Generalization of Pólya's fundamental theorem in enumerative combinatorial analysis

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MATHEMATICS

GENERALIZATION OF POLYA'S FUNDAMENTAL THEOREM IN ENUMERATIVE COMBINATORIAL ANALYSIS

BY

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1. *Introduction*

It is the task of enumerative combinatorial analysis to determine the number of solutions of any given combinatorial problem. In many cases the question can be put into the following form: Find the number of functions which map a given finite set D (the domain) into another given finite set R (the range) and which satisfy certain conditions. Every such function represents a solution of the combinatorial problem under consideration.

It often happens, however, that we are not interested in the number of solutions as such, but in the number of classes of solutions. This happens if there is an equivalence notion in the set of all solutions, and if we want to consider two equivalent solutions as one and the same possibility. A simple example is the question of the number of different dice. A die is a cube, the faces of which are numbered with the numbers 1, 2, 3, 4, 5, 6, each number occurring just once. The set D is the set of faces, the set R is $\{1, 2, 3, 4, 5, 6\}$. There are obviously $6!$ solutions. But we wish to consider two solutions f_1, f_2 as equivalent if f_1 can be transformed into f_2 by a rotation of the cube. For, when two such equivalent dice are thrown on the table, and this is what one usually does with dice, we can see no difference between them. One easily finds out that there are 30 classes, by the following argument. If we apply the 24 different rotations which transform the cube into itself, a single solution produces 24 different solutions, and we infer that each equivalence class contains exactly 24 solutions. Therefore, the number of classes is $6!/24=30$.

This simple argument does not succeed, however, in the following problem, where the 24 solutions which are obtained from a given solution by rotations of the cube, are not always all different: The faces of the cube are to be painted with the colours red, white and blue, in such a way that two faces are red, two are white and two are blue. With the equivalence relation induced by the rotations of the cube, the number of solutions is required (the answer is 6).

In a general form the problem was proposed and solved by Polya [1]. Pólya considered a finite set D and a permutation group G operating on D. Two functions f_1, f_2, defined on D are called equivalent if there is a permutation $g \in G$ such that $f_1(d) = f_2(gd)$ for all $d \in D$ (gd denotes the effect of the permutation g on the element d, so $gd \in D$). Furthermore, there is a collection of disjoint finite sets $R_1, R_2, \ldots$. If $k_1, k_2, \ldots$ are given integers $\geqslant 0$, whose sum equals the number of elements of D, the problem is to find the number of classes of mappings f of D into the union R of $R_1, R_2, \ldots$, with the property that for k_i values of $d \in D$ the image $f(d)$ lies in R_i ($i = 1, 2, \ldots$).

The solution uses generating functions. It attaches a variable w_1 to R_1, w_2 to $R_2, \ldots$. If n_i is the number of elements of R_i, the expression

$$\sum n_i w_i$$

is called the store enumerator of R (R is called the store; the terminology can be found in RIORDAN [2]). Now Pólya's fundamental theorem asserts that the required number of classes equals the coefficient of $w_1^{k_1} w_2^{k_2} \ldots$ in the expansion of

$$P_G(\sum n_i w_i, \sum n_i w_i^2, \sum n_i w_i^3, \ldots). \tag{1.1}$$

P_G represents the so-called *cycle index* (Zyklenzeiger) of G. Its definition depends on the cyclic representation of the elements of G, considered as permutations of D. The element $g \in G$ is said to have the type $\{b(1), b(2), b(3), \ldots\}$ if there are $b(1)$ cycles of length 1, $b(2)$ cycles of length 2, etc. (A *cycle of length* k is a set of k elements of D which are cyclically permuted by g, so $b(1) + 2b(2) + 3b(3) + \ldots$ equals the number of elements of D). To this element g we attach the term $x_1^{b(1)} x_2^{b(2)} \ldots$. The sum of these terms (if g runs through G), divided by the number of elements of G, is denoted by $P_G(x_1, x_2, \ldots)$. Notice that P_G is not determined by the structure of G itself, but by its actual representation as a group of permutations of D.

In this paper, Pólya's theorem will appear as a special case of a more general theorem (see the beginning of sec. 3).

Pólya's result holds true if there are infinitely many R_i. It is, however, sufficient to show it for the case of finitely many R_i, for then the infinite case can be obtained by remarking that in every term $w_1^{k_1} w_2^{k_2} \ldots$ only a finite number of w_i have a positive exponent, so that all but a finite number of R_i can be discarded as far as this term is concerned. Accordingly, the results in the present paper will be stated and proved for a finite number of R_i, but will remain true if there are infinitely many of them.

For future reference we quote some examples. Let D have n elements. If G consists of the unit element only, we obviously have

$$P_G(x_1, x_2, \ldots) = x_1^n.$$

We also consider the case that G is the full symmetric group, i.e. the group of all $n!$ permutations of D. Then $P_G(x_1, x_2, \ldots)$ equals (see [2], p. 68) the coefficient of y^n in the expansion of

$$\exp\,(yx_1 + {}^1/_2 y^2 x_2 + {}^1/_3 y^3 x_3 + \ldots).$$

The following result has some connection with this paper, though it is not used explicitly. Let D_1 and D_2 be disjoint sets, let H be a group of permutations of D_1, and K a group of permutations of D_2. Then every element $h \times k$ of the direct product $H \times K$ ($h \in H$, $k \in K$) defines a permutation of the union $D_1 \cup D_2$. So we have a group of permutations, $H \times K$ acting on $D_1 \cup D_2$, and it is easy to see that (see Pólya [1])

$$P_{H\times K}(x_1, x_2, \ldots) = P_H(x_1, x_2, \ldots)\, P_K(x_1, x_2, \ldots).$$

The same thing holds for direct products of more than two factors.

In this paper we shall give an extension of Pólya's theorem, and also a related theorem.

In the first place we generalize the equivalence relation by adding equivalences which arise, in an obvious way, from permutation groups acting on $R_1, R_2, \ldots$. An example of a problem of this type is the following. We have a tetrahedron and a cube. Every vertex of the tetrahedron has to be connected by a wire to a vertex of the cube. (It is allowed that two different wires lead to one and the same vertex of the cube, but just one wire is attached to each vertex of the tetrahedron). Two possibilities are called equivalent if the first one can be transformed into the other one by rotations of the solids. (Only the connections are taken into consideration, and not the way the wires are interlinked in the space between the solids). The problem is to find the number of equivalence classes of solutions (the answer is 27).

In the second place we generalize by attaching weights to the solutions, in order to obtain more detailed information. In particular, we can arrange the weights in such a way that we can determine the number of classes of one-to-one mappings.

In sec. 4 the case $D = R$ is considered, and equivalence is obtained by application of one and the same permutation of G both to the domain and the range.

2. *Generalization of Pólya's theorem*

Let D and R be finite sets, and let G and H be groups acting on D and R, respectively. By $\mathfrak{f}$ we denote the collection of all mappings of D into R.

In $\mathfrak{f}$ there is an equivalence notion induced by G and H. The elements $f_1 \in \mathfrak{f}$, $f_2 \in \mathfrak{f}$ are called equivalent if there exist elements $g \in G$, $h \in H$ such that $f_1(d) = hf_2(gd)$ for all $d \in D$ (gd denotes the effect of the permutation g on the element $d \in D$, so $gd \in D$, and hr is the effect of h on the element $r \in R$, so $hr \in R$).

This equivalence notion gives rise to a dissection of $\mathfrak{f}$ into classes. We shall denote the set of all classes by $\mathfrak{F}$, and for the elements of $\mathfrak{F}$ we shall use the letter F (so every F is an equivalence class of mappings of D into R).

Next we shall attach a value $W(f)$ to every $f \in \mathfrak{f}$, to be called the *weight* of f, in such a way that equivalent functions have the same weight. Accordingly, if $F \in \mathfrak{F}$, we shall denote by $W(F)$ the common value of all $W(f)$ arising from functions f belonging to the class F.

The number of classes is given by the following

Lemma. If $|G|$ denotes the number of elements of G, and $|H|$ denotes the number of elements of H, then we have

$$\sum_{F \in \mathfrak{F}} W(F) = |G|^{-1} |H|^{-1} \sum_{g \in G} \sum_{h \in H} \sum\nolimits_f^* W(f). \tag{2.1}$$

Here $\sum_f^*$ denotes that we only sum for those $f \in \mathfrak{f}$ which satisfy

$$f(gd) = hf(d) \text{ for all } d \in D.$$

Proof. We shall use a well-known device from the theory of permutation groups, used by Pólya in his proof of the fundamental theorem. Let the elements of a group K act as permutations of a set S. That is to say, to each element $k \in K$ there corresponds a permutation π_k of S, such that $\pi_k \pi_l = \pi_{kl}$ $(k \in K, l \in K)$, but different elements of K need not correspond to different permutations. Two elements $s_1 \in S$, $s_2 \in S$ are called equivalent if there is some $k \in K$ with $\pi_k s_1 = s_2$.

For every element $s \in S$ there is a subgroup K_s of K, consisting of all $k \in K$ which leave s invariant, i.e. which satisfy $\pi_k s = s$. The index o K_s in K (i.e. $|K|/|K_s|$) equals the number of elements in the equivalence class to which s belongs. It follows that the number of equivalence classes equals

$$|K|^{-1} \sum_{s \in S} |K_s| = |K|^{-1} \sum_{s \in S} (\text{number of } k \in K \text{ for which } \pi_k s = s).$$

Next assume that we have defined, for each $s \in S$, a value $w(s)$ (the *weight* of s), such that $w(s)$ is constant on every class. Then the argument used above shows that the sum of the weights of the classes equals

$$|K|^{-1} \sum_{s \in S} w(s) \cdot (\text{number of } k \in K \text{ for which } \pi_k s = s),$$

and this can be written as

$$|K|^{-1} \sum_{k \in K} \sum\nolimits^* w(s). \tag{2.2}$$

The * indicates that we only sum for those $s \in S$ which satisfy $\pi_k s = s$.

We shall apply this to our case. For S we take the set $\mathfrak{f}$, and K will be the direct product $G \times H$. If $g \in G, h \in H$, then the effect of $g \times h$ on $f \in \mathfrak{f}$ is defined by

$$\pi_{g \times h} f = f_1, \text{ where } f_1(d) = hf(g^{-1}d) \text{ for all } d \in D,$$

so that indeed $\pi_k \pi_l f = \pi_{kl} f$ (if $k = g_1 \times h_1$, $l = g_2 \times h_2$).

And finally, for w we shall take the weight W, introduced above.

The equivalence classes in this S, defined by this K, are obviously the same classes F we considered before. Now (2.2) is easily translated into (2.1).

We shall now make some special assumptions about H and W. In the first place we assume that R is the union of a number of disjoint subsets $R_1, R_2, \ldots, R_k$ and that H is the direct product of groups $H_1 \times H_2 \times \ldots \times H_k$, where H_i is a group of permutations of R_i ($i=1, \ldots, k$). (This is no extra assumption, of course, since we can always take $k=1$). Next, we assume that, for $i=1, \ldots, k$; $n=1, 2, \ldots$, a value $\psi_i(n)$ is defined. If $r \in R_i$, we define $\psi(r, n)=\psi_i(n)$. We now construct a weight function as follows. If $f \in \mathfrak{f}$, $r \in R$, then $n(f, r)$ denotes the number of elements $d \in D$ which are mapped by f onto r. Then we put

$$W(f)=\prod_{r \in R} \psi(r, n(f, r)). \tag{2.3}$$

It is obvious that (2.3) is a weight function. For, if f_1 and f_2 are equivalent, then $f_1=hf_2g$ for some $h \in H$, $g \in G$. The permutation g has no effect on the numbers $n(f, r)$, and h only permutes the factors in the product.

The values $\psi(r, n)$ need not be numbers, but can represent elements of some commutative ring. In the applications it will be the ring of all formal power series in a number of indeterminates.

We now first introduce an abbreviation: If t is a positive integer, and if $r \in R$, then $\Psi_{i,t}(\zeta_1, \zeta_2, \ldots)$ denotes the power series which arises from

$$\exp\{t(\zeta_1 x+\zeta_2 x^2+\ldots)\}=\sum_{n=0}^{\infty} c_{nt}(\zeta_1, \zeta_2, \ldots)x^n$$

(the coefficients $c_{nt}(\zeta_1, \zeta_2, \ldots)$ are defined by this identity), if we replace x^n by $\{\psi_i(n)\}^t$, so

$$\Psi_{i,t}(\zeta_1, \zeta_2, \ldots)=\sum_{n=0}^{\infty} c_{nt}(\zeta_1, \zeta_2, \ldots)\{\psi_i(n)\}^t.$$

Theorem 1. If $W(f)$ is given by (2.3), then we have

$$\sum_{F \in \mathfrak{F}} W(F)=P_G\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots\right)\prod_{i=1}^{k} P_{H_i}(\eta_{i,1}, \eta_{i,2}, \ldots), \tag{2.4}$$

where

$$\eta_{i,t}=\Psi_{i,t}(z_t, z_{2t}, z_{3t}, \ldots) \qquad (i=1, \ldots, k,\ t=1, 2, \ldots),$$

and after the application of the differential operator we have to put $z_1=z_2=\ldots=0$.

Proof. We start from (2.1). Choosing $g \in G$, $h \in H$, we want to evaluate $\sum^* W(f)$. We have $h=h_1 \times h_2 \times \ldots \times h_k$, where $h_i \in H_i$. Represent the type of g by $\{b(1), b(2), \ldots\}$, and the type of h_i by $\{c(i, 1), c(i, 2), \ldots\}$.

By a "block of length l" in D we shall denote one of the $b(l)$ subsets of l elements which are cyclically permuted by g, and a similar terminology will be used in R_i. The length of a block B will be denoted by $\lambda(B)$.

It is not difficult to describe the functions f which have the property that $f(gd)=hf(d)$ for all $d \in D$. It is clear that if we consider a block B of length $\lambda(B)$ in D, then all $f(d)$ $(d \in B)$ belong to one and the same block C of R. The length $\lambda(C)$ has to be a divisor of $\lambda(B)$, and if d runs through B, then $f(d)$ runs $\lambda(B)/\lambda(C)$ times through C. So all admissible functions f are obtained as follows: to every block B of D we attach any block C of R whose length divides $\lambda(B)$, and then we still have the possibility to attach any arbitrary element of C to some pre-assigned element of B. Once this has been done the images of the other elements of B are uniquely determined by the condition that $f(d)=r$ implies $f(gd)=hr$.

It now follows from (2.3) that $\sum^* W(f)$ can be obtained as follows: Associate a variable w_C to each block C of R. Consider

$$\prod_B \{\sum_C \lambda(C) w_C^{\lambda(B)/\lambda(C)}\, \varepsilon(\lambda(B), \lambda(C))\}, \tag{2.5}$$

where B runs through the blocks of D, C runs through the blocks of R, and $\varepsilon(\lambda(B), \lambda(C))=1$ if $\lambda(C)$ divides $\lambda(B)$, $\varepsilon(\lambda(B), \lambda(C))=0$ otherwise. It is not difficult to see that if we put all w_C equal to 1, then (2.5) represents the total number of functions under consideration in the case that all weights are equal to 1. In order to deal with a general weight function (of the type (2.3), we write (2.5) as a polynomial in the variables w_C, i.e. as a linear combination of products of the form $\prod_C w_C^{\alpha(C)}$ (where the $\alpha(C)$ are integers $\geqslant 0$). In each term we replace

$$\prod_C w_C^{\alpha(C)} \text{ by } \prod_C \{\psi_{i(C)}(\alpha(C))\}^{\lambda(C)},$$

if $i(C)$ is the index with $R_{i(C)} \supset C$. (Since the R_i are transformed onto themselves by h, it is clear that each block C is a part of one of the R_i). This operation, which replaces the w-products by ψ-products, will be denoted by $\boldsymbol{T}$. It is not difficult to verify that $\sum^* W(f)$ is obtained by application of $\boldsymbol{T}$ to (2.5).

In (2.5) every B gives rise to a factor in the product, and this factor only depends on the length of B. Abbreviating the product to $\prod \chi(\lambda(B))$, we notice that it can be written as

$$\prod_{l=1}^{\infty} (\chi(l))^{b(l)} = \left(\frac{\partial}{\partial z_1}\right)^{b(1)} \left(\frac{\partial}{\partial z_2}\right)^{b(2)} \cdots \exp\{\sum_{l=1}^{\infty} \chi(l) z_l\},$$

evaluated at $z_1=z_2=\ldots=0$. So summing in (2.1) for g, we obtain

$$\sum_{F \in \mathfrak{F}} W(F) = P_G\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots\right) |H|^{-1} \sum_{h \in H} \boldsymbol{T} \exp\{\sum_{l=1}^{\infty} z_l \sum_C \lambda(C) w^{l/\lambda(C)}\, \varepsilon(l, \lambda(C))\}.$$

The double sum which occurs in the exponential can be rearranged, by splitting R into $R_1, R_2, \ldots$. For $i=1, \ldots, k$; $t=1, 2, \ldots$, we have $c(i, t)$ blocks of length t in R_i. We denote these blocks by C_{itj} $(j=1, \ldots, c(i, t))$, and if $C=C_{itj}$ we write w_{itj} instead of w_C. We obtain that

$$\sum_l \sum_C = \sum_{i=1}^{k} \sum_{t=1}^{\infty} \sum_{j=1}^{c(i,t)} t(z_t\, w_{itj} + z_{2t}(w_{itj})^2 + z_{3t}(w_{itj})^3 + \ldots).$$

We next evaluate

$$\boldsymbol{T} \exp \sum_{l} \sum_{C} = \prod_{i=1}^{k} \prod_{t=1}^{\infty} \prod_{j=1}^{c(i,t)} \boldsymbol{T} \exp \{t(z_t w_{itj} + z_{2t}(w_{itj})^2 + \dots)\} =$$

$$= \prod_{i=1}^{k} \prod_{t=1}^{\infty} \{\Psi_{i,t}(z_t, z_{2t}, z_{3t}, \dots)\}^{c(i,t)}.$$

If we sum this for all $h \in H$, we obtain, after division by $|H|$,

$$\prod_{i=1}^{k} P_{H_i}(\Psi_{i,1}(z_1, z_2, z_3, \dots), \Psi_{i,2}(z_2, z_4, z_6, \dots), \dots),$$

and the theorem follows.

We shall consider two special cases, obtained by specializing the weight function. In the first case we take variables $w_1, w_2, \dots,$ and we put $\psi_i(n) = w_i^n$. So the weight becomes

$$W_0(f) = w_1^{m_1} w_2^{m_2} \dots,$$

where m_i is the number of $f(d)$ $(d \in D)$ falling into R_i.

In the second case we take

$$W_1(f) \begin{cases} = W_0(f) & \text{if } f \text{ is a one-to-one mapping of } D \text{ into } R, \\ = 0 & \text{otherwise.} \end{cases}$$

This means that we take $\psi_i(0) = 1$, $\psi_i(1) = w_i$, $\psi_i(2) = \psi_i(3) = \dots = 0$. By substitution into theorem 1 we easily obtain the following two theorems.

Theorem 2. If we put

$$\exp \{t(z_t w_i^t + z_{2t} w_i^{2t} + \dots)\} = \varrho_{i,t},$$

then we have

$$\sum_{F \in \mathfrak{F}} W_0(F) = P_G\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \dots\right) \prod_{i=1}^{k} P_{H_i}(\varrho_{i,1}, \varrho_{i,2}, \dots),$$

evaluated at $z_1 = z_2 = \dots = 0$.

Theorem 3. For the weighted one-to-one mappings we have

$$\sum_{F \in \mathfrak{F}} W_1(F) = P_G\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \dots\right) \prod_{i=1}^{k} P_{H_i}(1 + w_i z_1, 1 + 2w_i^2 z_2, 1 + 3w_i^3 z_3, \dots)$$

evaluated at $z_1 = z_2 = \dots = 0$.

3. *Applications*

It is not difficult to see that Theorem 2 specializes to Pólya's theorem if we assume that each H_i consists of the identity only. Then we have

$$P_{H_i}(x_1, x_2, \dots) = x_1^{n_i},$$

where n_i is the number of elements of R_i. The product in Theorem 2 becomes

$$\exp \{\sum_i n_i(z_1 w_i + z_2 w_i^2 + \dots)\} = \exp \{z_1 \sum n_i w_i + z_2 \sum n_i w_i^2 + \dots\},$$

and the effect of the differential operator at $z_1=z_2=\ldots=0$ is easily seen to be equal to the expression (1.1).

We next specialize Theorem 2 to the case that there is only one R_i; we can denote this one by R. So we have groups G and H, operating on D and H, respectively. Let D have m elements, and let R have n elements. There is no loss if we replace w_1 by 1, since every f gets the same weight w^m in this case.

If G is the symmetrical group of all $m!$ permutations of D, then the differential operator P_G equals the coefficient of y^m in the expansion of

$$\exp\left(y\frac{\partial}{\partial z_1}+\frac{1}{2}y^2\frac{\partial}{\partial z_2}+\frac{1}{3}y^3\frac{\partial}{\partial z_3}+\ldots\right),$$

and the effect of this operator on a function $\varphi(z_1, z_2, \ldots)$ at $z_1=z_2=\ldots=0$ equals $\varphi(y, {}^1/_2y^2, {}^1/_3y^3, \ldots)$, according to Taylor's series theorem. Therefore, the required number of classes is the coefficient of y^m in $P_H(\varrho_1, \varrho_2, \ldots)$, where

$$\varrho_t=\exp\left\{t\left(\frac{y^t}{t}+\frac{y^{2t}}{2t}+\frac{y^{3t}}{3t}+\ldots\right)\right\}=(1-y^t)^{-1}.$$

So the number of classes equals the coefficient of y^m in

$$P_H((1-y)^{-1}, (1-y^2)^{-1}, (1-y^3)^{-1}, \ldots).$$

This can also be deduced from Pólya's theorem. For, the classes we are considering at present, can easily be brought into one-to-one correspondence with the classes of functions φ on R, with 0, 1, 2, ... as admissible values, with $\sum_{r\in R}\varphi(r)=m$, and with an equivalence relation defined by H. The correspondence is as follows: if f is a mapping of D into R, then to f we make correspond φ, where (for each $r\in R$) $\varphi(r)$ is the number of $d\in D$ with $f(d)=r$.

Again considering the case that there is only one R_i, but without specialization of either G or H, we notice that the result of Theorem 2 can be written in a form which can be more suitable for applications. We again take $w_1=1$. It is easily seen that any term $x_1^{c(1)}x_2^{c(2)}\ldots$ of $P_H(x_1, x_2, \ldots)$ gives rise to a contribution

$$P_G\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots\right)\varrho_{1,1}^{c(1)}\varrho_{1,2}^{c(2)}\ldots=P_G\left(\sum_{t/1}tc(t), \sum_{t/2}tc(t), \ldots\right).$$

($\sum_{t/k}$ denotes that t runs through all divisors of k).

For example, if $n=2$, and if H is the symmetrical group of order 2, we have $P_H={}^1/_2(x_1^2+x_2)$, and we obtain

$$\sum_{F\in\mathfrak{F}}1={}^1/_2P_G(2, 2, 2, \ldots)+{}^1/_2P_G(0, 2, 0, 2, \ldots). \tag{3.1}$$

If $n=3$, and if H is the symmetrical group of order 3!, we have $P_H={}^1/_6x_1^3+{}^1/_2x_1x_2+{}^1/_3x_3$, whence

$$(3.2)\quad \sum_{F\in\mathfrak{F}} 1={}^1/_6P_G(3,3,3,\ldots)+{}^1/_2P_G(1,3,1,3,\ldots)+{}^1/_3P_G(0,0,3,0,0,3,\ldots).$$

We mention a simple application of the latter formula. We want to colour the sides of a square in the plane, and there are three colours available. We are not interested in the colour schemes themselves, but only in their types: two colour schemes are said to be of the same type if they can be obtained from each other by rotation of the square and by permutations in the store of colours. The number of types is obtained from (3.2) if we take for G the cyclic group of order 4. Its cycle index is $P_G(x_1, x_2, \ldots)={}^1/_4(x_1^4+x_2^2+2x_4)$. So the number of types is

$${}^1/_4\{{}^1/_6(3^4+3^2+2\cdot 3)+{}^1/_2(1+3^2+2\cdot 3)\}=6.$$

This is easily verified. The six types are: (i) all sides have the same colour; (ii) three sides have the same colour, the fourth side has a second colour; (iii) two adjacent sides have the same colour, the remaining sides have a second colour; (iv) two opposite sides have the same colour, the remaining sides have a second colour; (v) two adjacent sides have the same colour, the remaining sides have a second and a third colour, respectively; (vi) two opposite sides have the same colour, the remaining sides have a second and a third colour.

Another application refers to the case $n=2$. We have two elements in R, and for H we take the identity only. Then the number of classes is just the number of classes of subsets of D (two subsets are said to be equivalent if the first can be transformed into the second by a permutation of G). The cycle index of H being x_1^2, we find $P_G(2, 2, 2, \ldots)$ for the number of classes. If we compare this to (3.1), we find information on the subsets which are equivalent to their complements. The result is that there are $P_G(0, 2, 0, 2, \ldots)$ classes which consist of subsets of this structure.

Applying this to the cube (with ordinary rotations), we obtain that there are 7 types of sets of vertices which are equivalent to their complements, and similarly 2 types of sets of faces and 10 types of sets of edges. This follows from the cycle index, which is

${}^1/_{24}(x_1^8+6x_4^2+9x_2^4+8x_1^2x_3^2)$ for the vertices,

${}^1/_{24}(x_1^6+6x_1^2x_4+3x_1^2x_2^2+6x_2^3+8x_3^2)$ for the faces, and

${}^1/_{24}(x_1^{12}+6x_4^3+3x_2^6+6x_1^2x_2^5+8x_3^4$ for the edges.

We shall give a single application of Theorem 3 We take for D a set with m elements, and for G the group consisting of he unit element only. We take only one R, permuted by an arbitrary group H. And we put $w_1=1$. We obtain that

$$(3.3)\qquad \sum_{F\in\mathfrak{F}} W_0(F)=\left\{\left(\frac{d}{dz}\right)^m P_H(1+z, 1, 1, \ldots)\right\}_{z=0}.$$

The left-hand-side enumerates the classes of *m-permutations* from R. An m-permutation (or variation, or labelled subset) is an ordered selection of m different elements from R, and two such m-permutations belong to the same class if the one can be transformed into the other by a permutation of H.

Notice that the right-hand-side of (3.3) can be read as

$$m!\,|H|^{-1}\sum_{h\in H}\binom{b_1(h)}{m},$$

where $b_1(h)$ is the number of cycles of length 1 in the permutation h, and the binomial coefficient represents 0 if $b_1(h)<m$.

We now return to the general situation to which Theorem 1 refers, but we give a special rôle to R_1.

We take for R_1 the set of positive integers, and for H_1 we take the group consisting of the identity only. And we simply take: $\psi_1(n)=1$ for all n. But we add an extra weight factor $\varrho(f)$ (and as it only depends on the equivalence class of f, we may also write $\varrho(F)$ if F is the class containing f). Choosing a variable v, we take $\varrho(f)=v^N/N!$ if there are N elements of D with $f(d)\in R_1$, and if moreover, these $f(d)$'s represent the numbers $1, \ldots, N$ in some order. We take $\varrho(f)=1$ if no $f(d)$ lies in R_1, but $\varrho(f)=0$ if there are two different elements d_1, d_2 of D with $f(d_1)=f(d_2)\in R_1$, and also $\varrho(f)=0$ if there are numbers $n_1\in R_1$, $n_2\in R_1$, $n_1<n_2$, where $n_2=f(d)$ has a solution in D but $n_1=f(d)$ has not.

In other words, what we are doing is this: We consider any mapping of any part of D into the union of $R_2, R_3, \ldots$, whereas the remaining elements of D have to be labelled with consecutive numbers $1, 2, \ldots$ (each number occurring just once). Two such situations are called equivalent if there are elements $g\in G$, $h\in H$ by which one of them is transformed into the other.

The answer is that $\sum_{F\in\mathfrak{F}} W(F)\,\varrho(F)$ is obtained from the right-hand-side of (2.4) if one replaces $P_H(\eta_{1,1}, \eta_{1,2}, \ldots)$ by $\exp(vz_1)$. It is not difficult to show this. If we choose a special integer N, and if we count the number of classes of functions (with the prescribed weights) of which N values fall into R_1, we can act as if R_1 had just the N elements $1, \ldots, N$ and as if we would only be interested in functions which attain each one of these values exactly once, provided that we multiply the number obtained this way by $v^N/N!$. For this new problem, the cycle index of R_1 (with the trivial group) is x_1^N. Accordingly, we have to replace P_{H_1} in (2.4) by $(N!)^{-1}(1+vz_1)^N$ (since we want one-to-one mappings as far as R_1 is concerned, see Theorem 3) and from the expression obtained this way we need the term with v^N (since we require that there are N elements of D which are mapped into R_1). This means that P_{H_1} has to be replaced by $(vz_1)^N/N!$. Summing for N, we obtain the result announced above.

The result we just obtained can be extended to more complicated arrangements about the labelling of functions. For the case that all H_i consist of the unit element only, such possibilities are indicated by RIORDAN ([2], p. 132–133).

4. *The case $D=R$ with simultaneous permutations*

Consider a finite set D with a permutation group G, and consider functions f which map D onto D, in other words, permutations of D. Two such permutations f_1, f_2 are called equivalent if there is an element $g \in G$ such that $g^{-1}f_1g=f_2$ (i.e. such that $g^{-1}f_1(gd)=f_2(d)$ for all $d \in D$). We ask for the number of classes.

Theorem 4. The number of classes (with respect to G) of permutations of a set D of m elements equals

$$\int_0^\infty \dots \int_0^\infty \exp\left(-(x_1+\dots+x_m)\right) P_G(x_1, 2x_2, 3x_3, \dots)\, dx_1 \dots dx_m =$$

$$= \left\{ P_G\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \dots\right) (1-z_1)^{-1}(1-2z_2)^{-1}(1-3z_3)^{-1} \dots \right\}_{z_1=z_2=\dots=0}.$$

Proof. By the device used in the beginning of the proof of the lemma in sec. 2, we obtain that the number is

$$|G|^{-1} \sum_{g \in G} (\text{number of permutations } f \text{ with } fg=gf).$$

If g has the type $\{b(1), b(2), \dots\}$, then the number of f with $fg=gf$ is easily seen to be

$$b(1)!\, 1^{b(1)}\, b(2)!\, 2^{b(2)} \dots,$$

and the theorem follows.

As a verification, we take for G the symmetric group. Then the number of classes is equal to the number of partitions of m. And indeed, $P_G(x_1, 2x_2, \dots)$ is the coefficient of y^m in $\exp(yx_1+y^2x_2+\dots)$, and both expressions can be verified to be equal to the coefficient of y^m in $(1-y)^{-1}(1-y^2)^{-1} \dots (1-y^m)^{-1}$.

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2. RIORDAN, J., An introduction to Combinatorial Analysis, New York–London (1958).

Added in proof: Pólya's theorem is also presented in Appendice V (by J. Riguet) of the recent book by C. Berge, Theorie des Graphes et ses Applications, Paris (1958).