

Solution to Problem 64-7 : An asymptotic series

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In view of (1) and (2), we then have

$$\frac{V}{M} = \frac{1}{u} F\left(-\frac{1}{2}, \frac{1}{2}; 2; \frac{a^2}{u^2}\right).$$

Since the attractive force F equals $-\partial V/\partial z$, the required force is then found to be given by

$$\frac{F}{M} \frac{z}{(z^2 + b^2)^{3/2}} F\left(-\frac{1}{2}, \frac{3}{2}; 2; \frac{a^2}{z^2 + b^2}\right).$$

Also solved by T. C. ANDERSON (Lockheed Missiles and Space Co.) in terms of elliptic functions directly from the force integral.

Problem 64-7, An Asymptotic Series, by N. G. DE BRUIJN (Technological University, Eindhoven, Netherlands).

Let $\phi(x)$ be infinitely often differentiable for $x \geq 0$, and let

$$\int_0^\infty |\phi^{(k)}(x)| dx$$

be convergent for each $k = 0, 1, 2, \dots$. Define

$$F(t) = \sum_{n=1}^{\infty} n^{-1} \phi(nt), \quad t > 0.$$

Show that $F(t) + \phi(0) \log t$ has an asymptotic development in the form of an asymptotic series $\sum_{n=0}^{\infty} c_n t^n$ if $t > 0$, $t \rightarrow 0$.

Solution by the proposer.

Introducing a positive constant λ , we put $\phi_1(x) = \phi(x) - \phi(0)e^{-\lambda x}$. Then ϕ_1 still has the properties attributed to ϕ , and moreover $\phi_1(0) = 0$. We put $x^{-1}\phi_1(x) = \eta(x)$, and we apply the Euler-Maclaurin sum formula to $\sum_0^\infty \eta(nt)$ (we can apply it to the infinite series since $\eta^{(k)}(x) \rightarrow 0 (x \rightarrow \infty)$ for each k , and $\int_0^\infty |\eta^{(k)}(x)| \cdot dx \rightarrow \infty$):

$$\begin{aligned} \sum_1^\infty \eta(nt) &= t^{-1} \int_0^\infty \eta(x) dx - \frac{1}{2} \eta(0) - \sum_{k=1}^m \frac{B_{2k} t^{2k-1} \eta^{(2k-1)}(0)}{(2k)!} \\ &\quad - t^{2m} \int_0^\infty \frac{\eta^{(2m)}(x) B_{2m}(tx - [tx])}{(2m)!} dx. \end{aligned}$$

Thus we obtain the following asymptotic series for $F(t) + \phi(0) \log t$:

$$\begin{aligned} \int_0^\infty x^{-1}(\phi(x) - \phi(0)e^{-\lambda x}) dx - \phi(0) \log \frac{1 - e^{-\lambda t}}{\lambda t} \\ - \frac{1}{2} t \eta(0) - \sum_{k=1}^{\infty} \frac{B_{2k} t^{2k} \eta^{(2k-1)}(0)}{(2k)!}. \end{aligned}$$

We finally want to get rid of λ . We note that $\int_0^\infty x^{-1}(e^{-x} - e^{-\lambda x}) dx = \log \lambda$ and

that the coefficients of the asymptotic development of $F(t) + \phi(0) \log t$ should not depend on λ . We now evaluate these coefficients by taking $\lambda = 0$. Then we have $\eta^{(k)}(0) = (k + 1)^{-1}\phi^{(k+1)}(0)$, and we obtain for the asymptotic series

$$\int_0^\infty x^{-1}(\phi(x) - \phi(0)e^{-x}) dx - \frac{1}{2}t\phi'(0) - \sum_{k=1}^\infty \frac{B_{2k} t^{2k} \phi^{(2k)}(0)}{(2k)(2k)!}.$$

We remark that there is strict equality if $\phi(x) = e^{-\lambda x}$ ($\lambda > 0$).

Also solved by L. A. SHEPP (Bell Telephone Laboratories).

Problem 64-8, A Definite Integral, by P. J. SHORT (White Sands Missile Range).

Evaluate the integral

$$I(x) = \int_0^x \frac{te^{t^2} \operatorname{erf} t dt}{\sqrt{x^2 - t^2}}, \quad x > 0.$$

The solutions of HERBERT B. ROSENSTOCK (U. S. Naval Research Laboratory), PERRY SCHEINOK (The Hahnemann Medical College) and SIDNEY SPITAL (California State Polytechnic College) were the same and are as follows:

Changing the variable t to z by $z = \sqrt{x^2 - t^2}$, we obtain

$$I(x) = \frac{2e^{x^2}}{\sqrt{\pi}} \int_0^x e^{-z^2} dz \int_0^{\sqrt{x^2 - z^2}} e^{-y^2} dy.$$

Transforming to polar coordinates, we then get

$$I(x) = \frac{2e^{x^2}}{\sqrt{\pi}} \int_0^{\pi/2} d\theta \int_0^x e^{-r^2} r dr = \frac{\sqrt{\pi}}{2} (e^{x^2} - 1).$$

Also solved by DONALD E. AMOS (University of Missouri), A. D. ANDERSON, J. B. LANGWORTHY, and A. W. SAENZ, jointly (U. S. Naval Research Laboratory), C. J. BOUWKAMP (Technological University, Eindhoven, Netherlands), J. L. BROWN, JR. and H. S. PIPER, JR., jointly (Ordnance Research Laboratory), R. G. BUSCHMAN (University of Buffalo), C. COMSTOCK, two solutions (Pennsylvania State University), C. R. DE PRIMA (California Institute of Technology), H. E. FETTIS (Wright-Patterson A. F. B.), WILLIAM D. FRYER (Cornell Aeronautical Laboratory), M. LAWRENCE GLASSER (University of Wisconsin), ELDON HANSEN (Lockheed Missiles and Space Co.), RICHARD P. KELISKY (IBM Watson Research Center), ANTHONY J. STRECKO (Argonne National Laboratory), ANDREW H. VAN TUYL, two solutions (Naval Ordnance Laboratory), and the proposer.

Editorial Note: Most of the other solutions were obtained by either expanding out $\operatorname{erf} t$ in a power series and integrating termwise or else by first showing that

$$I'(x) = x\sqrt{\pi} + 2xI(x).$$

Fryer also notes that

$$\int_0^x \frac{te^{t^2} \operatorname{erfc} t dt}{\sqrt{x^2 - t^2}} = \frac{\sqrt{\pi}}{2} \{1 - e^{x^2} \operatorname{erfc} x\}$$

with the limit $\sqrt{\pi}/2$ as $x \rightarrow \infty$.