Existence of general equilibria in infinite horizon economies with exhaustible resources : the continuous time case

EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computing Science

Memorandum COSOR 90-16

Existence of general equilibria
in infinite horizon economies with
exhaustible resources
(the continuous time case)

J. van Geldrop and C. Withagen

ABSTRACT

The existence problem of general equilibria in continuous-time economies with natural exhaustible resources and an infinite horizon cannot be solved by the classical Arrow/Debreu approach nor by the results from the more recent theory on infinite dimensional commodity spaces, without imposing conditions which, from a resource economic point of view, are hard to accept. An alternative two-step method is proposed, where, in the first step, existence of an equilibrium in the (artificially) truncated economy is established along the lines set out by Negishi, and, in the second step, the limiting allocation is shown to be an equilibrium for the infinite horizon economy.

1. Introduction

When studying economies endowed with natural exhaustible resources, dynamic considerations almost necessarily enter into the analysis because, typically, agents must decide to exploit the resources in the present or in the future, irrespective of the fact that one wants to deal with a centralized or a decentralized economy. For various reasons it seems appropriate to assume an infinite horizon, which entails an infinite dimensional commodity space. In this paper we focus on a general competitive equilibrium in an economy with exhaustible resources. Besides the presentation of a far more general model than those known in the literature, the main interest of the paper lies in the way of proving the existence of a general equilibrium.

Perhaps the most appealing way to sketch the model at hand is to view it in an international trade context although this is surely not the only possible interpretation. There is an arbitrary number of countries, each endowed with a natural exhaustible resource, which is costly to exploit, in the sense that capital has to be used as an input to extract the raw material. The raw material is traded on a competitive world market and serves as an input in production processes carried out by the individual countries, in which it is, together with capital, transformed into a so-called composite commodity. This commodity is malleable in the sense that it serves as the consumer good, as a means for capital accumulation and as a store of value as well. Balances of payments are not required to equilibriate. It is assumed that there exists a perfect world market for financial capital and no uncertainty; then the only budget condition is that total discounted expenditures do not exceed total discounted income. The countries involved in trade choose exploitation, investment and consumption patterns so as to maximize their (utilitarian) welfare functional. This model builds on earlier general equilibrium models by e.g. Kemp and Long (1980), Chiarella (1980), Elbers and Withagen (1984), which are surveyed in Withagen (1990). Our generalization lies in the fact that the number of countries is arbitrary, each country may possess resources and the technology to produce the composite commodity, and we allow for a non-Cobb-Douglas-like specification of the functions involved.

The second issue we address is the existence of a general equilibrium. The consequence of working with an infinite horizon is that traditional (eg. Arrow/Debreu) methods of proving the existence of a general equilibrium cannot be used because in the case at hand the commodity space is of an infinite dimension. Basically there are two ways of approach that can be chosen.

The first is to recognize that the model bears quite some similarity with neoclassical multisector growth models (Radner (1967), Gale (1967)) and to apply "Hamiltonian approach"-like methods (see e.g. Cass and Shell (1976)). An example of this is given by Mitra (1980), whose model however essentially contains one consumer and one aggregate production technology, whereas there are other differences as well. Nonetheless his method of analysis resembles ours in some respect. This will be clarified below.

The Hamiltonian approach departs from the existence of a solution to an infinite horizon control problem and then concentrates on issues like the asymptotic stability (local or global) of a steady state (if any). Hence there are two problems here. First it is well known that existence theorems for infinite horizon control problems assume strong boundedness conditions (as in Mitra (1980)), which, from an economic point of view, one does not wish to impose a priori and, second, it is against the nature of models with exhaustible resources that steady states exist.

A second possible approach is to rely on the growing literature on infinite dimensional commodity spaces (see Bewley (1972), Jones (1986), Mas-Colell (1986), Richard (1986), Zame (1987)). Van Geldrop et al. (1990) have put forward that this framework gives rise to several serious difficulties. Let us briefly sketch the discussion of these problems. In some of the contributions it is assumed that the initial endowment of each agent is strictly positive. When no new exhaustible resources become available in the future, as is assumed here, this condition is not satisfied. Admittedly this problem can be circumvened in a number of ways but one should be aware of it. Central is the choice of the commodity space. When there are only raw materials in the economy it seems right to choose l_1 as the commodity space (see Zame (1987)), and it would even be wrong to work in l_∞, of which Van Geldrop et al. give an example. However, when there are other commodities as well l_∞ is in general not eligible as the commodity space because there are many non-pathological examples from resource economics with an unbounded production set. One could hope that with positive rates of time preference equilibrium allocations are still bounded but Dasgupta and Heal (1974) provide examples where this is not the case. A second major problem refers to a type of assumptions which are commonly made in this literature and which are called "uniform properness" (Richard (1986), Mas-Colell (1986)), "boundedness of marginal efficiency" (Zame (1987)), "universal technical substitute" (Zame (1987)) and the like. Crudely speaking, these assumptions boil down to bounded marginal productivity of the primary factors in the economy. It is easy again to give examples from resource economics (for instance with Cobb-Douglas technologies) where these conditions do not hold, even when production sets are bounded. One could argue that there is an additional principal argument against imposing such a condition in resource economics. Let $f: \mathbb{R}^2_+ \to \mathbb{R}_+$ be a production function with the following properties: f is C^2 on $\mathbb{R}_{++}$, C^0 on $\mathbb{R}_+$, concave and $f(0, y) = f(x, 0) = 0$ (necessity of inputs). It can be shown that if f is differentiable in $(0, 0)$, f is identically zero. The properties of f listed above are quite popular and the conclusion one is tempted to draw is that it is hard to reconcile economically plausible or desirable assumptions with the application of existence results from the literature on infinite dimensional commodity spaces.

Does all this mean that there is no hope for existence results in resource economics beyond the cases where an equilibrium can actually be constructed? No, there is a third way, which is a combination of the former two. The basic idea is as follows. One first establishes the existence of a general equilibrium in the truncated, finite horizon economy. Subsequently one derives properties of these equilibria, to be more specific, uniform boundedness of equilibrium allocations. Then one invokes convergence results from functional analysis (e.g. Alaoglu) and shows that the limiting allocations constitute an equilibrium in the infinite horizon economy. This method has been successfully followed by Van Geldrop et al. (1990) for the case of a discrete- time model. Here

we deal with a continuous-time model, which is a serious complication because even for a finite horizon the commodity space is of an infinite dimension. Therefore the existence of an equilibrium with a finite horizon is not as straightforward as in a discrete-time setting. In the finite horizon continuous-time case the point made above on the boundedness of marginal efficiency and the like remains valid. On the other hand a control theoretic approach seems to be recommendable, also because there is a large variety of existence theorems for control problems in a finite horizon setting (see e.g. Cesari (1983)). These are however not sufficient to establish the existence of a general equilibrium because control theory typically deals with one optimizing agent, whereas we allow for an arbitrary number of agents. This is solved by applying the well-known Negishi approach: we first consider the problem of finding a Pareto-efficient allocation with arbitrary weights attached to the consumers and then show that there exist weights such that the corresponding shadow prices can be interpreted as equilibrium prices. In the present paper we assume that the production sets are convex cones, which facilitates the proof of equilibrium allocations being uniformly bounded to a large extent. This is not to say that without this assumption the method does not work, but the analysis would then become very tedious and would divert the attention from our principal aim, namely to advocate the simultaneous use of functional analysis and control theory.

A third issue of this paper could be the characterization of a general equilibrium in terms of the development over time of e.g. the (implicit) interest rate or capital intensity. However important, we shall not perform such an analysis here for reasons of space. For some of the results we refer to Van Geldrop and Withagen (1988).

The sequel of the paper is organized as follows. In section 2 the model is presented. Section 3 contains the existence proof. Section 4 discusses the results and concludes.

2. The model

Commodities

A commodity bundle in the economy will be represented by a pair $(\alpha; \beta)$ with $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_m) \in I\!R^{m+1}$ $(m \geq 1)$ and $\beta \in L := L[0, \infty]$, where $L[0, \infty)$ is the set of real-valued Lebesgue- measurable functions on $[0, \infty)$. Here α_0 refers to the stock of the composite commodity and $\alpha_k (k = 1, 2, \cdots, m)$ to the resource stock of type k. β associates with every $t \in [0, \infty)$ the rate of the composite commodity at t. A commodity bundle $(\alpha; \beta)$ will be called a commodity bundle in the economy with horizon $T (\in I\!N)$ if $\beta(t) = 0$ for $t > T$.

Consumption

There are n consumers, indexed by $h = 1, 2, \cdots, n$. The initial endowment of consumer h is denoted by $w^h = (w_0^h, w_1^h, \cdots, w_m^h; 0) \geq 0$ with $w_0^h > 0$. The $(m + 2)$-th element of w^h being identically zero means that no flow of the composite commodity becomes available in the future as endowment. It is furthermore implicit that no new resources are found in the future. Consumer h also holds a share $\theta^h \geq 0$ in the profits of the aggregate firm to be described below. The consumption set of consumer $h (X^h)$ is defined by

$$x = (\alpha; c^h) \in X^h \iff \alpha = 0 \text{ and } c^h \in L^+ .$$

Consumption sets in the finite horizon economies (X_T^h) are defined by

$$x = (\alpha; c^h) \in X_T^h \iff x \in X^h \text{ and } c^h(t) = 0 \text{ for } t > T .$$

Note that $X^h = X_\infty^h$. This choice of the consumption sets is justified by the assumption that the possession of stocks does not add to felicity per se: only from the flow of the composite commodity one can derive utility. Preferences are assumed to be represented in the following utilitarian way. Each consumer h has a positive constant rate of time preference ρ_h and an instantaneous utility function $u_h : I\!R_+ \to I\!R$ satisfying

U^1 u_h is continuous on $I\!R_+$ and continuously differentiable on $I\!R_{++}$, $u_h(0) = 0$,

U^2 u_h is strictly increasing and strictly concave .

U^3 $u'_h(0) = \infty; 0 < \eta_h \leq -c u''_h(c)/u'_h(c) = \eta_h(c) < \infty$ for all $c > 0$ and some constant η_h .

Assumptions $U^1 - U^3$ are commonly made in the growth literature and they facilitate the analysis, although they are of course by no means necessary to obtain interesting results.

Now we say that for $x := (\alpha; c^h) \in X^h$ and $\bar{x} := (\bar{\alpha}, \bar{c}^h) \in X^h$ $x \geq_h \bar{x}$ (x is preferred to $\bar{x}$ by h) if

$$U_h(x) := \int_0^\infty e^{-\rho_h s} u_h(c^h(s)) \, ds \geq U_h(\bar{x}) := \int_0^\infty e^{-\rho_h s} u_h(\bar{c}^h(s)) \, ds \ .$$

Here we adopt the convention: $\infty \geq \infty$ and $\infty > a$ all $a \in \mathbb{R}$.

Production

There is one aggregate producer, which exploits the resources and transforms the raw material together with capital (i.e. the stock of the composite commodity) into a flow of the composite commodity. Let K be the stock of capital available at some instant of time. In order to extract E_k from the k-th resource $a_k E_k$ units of capital are needed ($a_k \geq a_1 > 0$, $k = 1, 2, \cdots, m$). Let aE denote $\sum_{k=1}^m a_k E_k$ and $\sigma E := \sum_{k=1}^m E_k$.

The raw material once extracted is homogeneous. So the producer is left with $K - aE$ capital input and σE raw material input to produce the composite commodity. Gross output is then given by $F(K - aE, \sigma E)$. About F we assume

$F.^1$ F is continuous on $\mathbb{R}_+^2$, continuously differentiable on $\mathbb{R}_{++}^2$, concave and

       homogeneous of order 1 .

$F.^2$ F is strictly increasing on $\mathbb{R}_{++}^2$.

F^3 $F(K, 0) = F(0, R) = 0$ for all $K \in \mathbb{R}_+$ and all $R \in \mathbb{R}_+$.

F^4 $\lim_{K \to \infty} F(K, R)/K = 0$, all R; $\lim_{K \to 0} F(K, R)/K = \infty$ for all $R > 0$.

We shall comment on $F^1 - F^4$ below in Section 4.

The aggregate production set Y then defined as follows.

$$y = (-s \, ; z) \in Y \iff$$

i) $s \in \mathbb{R}_+^{m+1}$, $z \in L^+$.

ii)   There exist $E = (E_1, E_2, \cdots, E_m) \in (L^+)^m$, $K \in AC^+$ and $S = (S_1, S_2, \cdots, S_m) \in (AC^+)^m$ (where AC^+ denotes the class of nonnegative absolutely continuous functions on $[0, \infty)$) such that

(2.1) $aE \leq K$,

(2.2) $\dot{K} = F(K - aE, \sigma E) - z$, $K(0) = s_0$,

(2.3) $\dot{S} = -E$, $S(0) = (s_1, s_2, \cdots, s_m)$.

The production set for the finite horizon economy Y_T is defined by

$$y = (-s\,;\,z) \in Y_T \iff y \in Y \text{ and } z(t) = 0 \text{ for } t > T \ .$$

Note that Y and Y_T are convex cones due to the homogeneity of F and the fact that the extraction technology is linear.

Equilibrium

With a price system $\pi = (\pi_0, \pi_1, \cdots, \pi_m; p) \in I\!R_+^{m+1} \times L^+[0, \infty)\backslash\{0\}$ we associate the budget constraint $B^h(\pi)$ of consumer h:

$$B^h(\pi) = \{x^h \in X^h \mid \pi \cdot x^h \le \pi \cdot w^h\} \ .$$

Here profits are not included because they will be zero in an equilibrium anyway since Y is a convex cone. Note that $\pi \cdot x^h \le \pi \cdot w^h$ can be written as

$$\int_0^\infty p(s)c^h(s)\,ds \le \sum_{j=0}^m \pi_j w_j^h \ .$$

So the right-hand side represents the value of the stocks initially held by consumer h. The budget constraint of the finite horizon economy is denoted by $B_T^h(\pi_T)$.

We define a general equilibrium in the finite horizon economy E_T ($\infty > T > 0$) as a tuple

$$x_T^h \in X_T^h \ (h = 1, 2, \cdots, n), y_T \in Y_T, \pi_T = (\pi_{0T}, \pi_{1T}, \cdots, \pi_{mT}; p_T) \ge 0, \pi_T \ne 0 \ ,$$

with all functions measurable, such that

(2.4) $\qquad \pi_T \cdot y_T \ge \pi_T \cdot \bar{y}_T$ for all $\bar{y}_T \in Y_T$.

(2.5) $\qquad x_T^h \in B_T^h(\pi_T)$ and $x_T^h \underset{h}{\ge} \bar{x}_T^h$ for all $\bar{x}_T^h \in B_T^h(\pi_T)$, $h = 1, 2, \cdots, n$.

(2.6) $\qquad \sum_{h=1}^n x_T^h \le \sum_{h=1}^n w^h + y_T, \ \pi_T \cdot (\sum_{h=1}^n x_T^h - \sum_{h=1}^n w^h - y_T) = 0$.

The existence of an equilibrium in E_T has been established in an earlier paper (see Van Geldrop and Withagen (1990)), where some characteristics are given as well. These results are summarized in

Theorem 2.1

For all $T \in I\!N$ there exists a general competitive equilibrium $(x_T^h \mid_{h=1}^n, y_T, \pi_T)$ in economy E_T.
Furthermore

i) $p_T(0) = \pi_{0T} = 1$, p_T is absolutely continuous, $p_T(t) > 0$ for $t \in [0, T]$ and there exists $\gamma > 0$ such that $0 \le -\dot{p}_T/p_T = F_K(K_T - aE_T, \sigma E_T) \le \gamma$ a.e. on $[0, T]$

ii)   there exist constants $\lambda_T^h > 0$ such that for $h = 1, 2, \cdots, n$ and $t \in [0, T]$

$$e^{-\rho_h t} u'_h(c_T^h(t)) = \lambda_T^h p_T(t) \ .$$

☐

For a formal proof the reader is referred to Van Geldrop and Withagen but it may be instructive to sketch briefly the lines along which it runs. We start by attaching weights to the consumers and consider the problem of finding a Pareto-efficient allocation by means of optimal control theory for a fixed planning horizon T and with the stocks of capital and the exhaustible resources as state variables. It is shown that this problem has a solution with bounded states and bounded rates of consumption. Moreover the stock of capital is strictly positive for all $t < T$. This essentially allows for the application of the Pontryagin maximum principle, from which the (absolute) continuity of the rates of consumption can be derived, as well as shadow-prices. Subsequently the pseudo-budget constraints for the consumers are considered and by the Negishi approach (applied to a dynamic system) we establish the existence of weights such that all budget constraints are satisfied, thereby proving the existence of a general competitive equilibrium.

The properties of the finite horizon equilibria given in i) and ii) will play an important role in the sequel. To this must be added the uniform boundedness of production and consumption.

Lemma 2.1

There exist $B > 0$ such that $\|y_T\|_\infty \leq B$ and $\|x_T^h\|_\infty \leq B$ $(h = 1, 2, \cdots, n)$ for all $T \in I\!N$.

Proof

See the appendix.

☐

We shall also use some additional properties. Define

$$w_0 := \min_h w_0^h > 0; \ w := \min_{j \geq 0} \Sigma w_j^h > 0 \ .$$

$$u(B) := \max_h u_h(B) > 0; \ u'(B) := \min_h u'_h(B) > 0 \ .$$

$$M := \sum_h \int_0^\infty e^{-\rho_h s} u(B) \, ds \ .$$

Then we have

Lemma 2.2

i) $\displaystyle\sum_{j=0}^m \pi_{jT} \leq M / w u'(B).$

ii) $u'(B) \leq \lambda_T^h \leq M/w_0$ $(h = 1, 2, \cdots, n)$.

Proof

Given in the appendix   □

3. Equilibrium in the infinite horizon economy E

In this section we show that there exists an equilibrium in the infinite horizon E.

We start by studying the sequences $\dot{p}_T$ and p_T. By $\overset{*}{\rightarrow}$ and $\rightarrow$ we indicate weak $*$ and pointwise convergence respectively.

Lemma 3.1

There exists absolutely continuous $p \in L_\infty$ with $p(0) = 1$ such that

i) $p_T \overset{*}{\rightarrow} p$ on all $A \subset R$, bounded and measurable.

ii) $p_T \rightarrow p$.

iii) $\dot{p}_T \overset{*}{\rightarrow} \dot{p}$ on all $A \subset R$, bounded and measurable.

vi) $-\gamma p \le \dot{p} \le 0$.

Proof

p_T and $\dot{p}_T$ can on $[0, \infty)$ be considered as uniformly bounded L_∞ functions. By Alaoglu's theorem, there exists $q \in L_\infty$ and a subsequence (again denoted by T) such that $\dot{p}_T \overset{*}{\rightarrow} q$. Choose $t_0 \in [0, \infty)$. Then, for $T > t_0$, we have

$$p_T(t_0) = 1 + \int_0^{t_0} \dot{p}_T(s)\, ds$$

from which it follows that

$$p(t_0) := \lim_{T \to \infty} p_T(t_0) = 1 + \int_0^{t_0} q(s)\, ds \ .$$

Hence $p_T \rightarrow p$ and

$$p(t) = 1 + \int_0^t q(s)\, ds \ ,$$

whence $p(0) = 1$, $\dot{p} = q$ (a.e.) and $p \in AC^+$.

To see that $p_T \overset{*}{\rightarrow} p$, take $\phi \in L_1[0, \infty)$. $p_T \cdot \phi \rightarrow p \cdot \phi$ and $|p_T \cdot \phi| \le |\phi|$. By the Lebesgue dominated convergence theorem it follows that

$$\int_A p_T \cdot \phi\, ds \rightarrow \int_A p \cdot \phi\, ds, \text{ for all bounded measurable } A \ .$$

Part vi) is proved as follows, $\dot{p}_T + \gamma p_T \ge 0$ (theorem 2.1). Take $A \subset [0, \infty)$ measurable and

bounded. Then

$$\int\limits_A (\dot{p}_T + \gamma p_T)\, ds \geq 0 \text{ and hence } \int\limits_A (\dot{p} + \gamma p)\, ds \geq 0 \; . \qquad \qquad \square$$

Next we concentrate on the other prices and the equilibrium allocations.

Lemma 3.2

There exist $c^h \in L_\infty$ $(h = 1, 2, \cdots, n)$, $z \in L_\infty$, $\pi_k \in \mathbb{R}_+$ and $s_k \in \mathbb{R}_+$ $(k = 0, 1, 2, \cdots, m)$ such that

i) $c_T^h \overset{*}{\to} c^h$ on all bounded measurable $A \subset [0, \infty)$ and $c_T^h \to c^h$

ii) $z_T \overset{*}{\to} z$ on all bounded measurable $A \subset [0, \infty)$ and $z_T \to z$

iii) $\pi_{Tk} \to \pi_k$ $(k = 0, 1, 2, \cdots, m)$

iv) $s_{Tk} \to s_k$ $(k = 0, 1, 2, \cdots, m)$

v) $(-s_0, -s_1, \cdots, -s_m; z) \in Y$.

Proof

i) The sequences λ_T^h $(h = 1, 2, \cdots, n)$ are uniformly bounded (from below and from above). They have therefore convergent subsequences, denoted by T again, such that

$$\lambda_T^h \to \lambda^h \text{ with } u'(B) \leq \lambda^h \leq M / w_0 \; (h = 1, 2, \cdots, n) \; .$$

Since $e^{-\rho_h t} u'_h(c_T^h(t)) = \lambda_T^h p_T(t)$ it follows that $c_T^h(t) \to c^h(t)$ where $e^{-\rho_h t} u'_h(c^h(t)) = \lambda^h p(t)$. $c_T^h \overset{*}{\to} c^h$ is then trivially satisfied, on all bounded measurable A.

ii) This follows from $z_T(t) = \sum\limits_{h=1}^n c_T^h(t)$.

iii) and iv) The sequences π_{Tk} $(j \geq 1)$ and s_{Tk} $(k \geq 1)$ have "common" convergent subsequences.

v) See the appendix. $\qquad \qquad \square$

It remains to be shown that

$$\{x^h := (0, c^h) \,|\, _{h=1}^n, \; y := (-s; z), \; \pi := (1, \pi_1, \cdots, \pi_m; p)\}$$

constitutes a general equilibrium for the infinite horizon economy. It is evident that $x^h \in X^h$ for all h, $\sum\limits_h x^h \leq y + \sum\limits_h w^h$ and $\pi_j(\sum\limits_h w_j^h - s_j) = 0$. We also know $p(0) = \pi_0 = 1$.

The following lemmata show that all the other equilibrium conditions are satisfied.

Lemma 3.3

$\pi \cdot \bar{y} \leq 0$ for all $\bar{y} \in Y$.

Proof

Suppose $\bar{y} := (-\bar{s}, \bar{z}) \in Y$ and $\pi \cdot \bar{y} > 0$. Hence (omitting time arguments)

$$\int_0^\infty p\bar{z} \, dt > \sum_{j=0}^m \pi_j \bar{s}_j =: \hat{\pi}\bar{s} \ .$$

There exists then $\bar{T} > 0$ such that

$$\hat{\pi}\bar{s} = \lim_{T \to \infty} \hat{\pi}_T \bar{s} < \int_0^{\bar{T}} p\bar{z} \, dt = \lim_{T \to \infty} \int_0^{\bar{T}} p_T \bar{z} \, dt \ .$$

For T large enough and $T > \bar{T}$ we have

$$\int_0^{\bar{T}} p_T \bar{z} \, dt > \hat{\pi}_T \bar{s} \quad \text{and hence} \quad \int_0^T p_T \bar{z} \, dt > \hat{\pi}_T \bar{s} \ ,$$

a contradiction with the zero profit condition (in E_T). $\qquad\qquad\qquad\qquad$ ☐

Lemma 3.4

$\pi_T \cdot x_T^h \to \pi \cdot x^h \quad (h = 1, 2, \cdots, n)$.

Proof

Since $\sum_h x^h \leq \sum_h w^h + y$ and $y \in Y$, we know that $\pi \cdot x^h < \infty$. Take $\varepsilon > 0$ and choose $\bar{T}$ sufficiently large such that

$$\frac{u(B)}{u'(B)} \int_{\bar{T}}^\infty e^{-\rho_h t} \, dt \leq \tfrac{1}{3} \varepsilon \quad \text{and} \quad \int_{\bar{T}}^\infty p(t) c^h(t) \, dt \leq \tfrac{1}{3} \varepsilon \ .$$

For $T > \bar{T}$ we have (omitting time arguments)

$$\pi \cdot x^h - \pi \cdot x_T^h = \int_0^{\bar{T}} (pc^h - p_T c_T^h) \, dt + \int_{\bar{T}}^\infty pc^h \, dt - \int_{\bar{T}}^T p_T c_T^h \, dt$$

$$= \int_0^{\bar{T}} (p\,(c^h - c_T^h) + (p - p_T)\,c_T^h)\,dt + \int_T^\infty pc^h\,dt - \int_T^T p_T c_T^h\,dt \ .$$

$$\lim_{T \to \infty} \int_0^{\bar{T}} p\,(c^h - c_T^h)\,dt = \lim_{T \to \infty} \int_0^{\bar{T}} (p - p_T)\,c_T^h\,dt = 0$$

because of the weak $*$ convergence and the fact that $0 \le c_T^h \le B$.

Furthermore

$$\int_T^T p_T c_T^h\,dt = \frac{1}{\lambda_T^h} \int_T^T e^{-\rho_h t} c_T^h u'_h(c_T^h)\,ds \le \frac{u(B)}{u'(B)} \int_T^\infty e^{-\rho_h t}\,dt \ .$$

(See theorem 2.1 and lemma 2.2). So, for $T > \bar{T}$ and T large enough,

$$|\pi \cdot x^h - \pi_T \cdot x_T^h| \le \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon \ . \qquad \qquad \square$$

Theorem 3.1

$\{x^h \mid {}_{h=1}^n,\, y,\, \pi\}$ is a general equilibrium for economy E.

Proof

It follows from lemma 3.4. that $\pi_T \cdot x_T^h \to \pi \cdot x^h$ for all h. Furthermore $\pi_T \cdot x_T^h = \pi_T \cdot w^h \to \pi \cdot w^h$. Hence $\pi \cdot x^h = \pi \cdot w^h$.

Take $\bar{x}^h \in X^h$ for some h such that $\pi \cdot \bar{x}^h \le \pi \cdot w^h$. In view of the concavity of u_h we have

$$\int_0^\infty e^{-\rho_h t} u_h(c^h)\,dt - \int_0^\infty e^{-\rho_h t} u_h(\bar{c}_h)\,dt \ge \int_0^\infty \lambda^h p\,(c^h - \bar{c}^h)\,dt \ge 0$$

because $\pi \cdot w^h = \pi \cdot x^h \ge \pi \cdot \bar{x}^h$.

So all the conditions for a general competitive equilibrium are satisfied. $\qquad \square$

4. Conclusion

The model treated in the previous section can be deemed to be rather special in several respects and the question in how far the existence result remains valid for a broader class of models, is therefore justified although our model is already far more general than existing ones.

It should be noted that uniform boundedness of equilibria in the truncated economy together with convexity have played a crucial role in establishing the existence of a general equilibrium in the infinite horizon economy. Therefore it is tempting to conclude that these properties are sufficient for the existence of an equilibrium, irrespective of the underlying economic model. Let us therefore concentrate on the structure of the economy we have in mind. We wish to stress first that the high level of aggregation on the supply side of the economy is by no means essential. We have chosen for this level of aggregation only for notational convenience and to avoid unpleasant details in the mathematical treatment. So there is no problem in distinguishing m separate resource sectors and an arbitrary number of other productive sectors, as long as each sector is characterized by constant returns to scale. The only difference would be that, since the technologies differ, exploitation and production would switch over time from one sector to another. What can there be said about the relaxation of the constant returns to scale assumption? It has played an important role in proving the uniform boundedness. Intuition says that with decreasing returns to scale boundedness will hold a fortiori. The formalization of this idea is presently subject to further research. It is not too difficult to think of economies which have more physically distinguishable commodities than ours, where uniform boundedness is likely to hold. The precise conditions such economies must then satisfy are also subject to further research.

With these qualifications in mind we conclude that the existence proof given here goes beyond the model actually presented.

Appendix

Proof of Lemma 2.1

It will be shown that there exists some $A > 0$ such that $c_T^h(t) \leq A$ for all $T \in \mathbb{N}$, all h, all $t \, (\leq T)$.

To this end we fix some T and some h and we omit the subscript T and index h.

It follows from Theorem 2.1 that, with $r(t) := -\dot{p}(t)/p(t)$,

(A1.1) $\dot{K} \leq F(K - aE, \sigma E) - c$,

(A1.2) $\dfrac{\dot{c}}{c} = \dfrac{r(t) - \rho}{\eta(c)}$,

(A1.3) $r(t) = \dfrac{-\dot{p}}{p} = F_K(K - aE, \sigma E)$ and $0 \leq r(t) \leq \gamma$.

It is also to be noted that, due to the homogeneity of F, r is decreasing (see Van Geldrop and Withagen (1988)).

It follows from U^3 that

(A1.4) $\eta(c) = \dfrac{-cu''(c)}{u'(c)} \geq \eta > 0$.

We define $t_0 \in [0, T]$ by:

(A1.5) $\begin{cases} 0 \leq t < t_0 \;\Rightarrow\; r(t) > \rho \\ t_0 \leq t \leq T \;\Rightarrow\; r(t) \leq \rho \end{cases}$ ·

From (A1.2) we derive:

(A1.6) $e^{\delta_1 t} c$ is increasing and $e^{-\delta_2 t} c$ is decreasing

where $\delta_1 = \dfrac{\rho}{\eta}$ and $\delta_2 = \dfrac{\gamma}{\eta}$.

From (A1.5) it follows that $c(t) \leq \bar{C}$ where $\bar{C} = c(t_0)$.

We consider the interval $[0, t_0)$, where $r(t) > \rho$.

Homogeneity of F implies that there is some $d > 0$ such that $F_K(K - aE, \sigma E) > \rho \Rightarrow K - aE \leq d\sigma E$.

So, on $[0, t_0)$ we have $F(K - aE, \sigma E) \leq \sigma E \, F(d, 1)$ and $\dot{K} \leq \sigma E F(d, 1) - c$. Straightforward integration yields

(A1.7) $\qquad \int\limits_{0}^{t_0} c(t)\, dt \leq K(0) + F(d, 1)\, \bar{S}$

where $\bar{S}$ is the total resource stock at $t = 0$.

From $aE \leq K$ we derive $F(K - aE,\, \sigma E) \leq bK$ where $b = F(1, \frac{1}{a_1})$. So, for all $t \in [0, T]$, we

have $\dot{K} \leq bK - c$. From this it readily follows that $e^{-bT}K(T) - K(0) + \int\limits_{0}^{T} e^{-bt} c(t)\, dt \leq 0$ and we

conclude

(A1.8) $\qquad \int\limits_{t_0}^{T} e^{-bt} c(t)\, dt \leq K(0)$.

Now we apply (A1.6).

Inserting $e^{-\delta_2 t} c(t) \leq e^{-\delta_2 t_0}\, \bar{C}$ in (A1.7) and

$$e^{\delta_1 t} c(t) \geq e^{\delta_1 t_0}\, \bar{C} \quad \text{in (A1.8)}$$

gives $0 \leq \bar{C}(1 - e^{-\delta_2 t_0}) \leq \delta_2(K(0) + F(d, 1)\, \bar{S})$

$\qquad 0 \leq \bar{C}(e^{-bt_0} - e^{-(b+\delta_1)T + \delta_1 t_0}) \leq (b + \delta_1)\, K(0)$.

Recall that the right-hand sides of these inequalities are independent of T.

The rest is simple. A "great" value of $\bar{C}$ forces t_0 to zero, but then the second in equality keeps $\bar{C}$

bounded, since $1 - e^{-(b+\delta_1)T} \geq 1 - e^{-(b+\delta_1)} > 0$, for $T \in \mathbb{N}$. $\qquad \square$

Proof of Lemma 2.2

It follows from theorem 2.1 that

$$\lambda_T^h = u'_h(c_T^h(0))/p_T(0) = u'_h(c_T^h(0)) \geq u'_h(B) \geq u'(B)$$.

Using this we have

$$M \geq \int\limits_{0}^{T} \Sigma e^{-\rho_h t} u_h(c_T^h)\, dt \geq \int\limits_{0}^{T} \Sigma e^{-\rho_h t}\, u'_h(c_T^h)\, c_T^h\, dt$$

$$= \int\limits_{0}^{T} \Sigma \lambda_T^h p_T c_T^h\, dt \geq u'(B) \int\limits_{0}^{T} p_T z_T\, dt$$

$$= u'(B) \sum_{j=0}^{m} \pi_{jT} s_{jT} = u'(B) \, (\sum_h w_0^h + \pi_{1T} \sum_h w_1^h \cdots + \pi_{mT} \sum_h w_m^h)$$

$$\geq u'(B) \, w \, (\sum_{j=0}^{m} \pi_{jT}) \; .$$

This proves i).

It remains to show that $\lambda_T^h \leq M/w_0 \quad (h = 1, 2, \cdots, n)$.

Since

$$\int_0^T p_T c_T^h \, dt = \sum_{j=0}^{m} \pi_{jT} w_j^h$$

we have

$$\lambda_T^h \int_0^T p_T c_T^h \, dt = \lambda_T^h \sum_{j=0}^{m} \pi_{jT} w_j^h \geq \lambda_T^h w_0 \; .$$

Hence

$$\int_0^T e^{-\rho_h t} u'_h(c_T^h) \, c_T^h \, dt \geq \lambda_T^h w_0 \; .$$

But we also have

$$M \geq \int_0^T e^{-\rho_h t} u_h(c_T^h) \, dt \geq \int_0^T e^{-\rho_h t} u'_h(c_T^h) \, c_T^h \, dt \; .$$

Proof of Lemma 3.2(v)

We consider the sequence $y_T \in Y$, where $y_T = (-s_{0T}, -s_{1T}, \cdots, -s_{mT}; z_T)$, uniformly bounded by $\|y_T\|_\infty \leq B$, all $T \in I\!N$. Moreover $z_T(t) = 0, t > T$. It has been established in the previous statements of this lemma that there are reals $(s_0, s_1, \cdots, s_m)$ and an L_∞^+ function $z: [0, \infty) \to I\!R^+$ such that $z_T(t) \to z(t)$ all $t \in [0, \infty)$.

By definition of Y there are, for all T, AC^+ functions $K_T(t)$, $S_{1T}(t)$, $\cdots$, $S_{mT}(t)$ and non-negative measurable functions $E_{1T}(t) \cdots E_{mT}(t)$, all of them defined on $[0, T]$ satisfying:

$$\left. \begin{array}{ll} \text{satisfying: } aE_T \leq K_T & \\[4pt] \dot{K}_T = F(K_T - aE_T, \sigma E_T) - z_T \; ; K_T(0) = s_{0T} \; . & \\[4pt] \dot{S}_{jT} = -E_{jT} & ; S_{jT}(0) = s_{jT} \; . \end{array} \right\} \begin{array}{l} (\alpha) \\[4pt] (\beta) \\[4pt] (\gamma) \end{array}$$

From (α) and $a_j \geq a_1 > 0 \; (k = 1, \cdots, m)$ we derive $\sigma E_T \leq \dfrac{1}{a_1} K_T$ and $F(K_T - aE_T, \sigma E_T) \leq bK_T$,

where $b = F(1, \dfrac{1}{a_1})$. See also the proof of lemma 2.1.

Then it follows from (β) that $\dot{K}_T \leq bK_T$, hence $e^{-bt}K_T(t)$ is decreasing.

We introduce new functions by the transformations:

$$
\left.
\begin{aligned}
H_T(t) &:= e^{-bt}K_T(t) \\
U_{jT}(t) &:= e^{-bt}E_{jT}(t);\, j = 1, \cdots, m \quad. \\
X_{jT}(t) &:= e^{-bt}S_{jT}(t);\, j = 1, \cdots, m \quad. \\
W_T(t) &:= e^{-bt}Z_T(t) \quad.
\end{aligned}
\right\}
$$

It is easily seen that there is some uniform upperbound for all of these new functions.

Moreover, due to the homogeneity of F, we can rewrite the differential equations (β) and (γ) finding:

$$
\left.
\begin{aligned}
\dot{H}_T + bH_T &= F(H_T - aU_T,\, \sigma U_T) - W_T;\, H_T(0) = s_{0T} \\
\dot{X}_{jT} + bX_{jT} &= -U_{jT};\, X_{jT}(0) = S_{jT}(0);\, j = 1, 2, \cdots, m
\end{aligned}
\right\}
\begin{array}{l} (\beta') \\ (\gamma') \end{array}
$$

So, $\dot{H}_T$ and each $\dot{X}_{jT}$ may be considered as a uniformly bounded function in $L_\infty[0, \infty)$.

We denote the subsequences of weakstar convergence for these functions by T (again). From this it follows that H_T and X_{jT} are pointwise convergent. Concluding, we find that there are functions H, U_j, W, X_j satisfying

$$
\left.
\begin{aligned}
\dot{H} + bH &= F(H - aU,\, \sigma U) - W;\, H(0) = s_0 \\
\dot{X}_j + bX_j &= -U_j;\, X_j(0) = S_j(0) = s_j
\end{aligned}
\right\}
\begin{array}{l} (\beta'') \\ (\gamma'') \end{array}
$$

where

$$
\left\{
\begin{aligned}
\dot{U}_T &\overset{*}{\to} \dot{H} \quad \text{(weak star convergence)} \\
H_T &\to H \quad \text{(pointwise convergence)} \\
U_{jT} &\overset{*}{\to} U_j \\
W_T &\to W \\
\dot{X}_{jT} &\overset{*}{\to} \dot{X}_j \\
X_{jT} &\to X_j
\end{aligned}
\right.
$$

Here we exploit the Lebesgue dominated convergence theorem on each interval $[0, t]$. Now, performing the inverse transformation, introducing K, E_j, S_j, $\hat{z}$, we find

$$\dot{K} = F(K - aE, \sigma E) - \hat{z} \;\; ; K(0) = s_0 \;\; ; aE \le K \;\; .$$
$$\dot{S}_j = -E_j \qquad\qquad ; S_j(0) = s_j \;\; ; j = 1 \cdots m \;\; .$$

where $\hat{z}(t) := e^{bt} W(t)$.

From $e^{bt} W_T(t) = z_T(t)$ and $z_T(t) \to z(t)$ we derive $e^{bt} W_T(t) \to z(t)$. Hence $\hat{z}(t) = z(t)$. $\qquad$ ☐

References

Bewley, T. (1972) Existence of Equilibria with infinitely many Commodities, *Journal of Economic Theory*, 4, pp. 514-540.

Cass, D. and K. Shell (1976) The Hamiltonian Approach to Dynamic Economics, Academic Press, New York.

Cesari, L. (1983) Optimization - Theory and Applications, Springer Verlag, Berlin.

Chiarella, C. (1980) Trade between resource-poor and resource-rich economies as a differential game. In M. Kemp and N. Long (eds.), *Exhaustible Resources, Optimality and Trade*, Amsterdam, North-Holland, pp. 219-246.

Dasgupta, P. and G. Heal (1974) The optimal depletion of exhaustible resources, *Review of Economic Studies*, Symposium Issue, pp. 3-28.

Elbers, Ch. and C. Withagen (1984) Trading in exhaustible resources in the presence of conversion costs, a general equilibrium approach, *Journal of Economic Dynamics and Control*, 8, pp. 197-209.

Gale, D. (1967) On Optimal Development in a Multi-Sector Economy, *Review of Economic Studies*, 34, pp. 1-18.

Van Geldrop, J. and C. Withagen (1988) A General Equilibrium Model of International Trade with Exhaustible Natural Resource Commodities, Memorandum COSOR 88-04, Eindhoven University of Technology.

Van Geldrop, J., Shou Jilin and C. Withagen (1989) Existence of general equilibria in economies with natural exhaustible resources and an infinite horizon, Memorandum COSOR 89- 32, Eindhoven University of Technology, forthcoming in *Journal of Mathematical Economics*.

Van Geldrop, J. and C. Withagen (1990) On the Negishi-approach to dynamic economic systems, *Nieuw Archief voor Wiskunde*, forthcoming.

Jones, L. (1986) Special problems arising in the study of economies with infinitely many commodities, in Sonnenschein, H. (ed.), *Models of Economic Dynamics*, Springer Verlag, Berlin, pp. 184-205.

Kemp, M.C. and N.V. Long (1980) The interaction of resource-rich and resource-poor economies. In M. Kemp and N. Long (eds.), *Exhaustible Resources, Optimality and Trade*, Amsterdam, North-Holland, pp. 197-210.

Mas-Colell, A. (1986) Valuation Equilibrium and Pareto Optimum Revisited, in W. Hildenbrand and A. Mas-Colell (eds.), *Contributions to Mathematical Economics*, in Honor of Gerard Debreu, North-Holland, pp. 317-331.

Mitra, T. (1980) On optimal depletion of exhaustible resources: existence and characterization results, *Econometrica*, 48, pp. 1431-1450.

Radner, R. (1967) Efficiency Prices for Infinite-Horizon Production Programmes, *Review of Economic Studies*, 34, pp. 51-66.

Richard, S. (1986) Competitive Equilibria in Riesz Spaces, Carnegie-Mellon University Working paper.

Withagen, C. (1990) Topics in resource economics, in Van der Ploeg, F. (ed.), *Advanced Lectures in Quantitative Economics*, Academic Press, London, pp. 381-419.

Zame, W. (1987) Competitive equilibria in production economies with an infinite-dimensional commodity space, *Econometrica*, **55**, pp. 1075-1108.

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics and Computing Science

PROBABILITY THEORY, STATISTICS, OPERATIONS RESEARCH AND SYSTEMS THEORY

P.O. Box 513

5600 MB Eindhoven - The Netherlands

Secretariate: Dommelbuilding 0.03

Telephone: 040 - 47 3130

List of COSOR-memoranda - 1990

Number	Month	Author	Title
M 90-01	January	I.J.B.F. Adan J. Wessels W.H.M. Zijm	Analysis of the asymmetric shortest queue problem Part 1: Theoretical analysis
M 90-02	January	D.A. Overdijk	Meetkundige aspecten van de productie van kroonwielen
M 90-03	February	I.J.B.F. Adan J. Wessels W.H.M. Zijm	Analysis of the assymmetric shortest queue problem Part II: Numerical analysis
M 90-04	March	P. van der Laan L.R. Verdooren	Statistical selection procedures for selecting the best variety
M 90-05	March	W.H.M. Zijm E.H.L.B. Nelissen	Scheduling a flexible machining centre
M 90-06	March	G. Schuller W.H.M. Zijm	The design of mechanizations: reliability, efficiency and flexibility
M 90-07	March	W.H.M. Zijm	Capacity analysis of automatic transport systems in an assembly factory
M 90-08	March	G.J. v. Houtum W.H.M. Zijm	Computational procedures for stochastic multi-echelon production systems

Number	Month	Author	Title
M 90-09	March	P.J.M. van Laarhoven W.H.M. Zijm	Production preparation and numerical control in PCB assembly
M 90-10	March	F.A.W. Wester J. Wijngaard W.H.M. Zijm	A hierarchical planning system versus a schedule oriented planning system
M 90-11	April	A. Dekkers	Local Area Networks
M 90-12	April	P. v.d. Laan	On subset selection from Logistic populations
M 90-13	April	P. v.d. Laan	De Van Dantzig Prijs
M 90-14	June	P. v.d. Laan	Beslissen met statistische selectiemethoden
M 90-15	June	F.W. Steutel	Some recent characterizations of the exponential and geometric distributions
M 90-16	June	J. van Geldrop C. Withagen	Existence of general equilibria in infinite horizon economies with exhaustible resources. (the continuous time case)