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The capacitated multi-echelon inventory
system with serial structure:
1. The 'push ahead'-effect

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THE CAPACITATED MULTI-ECHELON INVENTORY SYSTEM WITH SERIAL STRUCTURE :
1. THE 'PUSH AHEAD'-EFFECT

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This paper considers a multi-echelon, periodic review inventory model with discrete demand. We assume finite capacities on various production/order sizes and backordering of excess demand. We show that under the average cost criterion the optimal order strategy may be characterized by a so-called 'push ahead'-effect. Further we shall find that a modified base-stock policy approximates the optimal policy quite well.

1. Introduction. In this paper we consider a basic production (or inventory) model, in which the stock of a single item must be controlled under periodic review. We assume demands in each period to be independent, nonnegative, and integer-valued. Further, all stockouts are backordered and production, holding, and shortage costs are linear. There are no fixed order costs. Handling an infinite planning horizon we operate the long-run-average cost criterion.

In the past few decades several excellent results have been achieved concerning production/inventory systems. Assuming infinite production capacity Langenhoff and Zijm[1990] and Van Houtum and Zijm[1991] e.g. have shown in a Clark and Scarf[1960]-like approach that a so-called base-stock, or critical-number, policy is the optimal production strategy under the average cost criterion for a multi-echelon inventory system with serial or assembly structure: If the echelon stock has dropped below a certain level, enough should be produced to raise total stock to that level; otherwise, nothing should be produced.

Addition of finite production capacities to these models gives rise to complications: In the uncapacitated case large demands in some periods can be corrected immediately in the next period; now a buildup of backorders is possible due to the finite production capacity. Another issue induced by the limited production sizes is the state-dependency on formerly taken

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decisions. Nevertheless, Federgruen and Zipkin have established a modified base-stock policy to be the optimal policy under the average cost criterion in a discrete demand model ([2]), as well under the discounted cost criterion in a continuous demand model ([3]) for the capacitated N-echelon inventory system with $N = 1$.

Exact calculation of the average cost respectively discounted cost associated with the optimal modified base-stock policy happens to be a very difficult issue. But already a few approximation methods have been developed, that successfully cope with the incomplete convolutions arising from the cost calculation, such as an application of the moment-iteration method of De Kok[1989], and a related method of Zijm (unpublished manuscript). Both methods however are as yet only applicable to the capacitated 1-echelon serial system. The development of an average cost approximation method for the N-echelon serial system with $N \geq 2$ will be presented in a companion paper (Speck and Van der Wal[1991]).

In this paper the capacitated N-echelon production/inventory system with serial structure and $N \geq 2$ is regarded, see figure.

We had in mind to demonstrate the Federgruen and Zipkin result still to be valid for the N-echelon serial system with $N \geq 2$. But slowly we got the conviction that a combination of former executed replenishments yet to arrive, a certain demand course, and finite ordering/production capacities might induce a so-called 'push ahead'-effect to be observable within the optimal periodic review policy. That is, it might be profitable in certain circumstances to ship more items than prescribed by a modified base-stock policy in order to prevent being restricted next period. We will implement successive approximations in order to give numerical evidence to our conjecture.

In section 2 we introduce notation, definitions, and required assumptions, by means of which in section 3 the central conjecture of this paper will be demonstrated on the basis of the capacitated 2-echelon serial system. In section 4 finally this conjecture will be confirmed by a numerical proof.

2. Definitions and assumptions. We will restrict ourselves to the capacitated 2-echelon serial system. The results however can be extended

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straightforwardly to the capacitated N-echelon problem with $N > 2$.

First of all we shall give some definitions:

1. We define the *echelon stock* of a given installation as the stock at that installation plus all the stock in transit to or on hand at any installation downstream minus the backlogs at the most downstream installation.

2. Next, the *echelon inventory position* of an installation denotes the echelon stock plus all items heading for that installation, that already left the preceding installation (or the external supplier).
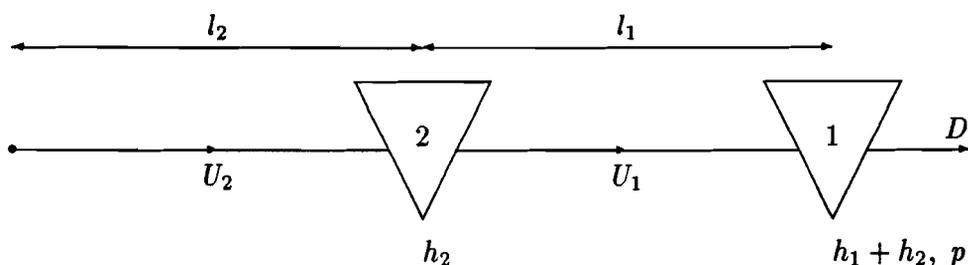
Let us consider the model for the capacitated 2-echelon serial system as depicted above, in which

U_i = maximal production/order size, $i = 1, 2$.
l_1 = leadtime of the route from installation 2 to installation 1.
l_2 = leadtime of the route from external supplier to installation 2.
p = shortage cost per unit per period at echelon 1.
h_i = additional holding cost per unit per period at echelon i, $i = 1, 2$.
x_i = stock at echelon i at the beginning of a period, $i = 1, 2$.
y_i = inventory position at echelon i at the beginning of a period, $i = 1, 2$.
D_t = demand in period t, $t \in I\!N$.
$q(w)$ = $I\!P\{D_t = w\}$, $w = 0, 1, \ldots$, the demand probabilities.
$F(u)$ = $\sum_{w=0}^{u} q(w)$, $u \in I\!N$, the demand distribution.

The leadtimes l_i are deterministic and equal to an integer. The capacities U_i are assumed to be finite for $i = 1, 2$. Furthermore, x_i and y_i are always integer-valued and may be negative since stockouts are backordered.

In every period the following actions take place: At the beginning of the period ordered items arrive at installations and after inspecting each echelon inventory position every installation places an order with the preceding

installation or external supplier; next the external demand in that period is met, followed by a cost determination at the end of the period.

The expected costs at the end of a period consist of linear holding and shortage costs. By the way, linear production/ordering costs are not taken into account; they can be disregarded because of the average cost criterion. Hence, the expected costs at the end of a period associated with an echelon stock x_i, $i = 1, 2$, at the beginning of that period are described by the well-known *Newsboy*-formulas, here displayed in the discrete form:
For $x_i \in \mathbb{Z}$, $i = 1, 2$,

$$L_1(x_1) := \sum_{w=0}^{\infty} q(w)[(h_1 + h_2)(x_1 - w)^+ + p\,(x_1 - w)^-] - h_2 x_1 \quad (1)$$

$$L_2(x_2) := h_2 x_2. \quad (2)$$

Notice the shifting of the term $h_2 x_1$ from the expected cost associated with the stock of echelon 2 to the expected cost associated with the stock of echelon 1, thus creating L_2 to be independent of x_1.

3. Analysis. A first observation we make is that, while searching for an optimal periodic review policy under the average cost criterion, the optimal replenishment amount for the inventory position of echelon 2 will not exceed the finite capacity U_2 (rather trivial), neither the finite capacity U_1! Clearly, when a replenishment larger than U_1 on the echelon inventory position 2 is carried out at the beginning of a period t, then this amount arrives at installation 2 at the beginning of period $t + l_2$. Since at most U_1 can be shipped per period from installation 2 to installation 1, that replenishment larger than U_1 definitely implies a positive stock at installation 2 during period $t + l_2$, and thus extra holding costs.
We have thus stated a necessary optimality condition for the optimal policy:

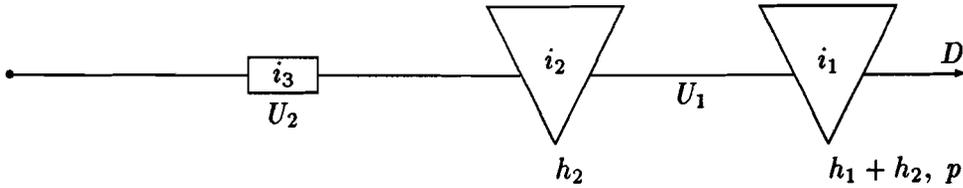> The optimal replenishments of echelon inventory position 2 are restricted to the upper bound $\min(U_2, U_1)$.

Therefore, suppose without loss of generality

$$0 < U_2 \le U_1. \quad (3)$$

Now we are about to advance our central conjecture which we illustrate on the basis of the capacitated 2-echelon serial system in which $l_2 = 2$ and $l_1 \approx 0$. We characterize states as the possible system states at the beginning

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of a period before meeting external demand: Let i_3 denote the shipment that arrives next period at installation 2, i_2 the fysical stock at installation 2, and i_1 the echelon stock of installation 1 (which of course may be negative).

At the beginning of every period two decisions have to be taken concerning the replenishments of both echelon inventory positions. The first one, denoted by k_1, regards the shipment from installation 2 to installation 1, whereas the second one, indicated by k_2, is the amount to be shipped from the external supplier to installation 2.

A possible conjecture is that a modified base-stock policy is optimal for the capacitated N-echelon serial system with $N \geq 2$ as well. That means that the replenishments of both echelon inventory positions depend on an aggregated state description: Decision k_1 would depend on i_1 only, and decision k_2 would depend on the sum of i_1, i_2, and i_3 only.

Due to a combination of the already known shipment i_3, demand fluctuations, and the finite capacity U_1 however, decision k_1 might as well be influenced by the shipment i_3 arriving at installation 2 next period. So another conjecture would be that a so-called 'push ahead'-effect might be perceptible within the optimal periodic review policy. Let us illustrate this by an example.

Example

Suppose demand per period varies from 17 to 23 items, while the holding and shortage costs are such that the ideal inventory level of echelon 1 (or installation 1 in this case) equals 20. Now assume the system is in state (i_3, i_2, i_1)=(20,20,1). Further $U_2 = U_1 = 20$.
Suppose we would act according to a base-stock policy and therefore ship 19 items from installation 2 to installation 1. Then at the beginning of the next period 21 items are in stock at installation 2: The one left plus the 20 arriving next period. If during that period at most 20 items are sold, then there is no problem. But when more than 20 items are sold we are handi-

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capped in the next period by U_1: Only 20 items can be shipped, while at least 21 items are needed in this case. If we had shipped 20 items instead of those 19 we would have achieved a better situation by having the possibility of shipping all 20.

Comparing this situation with the situation in which only i_3 has a different value (i_3=19) the latter cannot lead to any difficulty coming from the finite capacity U_1.

4. A numerical proof. In order to confirm the possible appearance of a 'push ahead'-effect we have implemented a successive approximation method tailored to the considered capacitated 2-echelon model. To that end we have to arrange by truncation a finite state space $\mathcal{S}$ of triples $\underline{i} = (i_3, i_2, i_1)$, a set $\mathcal{A}(\underline{i})$ of actions $\underline{k} = (k_2, k_1)$ per state $\underline{i} \in \mathcal{S}$, a collection of feasible transitions per state, and the expected costs attached to an action in a state.

We suffice by presenting the algorithm we applied (for further information concerning successive approximation, see e.g. Heyman and Sobel[1984]).

Algorithm 1 (Successive Aproximation)

Let there be given a finite state space $\mathcal{S}$ and an action set $\mathcal{A}(\underline{i})$ for every state $\underline{i} \in \mathcal{S}$.

Initialisation: *Choose v_0.*

Iteration: *Determine for $n = 0, 1, \ldots$ and for any $\underline{i} \in \mathcal{S}$ an action $\underline{k} \in \mathcal{A}(\underline{i})$ and a value $v_{n+1}(\underline{i})$ such that*[1]

$$v_{n+1}(\underline{i}) = \min_{\underline{k} \in \mathcal{A}(\underline{i})} \left\{ r(\underline{i}, \underline{k}) + \sum_{w=0}^{D_{max}} q(w) \, v_n(k_2, i_3 + i_2 - k_1, i_1 + k_1 - w) \right\}$$

Termination: *Stop as soon as*

$$\max_{\underline{i} \in \mathcal{S}} (v_{n+1} - v_n)(\underline{i}) - \min_{\underline{i} \in \mathcal{S}} (v_{n+1} - v_n)(\underline{i}) \leq \varepsilon_{sp}$$

for some prespecified $\varepsilon_{sp} > 0$.

$\square$

In every iteration of the successive approximation algorithm a lower bound

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[1] Possibly invoking feasibility measures on the edges.

$\tilde{g}_{low}^*$ and an upper bound $\tilde{g}_{up}^*$ are computed for the average cost $\tilde{g}^*$ going with the generated policy; in its turn $\tilde{g}^*$ serves as an upper bound for the minimum average cost g^*:

$$\tilde{g}_{low}^* \le g^* \le \tilde{g}^* \le \tilde{g}_{up}^*$$

in which

$$\tilde{g}_{up}^* = \max_{\underline{i} \in \mathcal{S}} (v_{n+1} - v_n)(\underline{i})$$
$$\tilde{g}_{low}^* = \min_{\underline{i} \in \mathcal{S}} (v_{n+1} - v_n)(\underline{i}).$$

In order to keep the instance manageable we have chosen the following model parameters:

$$U_1 = 2$$
$$U_2 = 2$$
$$l_1 \approx 0$$
$$l_2 = 2$$
$$\varepsilon_{sp} = 10^{-4}$$
$$D_{max} = 4,$$

where ε_{sp} indicates the maximum length of the calculated interval after termination, and D_{max} specifies the largest possible demand per period.

We have already characterized the states $\underline{i} = (i_3, i_2, i_1) \in \mathbb{Z}^3$. Keeping in mind the influence of the number of states on the calculation time and taking into account the values of the model parameters we take as finite state space

$$\mathcal{S} = \{ \underline{i} \in \mathbb{Z}^3 \mid 0 \le i_3 \le U_2 ,\ 0 \le i_2 \le 6 ,\ -10 \le i_3 \le 10 \}. \tag{4}$$

Next, the set of allowed decisions per state $\underline{i} \in \mathcal{S}$ is given by

$$\mathcal{A}(\underline{i}) = \{ \underline{k} \in \mathbb{N}^2 \mid\ 0 \le k_2 \le \min(U_2, U_1),$$
$$\max(0, i_3 + i_2 - 6) \le k_1 \le \min(U_1, i_2) \},$$

in which we necessarily imposed an artificial lower bound on k_1 to avoid finding the system itself outside its state space $\mathcal{S}$. This manipulation is accompanied by a penalty cost; one could think of destroying stock, which is expensive. Notice also the modified upperbound on k_2, coming from our preliminary result.

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Further, to avoid pathological situations we require D_{max} and the probabilities $q(0),\ldots,q(D_{max})$ to be chosen such that $D_{max} > \min(U_2, U_1)$, but $E\{D\} < \min(U_2, U_1)$.

State transitions are evident: Sitting in (i_3, i_2, i_1) and taking decision (k_2, k_1) in a period leads to state $(k_2, i_3 + i_2 - k_1, i_1 + k_1 - w)$ with probability $q(w)$ in the next period. Again we have to truncate the action set, but only for the extreme states, i.e. with i_1 close to -10: By eliminating absurd actions and/or absorbing probability mass we prevent transitions to states outside $\mathcal{S}$. In this case one could think of invoking emergency measures to quickly correct the backlog situation which again involves penalty costs. These inevitable truncations are made in such a way that they are of negligible influence on the optimal policy generating process. This has been checked by moving the range of i_1 in (4).

Finally, the expected costs associated with state $\underline{i} \in \mathcal{S}$ and action $\underline{k} \in \mathcal{A}(\underline{i})$ are given by

$$r(\underline{i}, \underline{k}) = L_2(i_2 + i_1) + L_1(i_1 + k_1).$$

In order to prove the existence of a 'push ahead'-effect we have to combine values concerning the parameters h_1, h_2, p, and $q(0),\ldots,q(D_{max})$. Most combinations we tried yielded problems with a modified base-stock policy being optimal. But we succeeded in constructing a few optimal periodic review policies exhibiting the 'push ahead'-effect. One of them is displayed in table 1. We see that as long as the total amount of items head-

(p_0, p_1, p_2, p_3, p_4) = (.1, .6, .2, 0, .1)			
(h_1, h_2, p) = (1, 1, 1)			
states	generated policy	$\tilde{g}^*_{low}$	$\tilde{g}^*_{up}$
$i_3 + i_2 < 3$	(4,1)	1.95358	1.95367
$i_3 + i_2 \geq 3$	(4,2)		

Table 1: Generated optimal policy

ing for and on hand at installation 2 does not exceed U_1, the optimal policy is formed by the modified base-stock policy (4,1). That is, at the beginning

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of each period the inventory position of echelon 2 and 1 has to be raised to 4 respectively 1, if possible. But as soon as $i_3 + i_2$ exceeds U_1 the optimal replenishments are according to the modified base-stock policy (4,2). It appears that whenever $i_3 + i_2 \geq 3$ and for instance $i_1 = 0$ it is profitable to exploit the possibility of transshipping 2 from installation 2 to installation 1, and thus creating a fysical stock of 2 items in this case instead of 3 (when handling (4,1)) at installation 2 at the beginning of next period. In other words, we obtained an optimal policy containing a 'push ahead'-effect.

Next we determined the average cost interval belonging to nearby modified base-stock policies, see table 2. Now, since all of these intervals do not

nearby modified base-stock policies				
(d_2, d_1)	$\tilde{g}_{low}^{(d_2, d_1)}$	$\tilde{g}_{up}^{(d_2, d_1)}$	$\tilde{g}_{low}^{*}$	$\tilde{g}_{up}^{*}$
(4,1)	1.96089	1.96099	1.95358	1.95367
(4,2)	1.98657	1.98665		

Table 2: Comparison to base-stock policies

intersect $[\tilde{g}_{low}^{*}, \tilde{g}_{up}^{*}]$ we numerically proved a modified base-stock policy not to be the optimal one.

5. **Conclusions** In this case, and also in many other parameter combinations we examined, the optimal periodic review policy is not a modified base-stock policy. In all cases we found a nearby modified base-stock policy which was nearly optimal. That is, the difference in average cost was marginal (in the case $l_1 = 1$ we even could not numerically prove a modified base-stock policy to be inoptimal). Therefore, we are interested in a method for approximating the average cost associated with a modified base-stock policy. Then an optimal modified base-stock policy can be obtained, which in its turn serves as an approximation of the optimal periodic review policy. This will be the topic of a forthcoming paper.

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