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ON PARTIAL SUMS OF $\sum_{d|M} \varphi(d)$

N. G. DE BRUIJN and J. H. VAN LINT

1. Introduction. We consider the well known identity

$$(1.1) \quad \sum_{d|M} \varphi(d) = M,$$

where φ denotes Euler's function.

We are interested in the question which partial sums are much smaller than M . For instance, if $\alpha < \frac{1}{2}$ then

$$\sum_{d|M, d < M^\alpha} \varphi(d) < \sum_{d < M^\alpha} d = O(M^{2\alpha}) = o(M)$$

for $M \rightarrow \infty$. Now, more generally, consider all functions f such that if $P = f(M)$ then

$$(1.2) \quad \sum_{d|M, d < M/P} \varphi(d) = o(M) \quad (M \rightarrow \infty).$$

We have seen that $f(x) = x^{1-\alpha}$ with $\alpha < \frac{1}{2}$ is a trivial example of such a function. The problem we wish to attack is finding functions f which are in some sense best possible for the relation (1.2) (where $P = f(M)$).

We shall prove the following statements :

Theorem 1. If $(\log f(x))/(\log \log x) \rightarrow \infty$ for $x \rightarrow \infty$ and $P = f(M)$, then (1.2) holds.

Theorem 2. If for some function f the relation (1.2) holds with $P = f(M)$, then

$$\limsup \frac{\log f(x)}{\log \log x} = \infty.$$

Actually relatively few numbers M (namely the numbers that have many different prime factors) require these relatively large values of P . This is illustrated by the following theorem regarding the mean value of

$$M^{-1} \sum_{d|M, d < M/P} \varphi(d).$$

Theorem 3. If $f(x) \rightarrow \infty$ for $x \rightarrow \infty$ and $P = f(M)$, then

$$\lim_{x \rightarrow \infty} x^{-1} \sum_{M \leq x} M^{-1} \sum_{d|M, d \leq M/P} \varphi(d) = 0.$$

Proof: We first replace P by a number y not depending on M . If μ denotes the Möbius function, we have

$$\begin{aligned} & \sum_{M \leq x} M^{-1} \sum_{d|M, d \leq M/y} \varphi(d) = \\ &= \sum_{M \leq x} \sum_{k|M, k \geq y} k^{-1} \sum_{t|(M/k)} t^{-1} \mu(t) = \\ &= \sum_{t \leq x} t^{-1} \mu(t) \sum_{y \leq k \leq x} k^{-1} \sum_{M \leq x, M \equiv 0 \pmod{kt}} 1 = \\ &= \sum_{t \leq x} t^{-1} \mu(t) \sum_{y \leq k \leq x} k^{-1} [x/(kt)]. \end{aligned}$$

We write $[x/(kt)] = (x/(kt)) + O(1)$. The main term gives

$$x \sum_{t \leq x} t^{-2} \mu(t) \sum_{y \leq k \leq x} k^{-2} = O(xy^{-1})$$

and the error term $O(1)$ gives at most

$$\sum_{t \leq x} t^{-1} \sum_{y \leq k \leq x} k^{-1} O(1) = O((\log x)^2).$$

Let M_0 be such that $P = f(M) > y$ for all $M > M_0$. We now have, if $x \rightarrow \infty$,

$$\begin{aligned} & \sum_{M \leq x} M^{-1} \sum_{d|M, d \leq M/P} \varphi(d) \leq \\ & \leq \sum_{M \leq M_0} M^{-1} \sum_{d|M} \varphi(d) + \sum_{M_0 < M \leq x} M^{-1} \sum_{d|M, d \leq M/y} \varphi(d) = \\ & = O(M_0) + O(xy^{-1}) + O((\log x)^2), \end{aligned}$$

and the theorem follows from the fact that y is arbitrary.

2. Proof of theorems 1 and 2.

The proof of theorem 1 depends on the following lemma.

Lemma 1. There is a constant C such that for every integer $M \geq 3$ and every $P > 1$

$$(2.1) \quad \sum_{d|M, d < M/P} \varphi(d) \leq CM (\log \log M) / (\log P).$$

Proof: Let $M = p_1^{\lambda(1)} p_2^{\lambda(2)} \dots p_k^{\lambda(k)}$ be the factorization of M .

Then we have

$$\begin{aligned} (\log P) \sum_{d|M, d < M/P} \varphi(d) & \leq \sum_{d|M, d < M/P} \varphi(d) \log(M/d) \\ & \leq \sum_{d|M} \varphi(d) \log(M/d). \end{aligned}$$

We define

$$\Pi(s) = M^s \sum_{d|M} \varphi(d) d^{-s} = \Pi_{i=1}^k \{ \sum_{j=0}^{\lambda(i)} p_i^{(\lambda(i)-j)s} \varphi(p_i^j) \}.$$

Then by differentiation we obtain

$$\begin{aligned} \sum_{d|M} \varphi(d) \log(M/d) &= \Pi'(0) = \\ &= \Pi(0) \left[\frac{d}{ds} \sum_{i=1}^k \log \left\{ \sum_{j=0}^{\lambda(i)} p_i^{(\lambda(i)-j)s} \varphi(p_i^j) \right\} \right]_{s=0} = \\ &= M \sum_{i=1}^k \frac{\log p_i}{p_i^{\lambda(i)}} \sum_{j=0}^{\lambda(i)} (\lambda(i)-j) \varphi(p_i^j) = \\ &= M \sum_{i=1}^k \frac{\log p_i}{p_i^{\lambda(i)}} \frac{p_i^{\lambda(i)} - 1}{p_i - 1} \leq M \sum_{p|M} \frac{\log p}{p-1} \leq \\ &\leq 2M \sum_{p|M} \frac{\log p}{p}. \end{aligned}$$

Now

$$(2.2) \quad \sum_{p \leq w} p^{-1} \log p = \log w + O(1)$$

and

$$(2.3) \quad \sum_{p \leq w} \log p \sim w \quad (w \rightarrow \infty).$$

Let w be such that $\pi(w) = k$, where $\pi(w)$ stands for the number of primes $\leq w$.

Then

$$\log M = \sum_{i=1}^k \lambda_i \log p_i \geq \sum_{p \leq w} \log p$$

and

$$\sum_{p|M} p^{-1} \log p \leq \sum_{p \leq w} p^{-1} \log p,$$

whence, by (2.2) and (2.3).

$$\sum_{p|M} p^{-1} \log p = O(\log \log M).$$

This completes the proof of (2.1).

Remark : If we replace the estimates in the proof by estimates with explicit constants (cf. J.B. ROSSER and L. SCHOENFELD [5]) we can prove, with a considerable amount of calculation, that (2.1) holds with $C = 2$. For $C = 1$ there are numbers M and P such that (2.1) is false.

Proof of theorem 1 : If $(\log f(x))/(\log \log x) \rightarrow \infty$ for $x \rightarrow \infty$ and $P = f(M)$, then the right-hand side of (2.1) is $o(M)$.

In the proof of theorem 2 we need the following lemma.

Lemma 2 : Let $d(u)$ be defined by

- (i) $d(u) = u$ for $0 \leq u \leq 1$,
- (ii) $d(u)$ is continuous for $u \geq 0$,
- (iii) $(u^{-1}d(u))' = -u^{-2}d(u-1)$ for $u \geq 1$.

Let $p(n)$ denote the largest prime factor of n . Then for $x > 1$, $y > 1$, $u = (\log x)/(\log y)$ we have

$$(2.4) \quad \sum_{n \leq x, p(n) < y} \frac{\mu^2(n)}{\varphi(n)} = d(u) \log y + O(1).$$

Furthermore, d is increasing and

$$(2.5) \quad \lim_{u \rightarrow \infty} d(u) = e^\gamma.$$

For a proof of (2.4) see [3]. (In [2] there is a weaker result which is also sufficient for our present purpose). For a proof of (2.5) we refer to [1] or to [2], theorem 2.2.

We now consider the numbers $M = M(y) = \prod_{p < y} p$. Let u be fixed and take $P = (\log M)^u$. We have $\log M = \sum_{p < y} \log p \sim y$ for $y \rightarrow \infty$, and therefore $\log \log M = \log y + O(1)$. It follows that

$$(2.6) \quad \begin{aligned} \sum_{d|M, d \geq M/P} \varphi(d) &= \varphi(M) \sum_{d|M, d \leq (\log M)^u} (\varphi(d))^{-1} \\ &= \varphi(M) \{d(u) \log y + O(1)\} \end{aligned}$$

by (2.4).

For $y \rightarrow \infty$ we have

$$(2.7) \quad \varphi(M) = M \prod_{p < y} (1 - p^{-1}) \sim M e^{-\gamma} (\log y)^{-1}$$

(cf. [4] page 81). For the left-hand member of (1.2) we now have (with $P = (\log M)^u$) :

$$(2.8) \quad \begin{aligned} \sum_{d|M, d < M/P} \varphi(d) &= M - \sum_{d|M, d \geq M/P} \varphi(d) \\ &\sim M (1 - e^{-\gamma} d(u)) \quad (y \rightarrow \infty) \end{aligned}$$

by (2.6) and (2.7).

We now prove theorem 2. Suppose that u is such that $(\log f(x))/(\log \log x) \leq u$ for all x . If in (1.2) we replace $P = f(M)$ by $(\log M)^u$, the sum on the left-hand side is certainly not enlarged, and what we obtain is the left-hand member of (2.8). The latter

expression is not $o(M)$ if $M \rightarrow \infty$, i.e. if $y \rightarrow \infty$. This proves that if (1.2) holds, $(\log f(x))/(\log \log x)$ is not bounded, and the proof of theorem 2 is complete.

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