}^{\infty} K_n}$ *defined by*

$$\Phi : (f_1, f_2, \ldots) \mapsto \left(x \mapsto \sum_{n=1}^{\infty} f_n(x) \right), \tag{B.12}$$

is unitary.

Proof:
The last statement easily follows by $\Phi : \psi_x \mapsto K_{\oplus;x}$ for all $x \in \mathbb{I}$ and

$$(\psi_x, \psi_y)_{\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}} = \sum_{n=1}^{\infty} K_n(x, y) = (K_{\oplus;x}, K_{\oplus;y})_{\mathbb{C}^{\mathbb{I}}_{\sum_{n=1}^{\infty} K_n}} = (\Phi\psi_x, \Phi\psi_y)_{\mathbb{C}^{\mathbb{I}}_{\sum_{n=1}^{\infty} K_n}},$$

for all $x, y \in \mathbb{I}$. Hence the mapping is unitary. □

As in the case of the sum of two functional Hilbert spaces we search for a condition such that $\overline{\langle\{\psi_x \mid x \in \mathbb{I}\rangle} = \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$. In that case Φ is a unitary mapping from $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$ onto $\mathbb{C}^{\mathbb{I}}_{\sum_{n=1}^{\infty} K_n}$.

Theorem B.4 $\overline{\langle\{\psi_x \mid x \in \mathbb{I}\rangle} = \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$ *if and only if for all* $f \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$ *the following condition is satisfied*

$$\forall x \in \mathbb{I} \left[\sum_{m=1}^{\infty} f_m(x) = 0 \right] \implies f = 0. \tag{B.13}$$

Proof:
The theorem easily follows by $(f, \psi_x)_{\oplus} = 0 \Leftrightarrow \sum_{n=1}^{\infty} f_n(x) = 0$, for all $f = (f_1, f_2, \ldots) \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$. □

Remind that in the case of the sum of two Hilbert spaces the condition $\mathbb{C}^{\mathbb{I}}_K \cap \mathbb{C}^{\mathbb{I}}_L = \{0\}$ was needed. The above condition is equivalent to a condition similar to $\mathbb{C}^{\mathbb{I}}_K \cap \mathbb{C}^{\mathbb{I}}_L = \{0\}$.

Lemma B.5 *The condition: If* $f = \{f_1, f_2, ...\} \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$, *then*

$$\forall x \in \mathbb{I} \left[\sum_{m=1}^{\infty} f_m(x) = 0 \right] \implies f = 0. \tag{B.14}$$

is equivalent to the condition:

$$\mathbb{C}^{\mathbb{I}}_{K_n} \cap \mathbb{C}^{\mathbb{I}}_{\sum_{m=1, m \neq n}^{\infty} K_n} = \{0\}, \tag{B.15}$$

for all $n \in \mathbb{N}$.

Proof:

$\Rightarrow$.

Let $n \in \mathbb{N}$ and $f \in \mathbb{C}^{\mathbb{I}}_{K_n} \cap \mathbb{C}^{\mathbb{I}}_{\sum_{m=1,m\neq n}^{\infty} K_m}$. Then write $f = f_n = \sum_{m=1,m\neq n}^{\infty} f_m$, where $f_m \in \mathbb{C}^{\mathbb{I}}_{K_m}$ for all $m \in \mathbb{N}$. Define the element $g \in \bigoplus_{m=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_m}$ by

$$g = (f_1, f_2, \ldots, f_{n-1}, -f_n, f_{n+1}, \ldots)$$

Then obviously

$$\forall x \in \mathbb{I} \left[\lim_{N\to\infty} \sum_{m=1}^{N} (g_m, K_{m;x}) = 0 \right],$$

hence $g_m = 0$ for all $m \in \mathbb{N}$. So $f = 0$.

$\Leftarrow$.

First we mention that $\mathbb{C}^{\mathbb{I}}_{K_n} \perp \mathbb{C}^{\mathbb{I}}_{K_m}$ for $m \neq n$ in $\mathbb{C}^{\mathbb{I}}_{\sum_{n=1}^{\infty} K_n}$. Indeed, let $f_1 \in \mathbb{C}^{\mathbb{I}}_{K_n}$ and $f_2 \in \mathbb{C}^{\mathbb{I}}_{K_m}$ for $m \neq n$, then by Theorem B.1

$$\|f_1 + f_2\|^2_{\mathbb{C}^{\mathbb{I}}_{K_n + \sum_{l=1,l\neq n} K_l}} = \|f_1\|^2_{\mathbb{C}^{\mathbb{I}}_{K_n}} + \|f_2\|^2_{\mathbb{C}^{\mathbb{I}}_{\sum_{l=1,l\neq n} K_l}}$$

$$= \|f_1\|^2_{\mathbb{C}^{\mathbb{I}}_{K_n}} + \|f_2\|^2_{\mathbb{C}^{\mathbb{I}}_{K_m + \sum_{l=1,l\neq m,l\neq n} K_l}} = \|f_1\|^2_{\mathbb{C}^{\mathbb{I}}_{K_n}} + \|f_2\|^2_{\mathbb{C}^{\mathbb{I}}_{K_m}}.$$

Now apply the polarization identity to get $(f_1, f_2)_{\mathbb{C}^{\mathbb{I}}_{\sum_{l=1}^{\infty} K_l}} = 0$. Since f_1, f_2 were arbitrary, we get $\mathbb{C}^{\mathbb{I}}_{K_n} \perp \mathbb{C}^{\mathbb{I}}_{K_m}$ for $m \neq n$.

Secondly, let $(f_1, f_2, \ldots) \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{I}}_{K_n}$ satisfy

$$\sum_{n=1}^{\infty} f_n(x) = 0,$$

for all $x \in \mathbb{I}$. Hence for all $k \in \mathbb{N}$

$$0 = \left(\sum_{n=1}^{\infty} f_n, f_k \right)_{\mathbb{C}^{\mathbb{I}}_{\sum_{m=1}^{\infty} K_m}} = \|f_k\|^2_{\mathbb{C}^{\mathbb{I}}_{K_k}}$$

So $f_k = 0$ for all $k \in \mathbb{N}$. □

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References

[A] N. Aronszajn, Theory of reproducing kernels, *Trans. A.M.S.*,**68** (1950), pp 337-404.

[An] T.Ando, *Reproducing Kernel Spaces and Quadratic Inequalities* (Sapporo, Hokkaido University, 1987)

[AS] M. Abramowitz and I.A. Stegun, *Handbook of mathematical funtions* , (Dover Publ. New York, 1970).

[B] V. Bargmann, On A Hilbert Space of Analytic Functions and an associated Integral Transform (I), *Communcations on pure and applied mathematics XIV* (1961) pp. 187-214

[D] J. Dieudonné, *Treatise on Analysis* (Academic Press New York, vol II & V, 1976).

[EG1] S.J.L. van Eijndhoven and J. de Graaf, A mathematical interpretation of Dirac's formalism part a-b-c, *Reports on Mathematical Physics* (22, no. 2,1985)

[EG2] S.J.L. Eijndhoven and J. de Graaf, A measure theoretical Sobolev Lemma, *Journal of Functional Analysis* (63(1),1985,pp. 74-85)

[Fa] R.C. Fabec, *Fundamentals of infinite dimensional representation theory*, (Chapman & Hall/CRC, 2000).

[FH] J. Faraut and K. Harzallah. *Deux cours d' analyse harmonique: Ecole d'Été d'Analyse Harmonique* (Birkhauser Basel,1987)

[FK] M. Fashandi, R.A. Kamyabi Gol, A. Nikman, and M. A. Pourabdollah, Continuous wavelet trnasform on a special homogeneous space, *Jour. of Math. Phys. 44(9)* (2003), pp 4260-4266.

[FM] Hartmut Führ, Matthias Mayer, Continuous Wavelet Transforms from Cyclic Representations: A General Approach Using Plancherel Measure, *Journal of Fourier Analysis and Applications* 8 (2002), pp. 375-398.

[Fo] Folland, G.B., *A course in abstract Harmonic analysis*, (CRC Press, Boca Raton, FL, 1995).

[GMP] A. Grosmann, J. Morlet and T. Paul, Transform associated to square integrable group representations, *Jour of Math. Phys. 23(10)* (1985), pp 2473-2479.

[KHV] S. N. Kalitzin, B.M. ter Haar Romeny and M.A. Viergever, Invertible apertured orientation filters in image analysis, (31(2/3): 145-158, 1999)

[Ma] F.J.L. Martens, *Spaces of analytic functions on inductive/projective limits of Hilbert Spaces*, (1988). [1]

[Mu] C. Mueller, *Analysis of spherical symmetries in Euclidean spaces*, (Springer Berline, 1998)

[R] H. Reiter and J.D.Stegeman, *Classical harmonic analysis and locally compact groups* (Oxford at the Clarendon press, 2000).

[T] M.E.Taylor, *Noncommutative Harmonic Analysis*, (American Mathematical Society , 1985)

[Y] K. Yosida, *Functional Analysis* (Springer Verlag , Edition 1980).

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