On partial sums of $\sum_{d|M} \varphi(d)$

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ON PARTIAL SUMS OF $\Sigma_{d|M}\, \varphi(d)$

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1. Introduction. We consider the well known identity

$$\Sigma_{d|M}\, \varphi(d) = M, \tag{1.1}$$

where φ denotes Euler's function.

We are interested in the question which partial sums are much smaller than M. For instance, if $\alpha < \frac{1}{2}$ then

$$\Sigma_{d|M, d<M^\alpha}\, \varphi(d) < \Sigma_{d<M^\alpha}\, d = O(M^{2\alpha}) = o(M)$$

for $M \to \infty$. Now, more generally, consider all functions f such that if $P = f(M)$ then

$$\Sigma_{d|M, d<M/P}\, \varphi(d) = o(M) \qquad (M \to \infty). \tag{1.2}$$

We have seen that $f(x) = x^{1-\alpha}$ with $\alpha < \frac{1}{2}$ is a trivial example of such a function. The problem we wish to attack is finding functions f which are in some sense best possible for the relation (1.2) (where $P = f(M)$).

We shall prove the following statements :

Theorem 1. If $(\log f(x))/(\log\log x) \to \infty$ for $x \to \infty$ and $P = f(M)$, then (1.2) holds.

Theorem 2. If for some function f the relation (1.2) holds with $P = f(M)$, then

$$\limsup \frac{\log f(x)}{\log\log x} = \infty .$$

Actually relatively few numbers M (namely the numbers that have many different prime factors) require these relatively large values of P. This is illustrated by the following theorem regarding the mean value of

$$M^{-1}\, \Sigma_{d|M, d<M/P}\, \varphi(d).$$

Theorem 3. If $f(x) \to \infty$ for $x \to \infty$ and $P = f(M)$, then

$$\lim_{x\to\infty} x^{-1} \Sigma_{M \leqslant x} M^{-1} \Sigma_{d|M, d \leqslant M/P} \varphi(d) = 0.$$

Proof: We first replace P by a number y not depending on M. If μ denotes the Möbius function, we have

$$\Sigma_{M \leqslant x} M^{-1} \Sigma_{d|M, d \leqslant M/y} \varphi(d) =$$

$$= \Sigma_{M \leqslant x} \Sigma_{k|M, k \geqslant y} k^{-1} \Sigma_{t|(M/k)} t^{-1} \mu(t) =$$

$$= \Sigma_{t \leqslant x} t^{-1} \mu(t) \Sigma_{y \leqslant k \leqslant x} k^{-1} \Sigma_{M \leqslant x, M \equiv 0 \pmod{kt}} 1 =$$

$$= \Sigma_{t \leqslant x} t^{-1} \mu(t) \Sigma_{y \leqslant k \leqslant x} k^{-1} [x/(kt)].$$

We write $[x/(kt)] = (x/(kt)) + O(1)$. The main term gives

$$x \Sigma_{t \leqslant x} t^{-2} \mu(t) \Sigma_{y \leqslant k \leqslant x} k^{-2} = O(xy^{-1})$$

and the error term $O(1)$ gives at most

$$\Sigma_{t \leqslant x} t^{-1} \Sigma_{y \leqslant k \leqslant x} k^{-1} O(1) = O((\log x)^2).$$

Let M_0 be such that $P = f(M) > y$ for all $M > M_0$. We now have, if $x \to \infty$,

$$\Sigma_{M \leqslant x} M^{-1} \Sigma_{d|M, d \leqslant M/P} \varphi(d) \leqslant$$

$$\leqslant \Sigma_{M \leqslant M_0} M^{-1} \Sigma_{d|M} \varphi(d) + \Sigma_{M_0 < M \leqslant x} M^{-1} \Sigma_{d|M, d \leqslant M/y} \varphi(d) =$$

$$= O(M_0) + O(xy^{-1}) + O((\log x)^2),$$

and the theorem follows from the fact that y is arbitrary.

2. Proof of theorems 1 and 2.

The proof of theorem 1 depends on the following lemma.

Lemma 1. There is a constant C such that for every integer $M \geqslant 3$ and every $P > 1$

$$\Sigma_{d|M, d < M/P} \varphi(d) \leqslant CM (\log \log M)/(\log P). \tag{2.1}$$

Proof: Let $M = p_1^{\lambda(1)} p_2^{\lambda(2)} \ldots p_k^{\lambda(k)}$ be the factorization of M.

Then we have

$$(\log P) \Sigma_{d|M, d < M/P} \varphi(d) \leqslant \Sigma_{d|M, d < M/P} \varphi(d) \log (M/d)$$

$$\leqslant \Sigma_{d|M} \varphi(d) \log (M/d).$$

We define

$$\Pi(s) = M^s \Sigma_{d|M} \varphi(d)\, d^{-s} = \Pi_{i=1}^{k} \{\Sigma_{j=0}^{\lambda(i)} p_i^{(\lambda(i)-j)s}\ \varphi(p_i^j)\}.$$

Then by differentiation we obtain

$$\sum_{d|M} \varphi(d) \log (M/d) = \Pi'(0) =$$

$$= \Pi\ (0)\ \left[\frac{d}{ds} \sum_{i=1}^{k} \log \left\{ \sum_{j=0}^{\lambda(i)} p_i^{(\lambda(i)-j)s}\ \varphi(p_i^j)\right\}\right]_{s=0} =$$

$$= M \sum_{i=1}^{k} \frac{\log p_i}{p_i^{\lambda(i)}} \sum_{j=0}^{\lambda(i)} (\lambda(i)-j)\ \varphi(p_i^j) =$$

$$= M \sum_{i=1}^{k} \frac{\log p_i}{p_i^{\lambda(i)}}\ \frac{p_i^{\lambda(i)}-1}{p_i-1} \leqslant M \sum_{p|M} \frac{\log p}{p-1} \leqslant$$

$$\leqslant 2\, M \sum_{p|M} \frac{\log p}{p}\,.$$

Now

(2.2) $$\Sigma_{p \leqslant w}\, p^{-1} \log p = \log w + O(1)$$

and

(2.3) $$\Sigma_{p \leqslant w} \log p \sim w \qquad (w \to \infty).$$

Let w be such that $\pi(w) = k$, where $\pi(w)$ stands for the number of primes $\leqslant w$.

Then

$$\log M = \Sigma_{i=1}^{k}\, \lambda_i \log p_i \geqslant \Sigma_{p \leqslant w} \log p$$

and

$$\Sigma_{p|M}\, p^{-1} \log p \leqslant \Sigma_{p \leqslant w}\, p^{-1} \log p,$$

whence, by (2.2) and (2.3).

$$\Sigma_{p|M}\, p^{-1} \log p = O\,(\log \log M).$$

This completes the proof of (2.1).

Remark : If we replace the estimates in the proof by estimates with explicit constants (cf. J.B. ROSSER and L. SCHOENFELD [5]) we can prove, with a considerable amount of calculation, that (2.1) holds with $C = 2$. For $C = 1$ there are numbers M and P such that (2.1) is false.

Proof of theorem 1 : If $(\log f(x))/(\log\log x) \to \infty$ for $x \to \infty$ and $P = f(M)$, then the right-hand side of (2.1) is $o(M)$.

In the proof of theorem 2 we need the following lemma.

Lemma 2 : Let $d(u)$ be defined by

(*i*) $d(u) = u$ for $0 \leqslant u \leqslant 1$,
(*ii*) $d(u)$ is continuous for $u \geqslant 0$,
(*iii*) $(u^{-1}d(u))' = -u^{-2}d(u-1)$ for $u \geqslant 1$.

Let $p(n)$ denote the largest prime factor of n. Then for $x > 1$, $y > 1$, $u = (\log x)/(\log y)$ we have

$$\Sigma_{n \leqslant x,\, p(n) < y} \frac{\mu^2(n)}{\varphi(n)} = d(u) \log y + O(1). \tag{2.4}$$

Furthermore, d is increasing and

$$\lim_{u \to \infty} d(u) = e^{\gamma}. \tag{2.5}$$

For a proof of (2.4) see [3]. (In [2] there is a weaker result which is also sufficient for our present purpose). For a proof of (2.5) we refer to [1] or to [2], theorem 2.2.

We now consider the numbers $M = M(y) = \Pi_{p<y}\, p$. Let u be fixed and take $P = (\log M)^u$. We have $\log M = \sum_{p<y} \log p \sim y$ for $y \to \infty$, and therefore $\log\log M = \log y + O(1)$. It follows that

$$\begin{aligned} \Sigma_{d|M,\, d \geqslant M/P}\, \varphi(d) &= \varphi(M)\, \Sigma_{d|M,\, d \leqslant (\log M)^u} (\varphi(d))^{-1} \\ &= \varphi(M)\{d(u) \log y + O(1)\} \end{aligned} \tag{2.6}$$

by (2.4).

For $y \to \infty$ we have

$$\varphi(M) = M\, \Pi_{p<y} (1-p^{-1}) \sim M\, e^{-\gamma} (\log y)^{-1} \tag{2.7}$$

(cf. [4] page 81). For the left-hand member of (1.2) we now have (with $P = (\log M)^u$) :

$$\begin{aligned} \Sigma_{d|M,\, d<M/P}\, \varphi(d) &= M - \Sigma_{d|M,\, d \geqslant M/P}\, \varphi(d) \\ &\sim M\,(1 - e^{-\gamma} d(u)) \qquad (y \to \infty) \end{aligned} \tag{2.8}$$

by (2.6) and (2.7).

We now prove theorem 2. Suppose that u is such that $(\log f(x)/(\log\log x) \leqslant u$ for all x. If in (1.2) we replace $P = f(M)$ by $(\log M)^u$, the sum on the left-hand side is certainly not enlarged, and what we obtain is the left-hand member of (2.8). The latter expression is not $o(M)$ if $M \to \infty$, i.e. if $y \to \infty$. This proves that if (1.2) holds, $(\log f(x))/(\log \log x)$ is not bounded, and the proof of theorem 2 is complete.

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