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by

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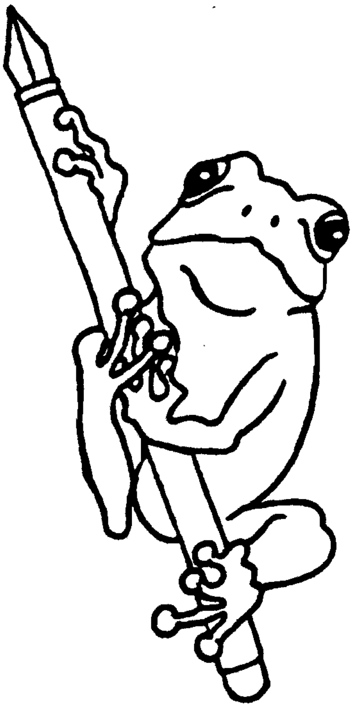

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Abstract. We consider multigrid methods applied to standard linear finite element discretizations of linear elliptic two-point boundary value problems. In the multigrid method damped Jacobi or damped Gauss-Seidel is used as a smoother. We show that the contraction number with respect to the maximum norm has an upperbound which is smaller than one and independent of the mesh size.

Key words. Multigrid, convergence, maximum norm, two-point boundary value problems.

AMS (MOS) subject classification. 65N20.

1. Introduction

If one considers elliptic boundary value problems in $\mathbb{R}^N$ ($N = 1, 2, 3$) then multigrid methods can be used to solve the large sparse linear systems that arise after discretization. If $N = 1$ then often the matrix involved is tridiagonal and thus many efficient solvers exist. If $N \geq 2$ then in general there are only few efficient solvers and often multigrid is one of them.
There is an extensive literature about the convergence analysis of multigrid methods. We refer to Hackbusch [3], McCormick [4] and the references given there. The main feature of multigrid is that the contraction number has an upperbound which is smaller than one and independent of the mesh size. In theoretical analyses this has been shown for a broad class of problems and for several variants of multigrid. In these analyses the contraction number is measured with respect to the energy norm (for symmetric problems) or the Euclidean norm (or sometimes some other exotic norm). However, there are no results with respect to the maximum norm. In this paper we present some first results about multigrid convergence in the maximum norm. We consider

multigrid applied to two-point boundary value problems and we prove the usual mesh-independent convergence of multigrid, but now with respect to the maximum norm. An important part of the analysis has a straightforward generalization to dimension $N = 2$ (cf. Remark 7.2 below). The analysis for the case $N = 2$ will be presented in a forthcoming paper.

The remainder of this paper is organized as follows: in §2 we introduce a class of two-point boundary value problems and we give some regularity results. In §3 we derive some properties of the usual linear finite element discretization. Our convergence analysis of the multigrid method is based on the Approximation Property and Smoothing Property as introduced by Hackbusch (cf. [3]). In §4 we prove the Approximation Property with respect to the maximum norm; our analysis is similar to the one used in Hackbusch [3]. In §5 we prove the Smoothing Property in the maximum norm; here a new approach is used. Based on the Approximation Property and Smoothing Property we prove convergence of the two-grid method and of the multigrid W-cycle in §6 and §7 respectively.

2. Continuous problem

Consider the linear two-point boundary value problem:

$$\text{(2.1)} \qquad \begin{aligned} &-(a(x)\varphi')' + b(x)\varphi' + c(x)\varphi = f(x) \qquad x \in I = (0,1) \\ &\varphi(0) = \varphi(1) = 0 \ , \end{aligned}$$

or, in weak form, the problem of finding $\bar{\varphi} \in H_0^1(I)$ such that for all $\psi \in H_0^1(I)$

$$\text{(2.2)} \qquad a(\bar{\varphi}, \psi) = \int_I f(x)\psi(x)dx$$

holds, with

$$\text{(2.3)} \quad a(\bar{\varphi}, \psi) := \int_I a(x)\bar{\varphi}'(x)\psi'(x)dx + \int_I b(x)\bar{\varphi}'(x)\psi(x)dx + \int_I c(x)\bar{\varphi}(x)\psi(x)dx \ .$$

We take $f \in L^2(I)$ and make the following assumptions about the coefficients a, b, c:

(2.4.a) $\quad a, b \in W^{1,\infty}(I) \ , \quad c \in L^\infty(I)$

(2.4.b) $\quad a(x) \geq a_0 > 0 \quad$ for all $x \in I$

(2.4.c) $\quad c(x) \geq c_0 \geq 0 \ , \quad |b(x)| \leq \delta \sqrt{a_0 c_0} \quad$ for all $x \in I$, with $\delta < 2$.

Remark 2.1. Due to the assumptions in (2.4) the bilinear form in (2.3) is H_0^1-elliptic. We note that for the conditions in (2.4.c) there are alternatives; for example, H_0^1-ellipticity is still guaranteed if the condition $|b(x)| \le \delta\sqrt{a_0 c_0}$ is replaced by $\|b'\|_{L^\infty} \le 2c_0$. Moreover, the conditions in (2.4.c) are not essential; if (2.4.c) is deleted our analysis is applicable with some technical modifications and the results still hold provided the discretizations we use are "fine enough".

The L^2-inner product is denoted by $(\cdot,\cdot)$. The following notation for Sobolev spaces and corresponding norms is used.

$$W^{1,p}(I) = \{\varphi \in L^p(I) \mid \varphi' \in L^p(I)\}$$
$$W^{2,p}(I) = \{\varphi \in W^{1,p}(I) \mid \varphi' \in W^{1,p}(I)\}$$
$$\|\varphi\|_{W^{m,p}} = \sum_{0\le r\le m} \|\varphi^{(r)}\|_{L^p}$$
$$H_0^1(I) \ : \text{closure of } C_0^\infty(I) \text{ in } W^{1,2}(I)\,.$$

It is well-known that for every $f \in L^2(I)$ the corresponding weak solution $\bar{\varphi} \in H_0^1(I)$ is an element of $W^{2,2}(I)$ too, and satisfies

$$\|\bar{\varphi}\|_{W^{2,2}} \le C\,\|f\|_{L^2}\,. \tag{2.5}$$

Note that for every $\psi \in C_0^\infty(I)$ we have

$$(a\bar{\varphi}'',\psi) = ((b-a')\bar{\varphi}' + c\bar{\varphi} - f,\psi)\,. \tag{2.6}$$

The (continuous) imbeddings $W^{2,2}(I) \hookrightarrow W^{1,2}(I) \hookrightarrow L^\infty(I)$ imply that for $\bar{\varphi} \in W^{2,2}(I)$ we have $\bar{\varphi},\bar{\varphi}' \in L^\infty(I)$. From (2.6) we then see that if $f \in L^\infty(I)$ then $a\bar{\varphi}'' \in L^\infty(I)$, and thus $\bar{\varphi} \in W^{2,\infty}(I)$ and

$$\|\bar{\varphi}''\|_{L^\infty} \le \frac{1}{a_0}\left\{(\|b\|_{L^\infty} + \|a\|_{W^{1,\infty}})\,\|\bar{\varphi}'\|_{L^\infty} + \|c\|_{L^\infty}\,\|\bar{\varphi}\|_{L^\infty} + \|f\|_{L^\infty}\right\}. \tag{2.7}$$

Combining the inequality in (2.7) with $\|\bar{\varphi}'\|_{L^\infty} \le C\,\|\bar{\varphi}\|_{W^{2,2}}$, $\|\bar{\varphi}\|_{L^\infty} \le C\,\|\bar{\varphi}\|_{W^{2,2}}$, $\|\bar{\varphi}\|_{W^{2,2}} \le C\,\|f\|_{L^2} \le C\,\|f\|_{L^\infty}$ (cf. (2.5)) yields that

$$\text{if } f \in L^\infty(I) \text{ then } \|\bar{\varphi}\|_{W^{2,\infty}} \le C\,\|f\|_{L^\infty}\,, \qquad \text{with } C = C(a,b,c)\,. \tag{2.8}$$

This regularity result will be used in the proof of the Approximation Property in §4.

3. Discretization and two-grid method

Let Φ_k be the n_k-dimensional space of functions φ with $\varphi(0) = \varphi(1) = 0$ that are piecewise linear on a mesh with nodes $x_{k,i}$ for which $0 = x_{k,0} < x_{k,1} < \dots < x_{k,n_k} < x_{k,n_k+1} = 1$. Φ_k is constructed from Φ_{k-1} by using mesh refinement, so we get a sequence of nested spaces

$$(3.1)\qquad \Phi_0 \subset \Phi_1 \subset \dots \subset \Phi_k \subset \dots \subset H_0^1(I)\ .$$

Let $h_{k,i} := x_{k,i} - x_{k,i-1}$ $(i = 1, \dots, n_k + 1)$ and $h_k := \max\limits_i\ h_{k,i}$. We assume quasi-uniformity of the meshes, i.e.:

$$(3.2)\qquad \max_{i,j}\ h_{k,i}h_{k,j}^{-1} \le \gamma_0 \qquad \text{with } \gamma_0 \text{ independent of } k\ .$$

Furthermore, the mesh refinement should be such that the following holds:

$$(3.3)\qquad h_k h_{k+1}^{-1} \le \gamma_1 \qquad \text{with } \gamma_1 \text{ independent of } k\ .$$

The standard basis on Φ_k is given by the hat functions $\varphi_i^{(k)}$ which satisfy $\varphi_i^{(k)}(x_{k,j}) = \delta_{ij}$. This basis induces a bijection

$$(3.4)\qquad P_k\ :\ U_k = I\!R^{n_k} \to \Phi_k\ , \qquad P_k(u_1, u_2, \dots, u_{n_k}) = \sum_{i=1}^{n_k} u_i \varphi_i^{(k)}\ .$$

On U_k we use a scaled Euclidean inner product:

$$(3.5)\qquad \langle u, v\rangle_k = h_k \sum_{i=1}^{n_k} u_i v_i\ .$$

The maximum norm on U_k is denoted by $\|\cdot\|_\infty$. Below, adjoints are always defined with respect to the L^2-inner product on Φ_k and the scaled Euclidean inner product on U_k.
The norms $\|\cdot\|_\infty$ (on $(U_k)_{k\ge0}$) and $\|\cdot\|_{L^\infty}$ (on $(\Phi_k)_{k\ge0}$) induce associated operator norms which are denoted by $\|\cdot\|_\infty$.

The sequences $(P_k)_{k\ge0}$, $(P_k^{-1})_{k\ge0}$, $(P_k^*)_{k\ge0}$, $((P_k^*)^{-1})_{k\ge0}$ are uniformly bounded with respect to $\|\cdot\|_\infty$:

LEMMA 3.1. *The following holds*:

$$(1)\qquad \|P_k u\|_{L^\infty} = \|u\|_\infty \qquad \textit{for all } u \in U_k$$

$$(2)\qquad (3\gamma_0)^{-1}\|\varphi\|_{L^\infty} \le \|P_k^*\varphi\|_\infty \le \|\varphi\|_{L^\infty} \qquad \textit{for all } \varphi \in \Phi_k\ \ (\gamma_0 \textit{ as in } (3.2))\ .$$

Proof. The result in (1) holds because $P_k u$ is piecewise linear and $(P_k u)(x_{k,i})$ equals the i-th component of u.
Due to (1) the statement in (2) is equivalent with:

$$(2')\qquad (3\gamma_0)^{-1}\|u\|_\infty \le \|P_k^* P_k u\|_\infty \le \|u\|_\infty \qquad \text{for all } u \in U_k\ .$$

The matrix $P_k^* P_k$ is the well-known mass matrix:

$$(P_k^* P_k)_{ij} = h_k^{-1} \langle P_k^* P_k e_j, e_i \rangle_k = h_k^{-1} (\varphi_j^{(k)}, \varphi_i^{(k)}) = h_k^{-1} \int_I \varphi_j^{(k)}(x) \varphi_i^{(k)}(x) dx \ .$$

So $P_k^* P_k$ is symmetric tridiagonal with elements

$$(P_k^* P_k)_{ii} = \tfrac{1}{3} h_k^{-1} (h_{k,i} + h_{k,i+1}) \ =: \ d_{k,i}$$
$$(P_k^* P_k)_{i,i-1} = \tfrac{1}{6} h_k^{-1} h_{k,i} \ =: \ e_{k,i} \ .$$

Because $|d_{k,i}| \leq \frac{2}{3}$ and $|e_{k,i}| \leq \frac{1}{6}$ we get $\|P_k^* P_k\|_\infty \leq 1$ and thus the second inequality in $(2')$ holds.
Let $D_k := \operatorname{diag}(P_k^* P_k)$, $R_k := P_k^* P_k - D_k$. Then $\|D_k^{-1}\|_\infty \leq \max_i d_{k,i}^{-1} \leq \frac{3}{2} \gamma_0$. Also $\|D_k^{-1} R_k\|_\infty = \max_i \left(d_{k,i}^{-1} (e_{k,i} + e_{k,i+1}) \right)$ (with $e_{k,1} := e_{k,n_k+1} := 0$), and thus $\|D_k^{-1} R_k\|_\infty \leq \frac{1}{2}$. So $\|(P_k^* P_k)^{-1}\|_\infty \leq \|D_k^{-1}\|_\infty (1 - \|D_k^{-1} R_k\|_\infty)^{-1} \leq 3\gamma_0$; this proves the first inequality in $(2')$. □

Galerkin discretization results in a stiffness matrix $L_k \ : \ U_k \to U_k$ defined by

(3.6) $$\langle L_k u, v \rangle_k = a(P_k u, P_k v) \qquad \text{for all } u, v \in U_k \ .$$

Also we have

(3.7) $$a(P_k L_k^{-1} g, \psi) = ((P_k^*)^{-1} g, \psi) \qquad \text{for all } g \in U_k \, , \ \ \psi \in \Phi_k \ .$$

In §5 we prove the Smoothing Property for matrices which are weakly diagonally dominant (i.e.: $\sum_{j \neq i} |A_{ij}| \leq |A_{ii}|$ for all i). It is well-known that often the stiffness matrices L_k are weakly diagonally dominant. For completeness we give a few criteria:

LEMMA 3.2. *Take k fixed and write $L_k = A + B + C$ with*

$$A_{ij} = (a\varphi_j', \varphi_i') \, , \ \ B_{ij} = (b\varphi_j', \varphi_i) \, , \ \ C_{ij} = (c\varphi_j, \varphi_i) \qquad (\varphi_m = \varphi_m^{(k)}) \ .$$

L_k is weakly diagonally dominant if one of the following conditions is satisfied:

(1) *all off-diagonal elements of L_k are nonpositive.*

(2) *$A + B$ is weakly diagonally dominant with $(A+B)_{ii} \geq 0$ for all i, and C is weakly diagonally dominant.*

(3) $h_k \, \frac{1}{2} \, a_0^{-1} (\|b\|_{L^\infty} + \frac{1}{3} \, h_k \, \|c\|_{L^\infty}) \leq 1$ *(a_0 as in* (2.4.b)*).*

(4) $h_k \, \frac{1}{2} \, a_0^{-1} \, \|b\|_{L^\infty} \leq 1$ *and* $c \in \Phi_k$.

Proof. First we consider (1). Let $I_m = [x_{k,m-1}, x_{k,m}]$ and $L_k^{(m)}$ the corresponding element stiffness matrix, i.e.:

$$(L_k^{(m)})_{ij} = a_{|I_m}(\varphi_j^{(k)}, \varphi_i^{(k)}) \; .$$

Note that due to the ellipticity of $a(\cdot,\cdot)$ all diagonal elements of $L_k^{(m)}$ are nonnegative. The p-th row of $L_k^{(m)}$ contains atmost one nonzero off-diagonal element; this element is of the form $a_{|I_m}(\varphi_j, \varphi_p)$ with $j \in \{p-1, p+1\}$, and

$$\begin{aligned} |a_{|I_m}(\varphi_j,\varphi_p)| = |a(\varphi_j,\varphi_p)| = -a(\varphi_j,\varphi_p) \; &= \\ = \; -a_{|I_m}(1-\varphi_p,\varphi_p) = a_{|I_m}(\varphi_p,\varphi_p) - a_{|I_m}(1,\varphi_p) &\leq (L_k^{(m)})_{pp} \; . \end{aligned}$$

So all element stiffness matrices are weakly diagonally dominant and have nonnegative diagonal elements. This implies that the (global) stiffness matrix is weakly diagonally dominant.
It is easy to show that L_k is weakly diagonally dominant if (2) holds. With respect to (3) and (4) we note that some elementary analysis yields that if the inequality in (3) holds then the condition (1) is fulfilled and if the conditions in (4) are fulfilled then (2) holds. □

Remark 3.3. Note that if c is small (compared with b) or if c is smooth, then the conditions in (3) and (4) in essence yield the usual bound for the Peclet number $h_k \, \frac{1}{2} \, a_0^{-1} \, \|b\|_{L^\infty}$. It is well-known that in general the standard finite element Galerkin discretization yields a poor approximation if the Peclet number is large. Other discretization techniques should be used in that situation.

For solving systems of the form $L_k u_k = g_k$ we use a standard multigrid method. The iteration matrix of the smoothing method is denoted by S_k. The prolongation $p = p_k \; : \; U_{k-1} \to U_k$ we use is the natural one:

$$p = P_k^{-1} P_{k-1} \; . \tag{3.8}$$

For the restriction $r = r_k \; : \; U_k \to U_{k-1}$ we take

$$r = p^* \; . \tag{3.9}$$

The iteration matrix of the two-grid method with ν pre-smoothing iterations is given by

$$T_k(\nu) = (I - pL_{k-1}^{-1} r L_k) S_k^\nu = (L_k^{-1} - pL_{k-1}^{-1} r) L_k S_k^\nu \; . \tag{3.10}$$

For convergence of the two-grid method we will prove the Approximation Property

$$\|L_k^{-1} - pL_{k-1}^{-1} r\|_\infty \leq C \, h_k^2 \; , \tag{3.11}$$

and the Smoothing Property

$$\|L_k S_k^{\nu}\|_{\infty} \leq \eta(\nu) h_k^{-2} \qquad (\text{with } \eta(\nu) \to 0 \text{ if } \nu \to \infty) \ . \tag{3.12}$$

These proofs will be given in §4 and §5 respectively.

4. Approximation Property

The proof of the Approximation Property is based on optimal L^∞ error estimates which can be found e.g. in Wheeler [10], Douglas-Dupont-Wahlbin [2] and on the uniform boundedness of the sequences $(P_k)_{k\geq 0}$, $((P_k^*)^{-1})_{k\geq 0}$.

LEMMA 4.1. *The following holds with a constant C independent of k:*

$$\|L_k^{-1} - pL_{k-1}^{-1} r\|_{\infty} \leq C\, h_k^2 \ . \tag{4.1}$$

Proof. Take $g \in U_k$. In the proof different constants c, all independent of k and g, are used.
Let $\varphi \in H_0^1(I)$ be such that

$$a(\varphi, \psi) = ((P_k^*)^{-1} g, \psi) \qquad \text{for all } \psi \in H_0^1(I) \ .$$

From (2.8) it follows that

$$\|\varphi\|_{W^{2,\infty}} \leq c\, \|(P_k^*)^{-1} g\|_{L^\infty} \ . \tag{4.2}$$

Let $\varphi_k \in \Phi_k$, $\varphi_{k-1} \in \Phi_{k-1}$ be such that

$$\begin{aligned} a(\varphi_k, \psi) &= ((P_k^*)^{-1} g, \psi) \qquad \text{for all } \psi \in \Phi_k \quad , \\ a(\varphi_{k-1}, \psi) &= ((P_k^*)^{-1} g, \psi) \qquad \text{for all } \psi \in \Phi_{k-1} \ . \end{aligned}$$

In [10], [2] it is shown that the following holds:

$$\|\varphi_m - \varphi\|_{L^\infty} \leq c\, h_m^2\, \|\varphi\|_{W^{2,\infty}} \qquad \text{for } m \in \{k, k-1\} \ . \tag{4.3}$$

Combination of (4.2), (4.3) and (3.3) yields

$$\|\varphi_k - \varphi_{k-1}\|_{L^\infty} \leq c(h_{k-1}^2 + h_k^2)\, \|(P_k^*)^{-1} g\|_{L^\infty} \leq c\, h_k^2\, \|(P_k^*)^{-1} g\|_{L^\infty} \ . \tag{4.4}$$

From (3.7) it follows that $\varphi_k = P_k L_k^{-1} g$, $\varphi_{k-1} = P_{k-1} L_{k-1}^{-1} r g$. Using Lemma 3.1 and (4.4) we finally get

$$\begin{aligned} \|(L_k^{-1} - pL_{k-1}^{-1} r) g\|_{\infty} &= \|P_k L_k^{-1} g - P_{k-1} L_{k-1}^{-1} r g\|_{L^\infty} \ = \\ &= \ \|\varphi_k - \varphi_{k-1}\|_{L^\infty} \leq c\, h_k^2\, \|(P_k^*)^{-1} g\|_{L^\infty} \leq c\, h_k^2\, \|g\|_{\infty} \ . \end{aligned}$$

□

5. Smoothing Property

A new approach for proving the Smoothing Property has been introduced in Reusken [7]. The results of this section can be found in a more general setting there. For completeness we give proofs here too.

Below we prove that the Smoothing Property, in the maximum norm, holds for damped Jacobi and for damped Gauss-Seidel.
Let $L_k = M_k - N_k$ be the splitting corresponding to the Jacobi method or the Gauss-Seidel method (both without damping). We consider a relaxation method with iteration matrix

$$(5.1) \qquad S_k = I - \tfrac{1}{2} M_k^{-1} L_k$$

(so damping with factor $\frac{1}{2}$). In our analysis we use that the splitting is such that $\|M_k^{-1} N_k\|_\infty \leq 1$ holds. Therefore we introduce the following

ASSUMPTION 5.1. *For every $k \geq 0$ the matrix L_k is weakly diagonally dominant.*

Note that in Lemma 3.2 some criteria with respect to diagonal dominance are given.

LEMMA 5.2. *Let A be an $n \times n$-matrix with $\|A\|_\infty \leq 1$. Then the following holds:*

$$(5.2) \qquad \|(I-A)(I+A)^\nu\|_\infty \leq 2 \binom{\nu}{[\frac{1}{2}\nu]} \leq 2^{\nu+1} \sqrt{\frac{2}{\pi\nu}} \qquad (\nu \geq 1)\,.$$

Proof.

$$\begin{aligned}(I-A)(I+A)^\nu &= (I-A) \sum_{k=0}^{\nu} \binom{\nu}{k} A^k \\ &= I - A^{\nu+1} + \sum_{k=1}^{\nu} \left(\binom{\nu}{k} - \binom{\nu}{k-1} \right) A^k\,.\end{aligned}$$

So

$$(5.3) \qquad \|(I-A)(I+A)^\nu\|_\infty \leq 2 + \sum_{k=1}^{\nu} \left| \binom{\nu}{k} - \binom{\nu}{k-1} \right|.$$

Using $\binom{\nu}{k} \geq \binom{\nu}{k-1} \Leftrightarrow k \leq \frac{1}{2}(\nu+1)$, and $\binom{\nu}{k} = \binom{\nu}{\nu-k}$ we get

$$\sum_{k=1}^{\nu} \left| \binom{\nu}{k} - \binom{\nu}{k-1} \right| =$$

$$= \sum_{1}^{[\frac{1}{2}(\nu+1)]} \left(\binom{\nu}{k} - \binom{\nu}{k-1} \right) + \sum_{[\frac{1}{2}(\nu+1)]+1}^{\nu} \left(\binom{\nu}{k-1} - \binom{\nu}{k} \right)$$

$$= \sum_{1}^{[\frac{1}{2}\nu]} \left(\binom{\nu}{k} - \binom{\nu}{k-1} \right) + \sum_{m=1}^{[\frac{1}{2}\nu]} \left(\binom{\nu}{m} - \binom{\nu}{m-1} \right)$$

$$= 2 \sum_{k=1}^{[\frac{1}{2}\nu]} \left(\binom{\nu}{k} - \binom{\nu}{k-1} \right) = 2 \left(\binom{\nu}{[\frac{1}{2}\nu]} - \binom{\nu}{0} \right) .$$

Combined with (5.3) this yields the first inequality in (5.2). Elementary analysis yields that

$$\binom{\nu}{[\frac{1}{2}\nu]} \leq 2^{\nu} \sqrt{\frac{2}{\pi\nu}} \qquad \text{for all } \nu \geq 1 .$$

For details we refer to [7]. □

LEMMA 5.3. *Suppose that Assumption* 5.1 *holds. Then for the damped Jacobi and for the damped Gauss-Seidel relaxation (cf.* (5.1)*) we have the following Smoothing Property:*

$$\|L_k S_k^{\nu}\|_{\infty} \leq C \, \frac{1}{\sqrt{\nu}} \, h_k^{-2} \qquad (C \textit{ independent of } k, \nu) .$$

Proof. Let $A := M_k^{-1} N_k = I - M_k^{-1} L_k$. Then due to Assumption 5.1 we have $\|A\|_{\infty} \leq 1$. Using Lemma 5.2 we get:

$$\begin{aligned}
\|L_k S_k^{\nu}\|_{\infty} &= \|L_k (I - \tfrac{1}{2} M_k^{-1} L_k)^{\nu}\|_{\infty} \\
&= \|M_k (I - A)(\tfrac{1}{2})^{\nu} (I + A)^{\nu}\|_{\infty} \\
&\leq \|M_k\|_{\infty} (\tfrac{1}{2})^{\nu} \, 2^{\nu+1} \sqrt{\frac{2}{\pi\nu}} \\
&= 2 \sqrt{\frac{2}{\pi}} \, \frac{1}{\sqrt{\nu}} \, \|M_k\|_{\infty} \\
&\leq C \, \frac{1}{\sqrt{\nu}} \, h_k^{-2} \qquad (C = C(a,b,c)) .
\end{aligned}$$

□

6. Convergence of the two-grid method

Lemma 4.1 and Lemma 5.3 together immediately yield the following result.

THEOREM 6.1. *Suppose that Assumption* 5.1 *holds. Let* $T_k(\nu) = (L_k^{-1} - pL_{k-1}^{-1}r)L_kS_k^\nu$ *be the iteration matrix of a two-grid method with* ν *pre-smoothing iterations of a damped Jacobi or a damped Gauss-Seidel relaxation (cf.* (5.1)*). Then the following holds*:

$$(6.1) \qquad \|T_k(\nu)\|_\infty \le \frac{C}{\sqrt{\nu}} \qquad \textit{with } C \textit{ independent of } k \textit{ and } \nu \,.$$

Remark 6.2. Clearly Theorem 6.1 shows that for the two-grid method with ν large enough (but fixed) the contraction number *with respect to the maximum norm* has an upperbound which is smaller than one and independent of the mesh size.
As a side issue we mention that Theorem 6.1 gives the first fairly general convergence result for the two-grid method with (damped) Gauss-Seidel as a smoother (cf. also Theorem 7.1 below for the multigrid convergence).

7. Convergence of the multigrid method

The analysis of the multigrid W-cycle follows the appraoch as given in Hackbusch [3].

The error iteration matrix $M_k(\nu)$ of the multigrid W-cycle with ν pre-smoothing iterations on each level is recursively defined as follows:

$$\begin{aligned} &M_1(\nu) = T_1(\nu) \\ &M_k(\nu) = T_k(\nu) + pM_{k-1}(\nu)^2 L_{k-1}^{-1} r L_k S_k^\nu \,, \qquad k \ge 2 \,. \end{aligned}$$

THEOREM 7.1. *Suppose that Assumption* 5.1 *holds. Let* $T_k(\nu)$ *be as in Theorem* 6.1, *and assume that* ν *is large enough such that* $\xi_\nu := \|T_k(\nu)\|_\infty < \frac{1}{2}(\sqrt{2}-1)$ *holds. Let* μ_ν *be the smallest root of the polynomial* $p(x) = (1+\xi_\nu)x^2 - x + \xi_\nu$. *Now the following holds*:

$$(7.1) \qquad \|M_k(\nu)\|_\infty \le \mu_\nu < 1 \,,$$

and also

$$(7.2) \qquad \mu_\nu \le \xi_\nu + 4\xi_\nu^2(1+\xi_\nu) \,.$$

Proof. Let $m_k(\nu) := \|M_k(\nu)\|_\infty$. Note that due to Assumption 5.1 we have $\|S_k\|_\infty = \|I - \frac{1}{2}M_k^{-1}L_k\|_\infty = \|\frac{1}{2}I + \frac{1}{2}M_k^{-1}N_k\|_\infty \le 1$. Note that $m_1(\nu) = \xi_\nu$, and for $k \ge 2$:

$$\begin{aligned} m_k(\nu) &\le \|T_k(\nu)\|_\infty + \|pM_{k-1}(\nu)^2 L_{k-1}^{-1} r L_k S_k^\nu\|_\infty \\ &\le \xi_\nu + \|M_{k-1}(\nu)\|_\infty^2 \, \|pL_{k-1}^{-1} r L_k S_k^\nu\|_\infty \qquad (\text{use } \|pv\|_\infty = \|v\|_\infty) \end{aligned}$$

$$= \xi_\nu + m_{k-1}(\nu)^2 \, \|S_k^\nu - T_k(\nu)\|_\infty$$
$$\leq \xi_\nu + m_{k-1}(\nu)^2(1+\xi_\nu) \, .$$

The iteration $x_1 := \xi_\nu$, $x_{i+1} := \xi_\nu + (1+\xi_\nu)x_i^2$ $(i \geq 1)$ has a fixed point $\mu_\nu := \frac{1}{2}(1+\xi_\nu)^{-1}\big(1-\sqrt{1-4\xi_\nu(1+\xi_\nu)}\big) < 1$, and for all i $x_i \leq \mu_\nu$ holds. So the inequalities in (7.1) hold. The inequality in (7.2) follows from

$$1-\sqrt{1-x} \leq \tfrac{1}{2}\,x + \tfrac{1}{2}\,x^2 \qquad \text{for all } x \in [0,1] \, . \qquad \square$$

Remark 7.2. With respect to a generalization of our analysis to the $2D$ situation we note the following. The arguments used in the proof of multigrid convergence (Theorem 7.1) can be used for the $2D$ case too, provided we have an upperbound for the two-grid contraction number as in (6.1). Such an upperbound is a direct consequence of the Approximation Property and Smoothing Property. The analysis of the Smoothing Property in §5 can also be used in $2D$. So, in essence it is only the Approximation Property that needs to be reconsidered. It is known from the literature (cf. e.g. [1], [5], [6], [8], [9]) that for linear finite element Galerkin approximations in $2D$ the optimal L^∞ error estimate is of the order $h_k^2\,|\log h_k|$ (instead of h_k^2). So in the Approximation Property we do not expect an upperbound $C\,h_k^2$ as in (4.1) but an upperbound $C\,h_k^2\,|\log h_k|$.
Multigrid convergence in the maximum norm for two dimensional elliptic boundary value problems will be analyzed in detail in a forthcoming paper.

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