

## Existence and robustness of minimal norm controllers

**Citation for published version (APA):**

Engwerda, J. C. (1987). *Existence and robustness of minimal norm controllers*. (Memorandum COSOR; Vol. 8716). Technische Universiteit Eindhoven.

**Document status and date:**

Published: 01/01/1987

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

**General rights**

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

[www.tue.nl/taverne](http://www.tue.nl/taverne)

**Take down policy**

If you believe that this document breaches copyright please contact us at:

[openaccess@tue.nl](mailto:openaccess@tue.nl)

providing details and we will investigate your claim.

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Faculty of Mathematics and Computing Science

Memorandum COSOR 87-16

Existence and Robustness of  
minimal norm controllers

by

J.C. Engwerda

Eindhoven, Netherlands

August 1987

## ABSTRACT

In this paper the possibility to stabilize linear discrete time-varying systems by state feedback is examined if the time dependency is totally unknown. We show that a minimum variance controller stabilizes the system in an optimal way in the sense that the norm of the closed-loop system matrix is minimized. We give some bounds for this minimal norm which can be easily calculated. Furthermore we present a method to calculate the sensitivity of the minimal norm if the controller is based on a small perturbation of the real system. At last a necessary and sufficient condition is given for the existence of a controller which with less control efforts than the minimum variance controller obtains the same minimal closed-loop norm.

### I. Introduction

In both economics and industry there exist a lot of processes which are time dependent, and from which the exact time dependency is unknown. If the goal is to stabilize such processes a well-known approach is to model this time dependency and design a controller based on this modelation. An additional requirement to this controller is then mostly that it should be robust to modelation errors, i.e. the controlled system should remain stable under small perturbations. (For a nice exposition on several robustness questions and modelation of perturbations we refer to Kwakernaak [3].)

However, if the only aim of the control process is to stabilize the system, it suffices to design a controller such that the norm of the closed-loop matrix becomes at any time smaller than one.

In this paper we derive state feedback which results in the smallest possible spectral norm that can be obtained by the closed-loop matrix.

Once we have derived this optimal controller, we consider the implications on the norm of the closed-loop matrix if the controller is not based on the real process parameters but on parameters which deviate slightly from them. The obtained results make use of a paper from Tzafestas et al [5].

The optimal controller is in general not unique. Therefore the question arises whether it is possible to obtain the same minimal norm with less control efforts. One way to model this desired behaviour is by introducing a quadratic cost criterium in which control efforts are penalized. Since the optimal minimum norm controller also minimizes a quadratic cost criterium in which the cost of control plays no role, this approach seems to be the most natural one. A necessary and sufficient condition is given for the existence of a less expensive control policy.

Although there are many good computerpackages to calculate the minimal obtainable norm, it may be still usefull to have some estimates.

Therefore we provide some upper and lower bounds for this norm.

## II. A minimal norm realizing state feedback

Before we turn to the main point of this section we introduce first the system and some notation.

The system we consider in this paper is described by the following first order difference equation:

$$\Sigma: x(k+1) = A(k) x(k) + B(k) u(k); x(0) = x \quad .$$

Here  $x(k) \in \mathbb{R}^n$  is the state of the system and  $u(k) \in \mathbb{R}^n$  the applied control. We assume throughout that  $B(k)$  is full column rank for any  $k$ .

By  $\|A\|_E := \left( \sum_{i,j} a_{ij}^2 \right)^{1/2}$  we shall denote the Euclidean norm of matrix  $A$  and by

$\|A\|_2$  the spectral norm (also known as the operator- or 2-norm).

$\sigma'(A) := \sigma_1(A) \geq \dots \geq \sigma_n(A)$  will denote the singular values of matrix  $A$ , and  $r(A)$  the spectral radius of it (i.e. the absolute largest eigenvalue of  $A$ ).

$A^T$  at last denotes the transposed of matrix  $A$ .

We give two equivalent definitions of the spectral norm which are both used in this paper. A proof of this equivalence can e.g. be found in Basilevsky [1] pp. 240.

### Proposition 1

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_E}{\|x\|_E} = \sigma'(A) \quad . \quad [1]$$

In this section we derive a linear state feedback which results in a minimal spectral norm of the closed-loop matrix.

The motivation to study the spectral norm is due to:

Proposition 2

For any matrix norm we have that  $\|A\| > r(A)$ .

Moreover, if  $A$  is normal (i.e.  $AA^T = A^T A$ ) then  $\|A\|_2 = r(A)$ .

Proof:

See Lancaster et al [4] pp. 359. []

Now a priori we have the following lower bound for the spectral norm of a closed-loop matrix.

Proposition 3

$\min_F \{ \|A - F\|_2 \mid \text{rank}(F) = s \} = \sigma_{s+1}(A)$ .

Proof:

This theorem is in literature known as the Eckart-Young theorem.

The existence of a matrix  $F$  obtaining this minimal norm is rather easy and illustrative. Making a singular value decomposition of matrix  $A$ , i.e.

$A = U^T \Sigma V$  with  $\Sigma = \text{diag}(\sigma_1(A), \dots, \sigma_n(A))$ , it is clear that  $F = U^T \Lambda V$  with  $\Lambda = \text{diag}(\sigma_1(A), \dots, \sigma_s(A), 0, \dots, 0)$  yields the minimal norm. []

In words: the spectral norm of the closed-loop matrix,  $\|A + BF\|_2$ , is for any matrix  $F$  at least  $\sigma_{m+1}(A)$ .

So, in general it will not be possible to design for a system  $\Sigma$  a feedback controller which has the property that the norm of the closed loop matrix is smaller than one, even when the system is controllable.

The next lemma is a preparation for the main result of this section.

Lemma 1

Let  $z$  be a given vector.

Then  $\min_y \|z + B y\|_E$  is obtained by  $y = -(B^T B)^{-1} B^T z$ .

Proof:

See Lancaster et al [4] pp. 436. []

Corollary 1

$$\min_y \| z + B y \|_E = \min_{y=Fz} \| z + B y \|_E, \text{ where } F = -(B^T B)^{-1} B^T.$$

The main result reads now as follows.

Theorem 1

$$\min_F \| A + BF \|_2 = \| (I - B(B^T B)^{-1} B^T) A \|_2.$$

Moreover, a linear state feedback which obtains this minimum is  $F = -(B^T B)^{-1} B^T A$ .

Proof:

According to proposition 1 and corollary 1 we have that

$$\begin{aligned} \| A + BF \|_2 &= \max_{x \neq 0} \frac{\| (A + BF)x \|_E}{\| x \|_E} \\ &\geq \max_{x \neq 0} \frac{\| I - B(B^T B)^{-1} B^T) Ax \|_E}{\| x \|_E} . \end{aligned}$$

$$\begin{aligned} \text{So, } \min_F \| A + BF \|_2 &\geq \min_F \| (I - B(B^T B)^{-1} B^T) A \|_2 \\ &= \| (I - B(B^T B)^{-1} B^T) A \|_2 . \end{aligned}$$

From this last inequality the statements of the theorem are immediate. []

In the sequel we will denote the matrix  $I - B(B^T B)^{-1} B^T$  by  $M$ .

Since for practical applications it may be handsome to have some bounds, that can be easily calculated, for this minimal operator norm we conclude this section with a proposition about this subject. The proposition contains also some bounds which are for theoretical purposes of more interest.

Proposition 4

The following bounds are valid for the minimal obtainable norm of the closed-loop matrix.

- i)  $\| MA \|_2 \geq \sigma_{m+1}(A)$
- ii)  $\| MA \|_2 \geq \frac{\| MA \|_E}{(n-m)^{\frac{1}{2}}}$  if  $m < n - 1$

iii)  $\| MA \|_2 = \| MA \|_E$  if  $m = n - 1$

iv)  $\| MA \|_2 = 0$  if  $m = n$

v) Let  $MA = [a_1, \dots, a_n]$  and  $\epsilon > 0$ . If  $\| a_i \|_E \leq \epsilon$  then  $\| MA \|_2 \leq n^{1/2} \epsilon$

vi)  $\| MA \|_2 \leq \sigma'(A)$ .

Proof:

i) was already stated in proposition 3.

ii), iii) and iv) result from the equality  $\| MA \|_E^2 = \sum_{i=1}^n \sigma_i^2(MA)$ .

More explicitly ii) results since  $\sum_{i=1}^{n-m} \sigma_i^2(MA) \leq (n-m) \sigma_1^2(MA)$ .

Now, let  $x = \sum_{i=1}^n x_i e_i$ , where  $e_i$   $i = 1, \dots, n$  is the standard basis of  $\mathbb{R}^n$ .

Then inequality v) follows from:

$$\begin{aligned} \| MA x \|_E &= \| MA \left( \sum_{i=1}^n x_i e_i \right) \|_E \leq \sum_{i=1}^n |x_i| \| MA e_i \|_E \\ &= \sum_{i=1}^n |x_i| \| a_i \|_E \leq \epsilon \sum_{i=1}^n |x_i| \leq \epsilon n^{1/2} \| x \|_E . \end{aligned}$$

The last inequality vi) is due to the fact that  $\| M \|_2 \leq 1$ , which is shown by straightforward multiplication. []

### III. About the robustness of the controller

In the previous section we assumed that at any time  $k$  the exact values of the system parameters  $A(k)$  and  $B(k)$  are known.

We shall drop now this assumption, and investigate what the consequences will be for the obtained closed-loop norm if the feedback matrix  $(B^T B)^{-1} B^T A$  is based on small disturbed model parameters.

So, we are investigating the effect of  $d(MA)$  on  $\| MA + d(MA) \|_2$  w.r.t.  $\| MA \|_2$ , if  $d(MA)$  is a small perturbation.

With a small perturbation is meant that products of differentials (perturbations) can be neglected.

Now according to proposition 1

$$\begin{aligned} \| MA + d(MA) \|_2^2 &= \sigma^2(MA + d(MA)) \\ &= r( (MA + d(MA))(MA + d(MA))^T ) \\ &\approx r( MAA^T M + MAd^T(MA) + d(MA)A^T M ) \end{aligned} \quad (1)$$

With the last wiggle equality sign,  $\approx$ , is meant almost equal to. This wiggle sign is based on the next fundamental result.

Proposition 5

The eigenvalues of a matrix depend continuously on the matrix parameters.

Proof:

Due to the fact that the determinant of  $A - sI$  is a continuous function of its parameters. []

Using now a result developed by Tzafestas et all in [5] we can analyze the first order disturbance effects on the norm. Their result reads as follows.

Theorem 2

Let  $\lambda(A)$  denote an eigenvalue of matrix  $A$ , and  $d\lambda$  the eigenvalue differential. Then, if the multiplicity of  $\lambda(A)$  is  $p$ ,  $d\lambda$  is given by:

$$\left[ \frac{1}{p!} [d^{p-1} \text{tr adj}(sI - A)]_\lambda \right]^{-1} \left[ \sum_{k=1}^p \frac{1}{k!} [d^{k-1} \text{adj}(sI - A)]_\lambda * d(A) \right]. \quad (2)$$

Here  $d^k$  means the  $k$ -th differential,  $\text{tr } A$  the trace of matrix  $A$ ,  $\text{adj } A$  its adjoint and  $C*D$  the inner product of two equidimensional square matrices, i.e.,  $C*D = \sum_i c_i d_i$  where  $c_i$  is the  $i$ -th row of  $C$  and  $d_i$  the  $i$ -th column of  $D$ .

$[ ]_\lambda$  denotes that the evaluation is to be done at  $\lambda$ . []



So, to determine the eigenvalue differential one has to solve an algebraic equation of degree  $p$ , where  $p$  is the multiplicity of the eigenvalue in question.

Applying this result to (1) yields:

Corrolary 2

Assume that the multiplicity of  $\sigma'^2(MA) = p$ .

Then  $d\sigma'(MA)$  is obtained as the square root of (2) in which  $A$  is substituted by  $MAA^T M$ ,  $d(A)$  by  $MA d^T(MA) + d(MA)A^T M$  and the evaluation takes place at  $\lambda = \sigma'^2(MA)$ .

In applications one can use this result to calculate the sensitivity of

$\| \overline{(MA)} \|_2 := \| I - \overline{B}(\overline{B}^T \overline{B})^{-1} \overline{B}^T \overline{A} \|_2$  to get insight into the question whether the norm of the really obtained closed-loop matrix, i.e.  $\| A - B(\overline{B}^T \overline{B})^{-1} \overline{B}^T A \|_2$ , will be smaller than one. Here a bar above a parameter indicates that the parameter has been estimated.

IV. Reduction of control efforts

In theorem 1 a controller was derived which always gives rise to a minimal spectral norm of the closed-loop matrix. The feedback gain was given by

$$-(B^T B)^{-1} B^T A.$$

However, this controller is also obtained when we take  $Q = I$  in the following minimum variance optimization problem.

Proposition 6

Consider  $J = x^T(k) Q x(k)$ , with  $Q > 0$ .

Then

$$\min_u J, \text{ subject to } \Sigma,$$

is obtained by  $u(k) = -(B^T(k)QB(k))^{-1} B^T(k)QA(k) x(k)$ .

Proof:

See e.g. Engwerda [2].

With  $Q > 0$  we mean that  $Q$  is a symmetric positive definite matrix. Instead of considering the cost functional  $J$  we will consider in the sequel the cost functional

$$J' = x^T(k) x(k) + u^T(k) R u(k), \text{ with } R > 0 .$$

The effect of minimizing  $J'$  instead of  $J$  w.r.t.  $\Sigma$  is that the amount of applied control is less the more positive matrix  $R$  is.

Therefore the question arises under which conditions there will exist a feedback, minimizing  $J'$ , which results in the minimal closed-loop norm  $\sigma'(M(k)A(k))$ .

Before we take a start in answering this question we give first the feedback which minimizes  $J'$ . Moreover, we shall provide two examples.

One example shows that it is sometimes possible to find a smoother control than  $-(B^T B)^{-1} B^T A$ , the other example deals with a situation in which this is not possible.

Proposition 7

$\min_u J'$  w.r.t.  $\Sigma$  is obtained by  $u(k) = -(R + B^T(k)B(k))^{-1} B^T(k) A(k) x(k)$ .

Proof:

See Engwerda [2].

The closed-loop matrix which results if the optimal control is applied is in the sequel abbreviated by  $M'(k)A(k)$ .

Example 1:

Take  $A = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$ ;  $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ;  $R = 1$ .

Then  $\|MA\|_2 = \|M'A\|_2 = \|A\|_2 = 3$ .

Example 2:

Take  $m = n$  and  $A \neq 0$ .

Then  $\| MA \|_2 = 0$  (see prop. 4iv) and  $\| M'A \|_2 \neq 0$  for any  $R > 0$ .

The following two lemmas are a preparation for the next theorem, which contains the main result of this section.

Lemma 2

Let  $D := \begin{pmatrix} E & | & F \\ \hline G & | & H \end{pmatrix} \geq 0$ , with  $E$  and  $H$  square matrices.

Assume that  $\max(\| E \|_2^2, \| H \|_2^2) = \| H \|_2^2 = \sigma^2$ .

Let  $\mathcal{H}$  be the eigenspace of  $H$  corresponding to  $\sigma$ .

Then  $\exists x \in \mathcal{H}$  such that  $x^T G \neq 0 \Rightarrow \| D \|_2^2 > \sigma^2$ .

Proof:

If  $\sigma = 0$ , the statement is trivial. Therefore assume  $\sigma \neq 0$ .

Let  $\bar{y}$  be an eigenvector from  $H$  such that  $\bar{y}^T G = 0$ . Take  $\bar{x}^T = \frac{1}{\sigma} \bar{y}^T G$ .

Then

$$\begin{aligned} \| D \|_2 &= \max_{(x,y) \neq 0} \frac{x^T E x + 2y^T G x + y^T H y}{\| x \|_E^2 + \| y \|_E^2} \\ &\geq \frac{\bar{x}^T E \bar{x} + 2\bar{y}^T G \bar{x} + \bar{y}^T H \bar{y}}{\| \bar{x} \|_E^2 + \| \bar{y} \|_E^2} \\ &= \frac{\frac{1}{\sigma^2} \bar{y}^T G E G^T \bar{y} + \frac{2}{\sigma} \bar{y}^T G G^T \bar{y} + \sigma \bar{y}^T \bar{y}}{\frac{1}{\sigma^2} \bar{y}^T G G^T \bar{y} + \bar{y}^T \bar{y}} \end{aligned}$$

Straightforward calculation shows that the last expression is greater than  $\sigma$  iff.  $\frac{1}{\sigma} \bar{y}^T G G^T \bar{y} + \frac{1}{\sigma^2} \bar{y}^T G E G^T \bar{y} > 0$ .

Since  $\bar{y}^T G \neq 0$  and  $E \geq 0$  the last inequality is satisfied, which completes the proof of the lemma. []

Lemma 3

Let  $D = \left( \begin{array}{c|c} E & F \\ \hline G & H \end{array} \right) \geq 0$  and  $\lambda$  an eigenvalue of  $H$  with multiplicity  $p$ .

Assume that  $H$  is the eigenspace of  $H$  corresponding to  $\lambda$ .

Then, if  $\forall x \in H \ x^T G = 0$ ,  $\lambda$  is also an eigenvalue of  $D$  with multiplicity greater or equal than  $p$ .

Proof:

Follows immediately from the establishment that if  $x$  is an eigenvalue of  $H$  that goes with  $\lambda$ , then  $\begin{pmatrix} 0 \\ x \end{pmatrix}$  is an eigenvalue of  $D$  that goes with  $\lambda$ . []

For the formulation of the theorem it is convenient to chose another orthonormal basis in  $\mathbb{R}^n$ . From now on we assume that  $\mathbb{R}^n$  is decomposed as  $\text{Im } B(k) \oplus X(k)$ .

With respect to this basis, matrix  $B^T(k)$  equals  $( B'^T(k) \mid 0 )$  where  $B'$  is a square invertible matrix, Furthermore, we write

$$A(k) = \left( \begin{array}{c|c} E(k) & F(k) \\ \hline G(k) & H(k) \end{array} \right), \text{ where } E(k) \in \mathbb{R}^{m \times m} \text{ and } H(k) \in \mathbb{R}^{(n-m) \times (n-m)}. \text{ Note that}$$

due to this choice of the basis, the costcriterium does not change.

The time index will be submitted again in the sequel.

The theorem reads as follows.

Theorem 3

Let  $\| MA \|_2 = \sigma^{\frac{1}{2}}$ , and  $H$  the eigenspace of  $MAA^T M$  belonging to  $\sigma$ .

Then, if  $\sigma \neq 0$

$$\exists R > 0: \| M'A \|_2 = \sigma^{\frac{1}{2}} \Leftrightarrow H(GE^T + HF^T) = 0, \text{ and}$$

if  $\sigma = 0$

$$\exists R > 0: \| M'A \|_2 = 0 \Leftrightarrow \text{i) } H(GE^T + HF^T) = 0$$

$$\text{ii) } EE^T + FF^T = 0 .$$

Proof:

" $\Rightarrow$ " Straightforward calculation shows that

$$\begin{aligned} \|M'A\|_2^2 &= \left\| \left( \begin{array}{c|c} (I + B^T R^{-1} B')^{-1} E & (I + B^T R^{-1} B')^{-1} E \\ \hline G & H \end{array} \right) \right\|_2^2 \\ &= r \left( \begin{array}{c|c} (I + B^T R^{-1} B')^{-1} (EE^T + FF^T) (I + B^T R^{-1} B')^{-1} & (I + B^T R^{-1} B')^{-1} (EH^T + FH^T) \\ \hline (GE^T + HF^T) (I + B^T R^{-1} B')^{-1} & GG^T + HH^T \end{array} \right) \end{aligned}$$

and

$$\|MA\|_2^2 = r \left( \begin{array}{c|c} 0 & 0 \\ \hline 0 & GG^T + HH^T \end{array} \right).$$

Applying now lemma 2 yields that  $H(GE^T + FF^T) = 0$  is a necessary condition that must be satisfied if  $\|MA\|_2^2 = \|M'A\|_2^2$ .

Moreover, if  $\sigma = 0$  also necessarily  $EE^T + FF^T$  must be zero.

" $\Leftarrow$ " First, we consider the case  $\sigma \neq 0$ .

Then the spectrum of  $MA$  equals  $\{0\} \cup \text{spectrum}(GG^T + HH^T)$ .

Since the spectrum of a matrix depends continuously on its parameters (prop. 5), the spectrum of the following matrix will be about the same provided  $S$ ,  $T$  and  $U$  are small enough:

$$\begin{pmatrix} S & T \\ U & GG^T + HH^T \end{pmatrix}.$$

Now choose  $R = e I$ , with  $e$  small enough, in  $M'AA^T M'$ .

Applying lemma 3 we see that the spectrum of this matrix will then consist of  $\{\sigma\}$ , small disturbances of  $\{0\}$  and small disturbances of the other (smaller!) eigenvalues of  $M'AA^T M'$ .

Therefore  $r(M'AA^T M) = \sigma$ , what had to be proved.

Finally we consider the case  $\sigma = 0$ .

In this case both  $EE^T + FF^T$  and  $GG^T + HH^T$  are zero.

Together with the first condition in the theorem it is clear that now any  $R > 0$  will give rise to a closed-loop matrix with zero norm. []

Note that the theorem only gives an existence result.

An interesting question which remains to be solved is of course how in practice the largest, if it exists,  $R$  matrix can be determined.

References:

- 1 Basilevsky A.: Applied Matrix Algebra in the Statistical Sciences; North Holland, 1983, New York.
- 2 Engwerda J.C.: On the set of obtainable reference trajectories using minimum variance control; submitted for publication.
- 3 Kwakernaak J.: Uncertainty models and the design of robust control systems; in Uncertainty and Control, Proceedings of an International Seminar Organized by DFVLR, pp. 84-107; Springer-Verlag, 1985, Berlin.
- 4 Lancaster P. and Tismenetsky M.: The Theory of Matrices, second edition; Academic Press, 1985, London.
- 5 Paraskevopoulos P.N., Tsonis C.A. and Tzafestas S.G.: Eigenvalue sensitivity of linear time-invariant control systems with repeated eigenvalues; I.E.E.E. Transactions on Automatic Control, 1974, pp. 610-612.