Probabilistic methods in problems of applied analysis

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1 Introduction and summary

The use of probability methods in analysis is not new. One of the most elegant examples is Bernsteins's proof of Weierstrass' theorem (Section 3); one of the more ill-famed the suspiciously accurate approximation of π by Lazzerini using Buffon's needle (cf. Kendall and Moran (1962)). For more recent applications we refer to Byrnes e.a. (1992).
In this paper we present some, mostly rather recent, situations where methods from probability theory have been used to prove results in non-stochastic problems of (applied) analysis. Sometimes a problem in analysis can be put in a probabilistic context and then solved by standard probability methods, sometimes the solution depends on analytic properties that are well-known in probability theory but not in analysis.
Here I shall give applications from three areas of probability that I have been involved with myself; brief applications of the law of large numbers and of infinite divisibility, and a more extensive application involving Poisson processes and the central limit theorem. The three areas of application are approximation theory, deconvolution of positive functions, and a description and solution of a problem concerning of the breakdown of insulator gases by ionization.
In Section 2 we briefly review some probability concepts, in Sections 3, 4 en 5 we discuss the three applications listed above.

2 Elements of probability theory

We just list some basic concepts and properties from probability theory; also to establish notations. We shall refer to the properties by the letters preceding them. For more detailed information we refer to Loève (1977).

(a) *Probability space*: normed neasure space $(\Omega, \mathcal{F}, P)$ with $P(\Omega) = 1$; the elements of $\mathcal{F}$ are called *events*, for $A \in \mathcal{F}$, $P(A)$ is the *probability* of A.

(b) *Random variable*: measurable function $X : \Omega \to I\!R$.

(c) *Distribution* of X: probability measure P_X defined on the Borel sets $\mathcal{B}(I\!R)$ by $P_X(B) = P(X^{-1}(B)) =" P(X \in B)"$; *distribution function* of X: $F_X(x) = P(X \leq x)$.

(d) *Random vector* $Y = (X_1, ..., X_n)$: vector of random variables; distribution of Y: $P_Y(B) = P(Y^{-1}(B)) =" P(Y \in B)"$ $(B \in \mathcal{B}(I\!R^n))$.

(e) $X_1, ..., X_n$ are independent if $P_Y = P_{X_1} \times ... \times P_{X_n}$ (product measure).

(f) *Expectation* of $g(Y)$: $Eg(Y) = \int\limits_{\Omega} g(y(w))dP = \int\limits_{\mathbb{R}^n} g(y)dP_Y$.

(g) *Variance* of X: var $X = EX^2 - (EX)^2$; if X_1 and X_2 are independent, then var $(X_1 + X_2) =$ var $X_1 +$ var X_2.

(h) If $X_1, ..., X_n$ are independent with $P(X_j = 1) = 1 - P(X_j = 0) = p$, then $P(X_1 + ... + X_n = k) = \binom{n}{k} p^k(1-p)^{n-k}$ for $k = 0, 1, ..., n$, a *binomial distribution.*

(i) X has an *exponential distribution* with $EX = \lambda^{-1}$ iff $F'_X(x) = \lambda e^{-\lambda x}$ $(x > 0)$.

(j) N has a *Poisson distribution* with $EN =$ var $N = \lambda$, iff $P(N = n) = e^{-\lambda}\lambda^n/n!$ $(n = 0, 1, 2, ..)$.

(k) If $X_1, X_2, ...$ are independent and exponentially distributed with $EX_j = \lambda^{-1}$, then $S_n := X_1 + ... + X_n$ has distribution function $F_n(x)$ with $F'_n(x) = \lambda^n x^{n-1} e^{-\lambda x}/(n-1)!$

(l) With $X_1, X_2, ...$ as above, the process $N(t)-1$ with $N(t) = \min\{n \in \mathbb{N}; S_n > t\}$, $t \geq 0$, is called a *Poisson-process*; $N(t)-1$ has a Poisson distribution with $EN(t)-1 =$ var $N(t) = \lambda t$.

(m) *Weak law of large numbers*: If $X_1, X_2, ...$ are independent with $EX_j = \mu$ and var $X_j = \sigma^2$, then $P\left(\left|\frac{X_1 + ... + X_n}{n} - \mu\right| < \varepsilon\right) \geq 1 - \frac{\sigma^2}{n\varepsilon^2}$.

(n) *Central limit theorem*: with $X_1, X_2, ...$ as above:

$$\lim_{n\to\infty} P\left(\frac{X_1 + ... + X_n - n\mu}{\sigma\sqrt{n}} \leq a\right) = \Phi(a) := \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{a} e^{-\frac{1}{2}x^2} dx \, .$$

(o) $P\left(\frac{N(t) - 1 - \lambda t}{\sqrt{\lambda t}} \leq a\right) \to \Phi(a)$ as $n \to \infty$.

(p) X is called infinitely divisible if for every $n \in \mathbb{N}$ independent random variables $X_{n,1}, ..., X_{n,n}$ exist with the same distribution and such that $P_X = P_{X_{n,1}+...+X_{n,n}} \equiv P^{*n}_{X_{n,1}}$ (n-fold convolution), or in terms of Fourier-transforms: $\widehat{P}_X = (\widehat{P}_{X_{n,1}})^n$.

(q) If K is integer valued and non-negative, and such that the sequence $P(K = k), k = 0, 1, ...,$ is log-convex, then K is infinitely divisible.

3 Approximating continuous functions

We briefly look at Bernstein's proof of the Weierstrass theorem, and of a variant; there are many others.

If $X_1, X_2, ...$ satisfy the conditions of (m), and if f is a continuous function its domain, then from (m) it easily follows that we have

$$Ef\left(\frac{X_1 + ... + X_n}{n}\right) \to f(\mu) \quad (n \to \infty) ,$$

uniformly on finite intervals.
As a special case, take X_j such that $P(X_j = 1) = 1 - P(X_j = 0) = x$; then $X_1 + ... + X_n$ has a binomial distribution (cf. (h)), and we obtain the well-known Bernstein polynomials:

$$B_n(f;x) = Ef\left(\frac{X_1 + ... + X_n}{n}\right) = \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} f\left(\frac{k}{n}\right) \to f(\mu) ,$$

as $n \to \infty$, uniformly on $[0,1]$; see also Schurer and Steutel (1977).

Similarly, using the fact that sums of independent Poisson random variables are Poisson distributed, we have using (i):

$$\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \to f(x) ,$$

as $n \to \infty$, uniformly on finite intervals.
For other example see e.g. Hahn (1980).

4 Two applications of infinite divisibility

Here we briefly consider two applications of infinite divisibility (see (p) and (q)). Very simply, a random variable is infinitely divisible if for every $n \in \mathbb{N}$ its distribution can be written as the n-fold convolution of another distribution, or if its Fourier-transform $\hat{f}$ has the property that $(\hat{f})^{1/n}$ is the transform of a probability measure for every $n \in \mathbb{N}$; similarly for related types of transforms. In probability theory these probability measures have been studied extensively. For more information we refer to Bondesson (1992) and Steutel (1979).

4.1 Robertson's conjecture

Robertson (1978) conjectured that all coefficients $d_{n,j}$ in

$$\left\{\left(\frac{1+z}{1-z}\right)^x - 1\right\}^{1/2} (2xz)^{-1/2} = \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} d_{n,j}x^j$$

would be nonnegative. It is easily verified that

$$(4.1) \qquad d_{n,j}\sqrt{2} = c_j r_{n-j}(j + \tfrac{1}{2}) ,$$

where c_j is the coefficient of u^j in $((e^u - 1)/u)^{1/2}$, and $r_m(p)$ is the coefficient of z^n in $\{2\sum_{n=0}^{\infty} z^{2n}/(2n+1)\}^p$. Since the sequence $(2n+1)^{-1}$, $n \geq 0$, is log-convex, it follows from (q), Section 2, that $r_m(p) > 0$ for m even. As, furthermore, some of the c_j are negative ($c_{13} < 0$), it follows from (4.1) that some of the $d_{n,j}$ are negative, i.e., that the conjecture is false.

4.2 Slow deconvolution

As a second application we mention the the work of Carasso (1987). The problem here is the recover the function g from the convolution equation

$$(4.2) \qquad b(t) = \int_0^t p(t-\tau)g(\tau)d\tau \ ,$$

where b is the response to a pulse p. However, not b is observed, but b_n, an output distorted by noise. The author solves the problem by deconvoluting (4.2) gradually; he constructs functions $u(x,t)$ with Fourier transforms satisfying

$$\widehat{u}(x,\cdot) = \widehat{b}/\widehat{p}^{1-x} \quad (0 \leq x \leq 1) \ ,$$

and such that $u(1,\cdot) = b$. To be able to do this he needs to know that $\widehat{p}^{1-x}$ is the Fourier transform of a *positive* function p_x, i.e., that p is infinitely divisible. Since large numbers of infinitely divisible probability densities are known (see e.g. Bondesson (1992) or Steutel (1979)), he has a variety of functions to choose from.

5 Electrons, Poisson processes and Bessel functions

In this section we discuss a problem from electrical engineering. Probability theory enters at three stages: 1. the problem, which originally was phrased deterministically, is put in a probabilitstic context; 2. an integral equation for the quantity of interest is derived by probabilistic interpretation; 3. the integral equation is solved by probabilistic means.
This section is divided into three parts. In Subsection 5.1 the technical problem is described, Subsection 5.2 contains some preliminary relations between Poisson processes and Bessel functions, and, finally, in Subsection 5.3 the problem is defined probabilistically, and solved.

5.1 Electrons in a gas

The technical problem concerns the breakdown by an avalanche of electrons of an insulator gas between two electrodes placed at a (small) distance w. The electrons are produced at the cathode and proceed towards the anode according to the following (non-stochastic) model.
Electrons move with constant speed one, but they may collide with the surrounding gas molecules; the collision rate is λ. A collision may lead to attachment of the electron to the

molecule or to ionization, i.e. the production of an extra free electron; the attachment and ionization rates are η and α respectively, so $\lambda = \alpha + \eta$. Attached electrons get detached at rate δ.
This mechanism can be translated into numbers of particles of the various types, or rather into number densities. This then leads to the following differential equations.

$$\frac{\partial \rho_e}{\partial t} + \frac{\partial \rho_e}{\partial x} = (\alpha - \eta) f_e + \delta \rho_n \ ,$$

$$(5.1) \qquad \frac{\partial \rho_p}{\partial t} + \frac{\partial \rho_p}{\partial x} = \alpha \rho_e \ ,$$

$$\frac{\partial \rho_n}{\partial t} = \eta \rho_e \ ;$$

here $\rho_e = \rho_e(x,t)$ denotes the density of free electrons at distance x from the cathode, ρ_p the density of positive ions (formed by ionization) and ρ_n the density of negative ions (formed by attachment).
The quantity of interest in the number of free moving electrons at time t, i.e. the current $I(t)$ given by

$$I(t) = n_0 C(t) = \int_0^w \rho_e(x,t)dx \ ,$$

with $n_0 = n_e(0,0)$ the number of free electrons, concentrated at the cathode at time zero. Equations (5.1) with the appropriate boundary conditions have been solved numerically; this leads to numerical result for $I(t)$. For technical information see Verhaart and van der Laan (1982). We shall be interested in $C(t)$, the current resulting from a single electron at $x = 0$ and $t = 0$. Our, equivalent, probablistic model will be given in Subsection 5.3; first we discuss a Bessel-function integral.

5.2 Poisson-processes and Bessel functions

Consider two persons, 1 and 2. Person 1 takes random steps of lenghts $X_1, X_2, \ldots$, person 2 takes random steps of lengths $Y_1, Y_2, \ldots$. By $N_j(z)$ we denote the number of steps person j needs to cover a distance z. If the X's and Y's are all independent and exponentially distributed with mean one, then N_1 and N_2 are independent with

$$(5.2) \qquad P(N_j(z) - 1 = n) = e^{-z}\frac{z^n}{n!} \ \ (n = 0, 1, \ldots) \ ,$$

i.e. $N_1 - 1$ and $N_2 - 1$ are Poisson processes (see (l)).
From (5.2) we easily obtain

Lemma 5.1

$$P(N_1(x) = N_2(y)) = e^{-x-y} I_0(2\sqrt{xy}) \ ,$$

where I_0 denotes the modified Bessel function of order zero.

Using the fact that (cf. (k))

$$P(N_1(x) > n) = P(X_1 + ... + X_n \leq x) = \frac{1}{(n-1)!} \int_0^x z^{n-1} e^{-z} dz ,$$

by straightforward calculation (see Steutel (1986)) we obtain the following theorem.

Theorem 5.2 Let N_1 and N_2 be as defined above. Then

$$P(N_1(x) \leq N_2(y)) = J(x, y) ,$$

where J is the well-known J-function, defined by (see e.g. Luke (1962)),

$$(5.3) \qquad J(x, y) = 1 - e^{-y} \int_0^x I_0(2\sqrt{yt}) e^{-t} dt .$$

We shall need (5.3) later, together with the following useful approximation, which is a consequence of the fact that $N_1(x) - N_2(y)$ is asymptotically normal (compare (o) in Section 2) with mean $x - y$ and variance $x + y$. The added $\frac{1}{2}$ is a so-called continuity correction, which improves the approximation.

Corollary 5.3

$$(5.4) \qquad J(x, y) = \Phi\left(\frac{y - x + \frac{1}{2}}{\sqrt{x + y}}\right) + O\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\right) .$$

The order term results from a refinement of the central limit theorem. It turns out that approximation (5.4) is more accurate then the approximations resulting from traditional asymptotic expansions for $I_0(x, y)$.

5.3 Probabilistic model and solution

We start with a single electron at the cathode at time zero. The probabilistic version of the model now is as follows. The electron moves with velocity one towards the anode during a random time X_1, after which it collides with a gas molecule. With probability $1 - p$ the collision leads to attachment of the electron, which reduces its velocity to zero. With probability p the election ionizes the molecule, i.e. produces a second free electron, which proceeds in the same way, independent of the other electron(s). The attached electron stays attached for a random time Y_1, after which it moves for a random time X_2, etc. The random times $X_1, Y_1, X_2, Y_2, ...$ are supposed to the independent and exponentially distributed, the X's with mean λ^{-1}, the Y's with mean μ^{-1}. The relations between these parameters and the ones introduced in Subsection 5.1 is as follows: $\lambda = \alpha + \eta, p = \alpha/(\alpha + \eta), \mu = \delta$. In Figure 1 the distance(s) $Z_p(t)$ of the electron(s) from the cathode is sketched as a function of time.

The current $C_p(t) = C_p(t; \lambda, \mu)$ we are interested in is (in the right units) equal to the number of moving electrons at time t. Since in practice the contributions of many electrons have to be added, we model the current at time t as the *expected* number of moving electrons, or in terms of Z_p:

$$C_p(t) = EZ'_p(t) \ ,$$

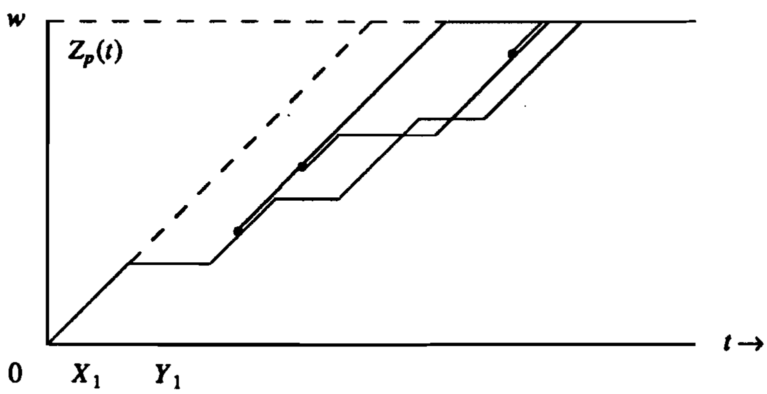

Figure 1:

where Z_p has to be counted with the appropriate multiplicities (see Figure 1). Now by conditioning on the value of the first free time X_1, on attachment or non-attachment, and on the value of the first attachment period Y_1, we can immediately write down the following integral equation for $C_p(t, w)$.

$$(5.5) \qquad C_p(t,w) = \begin{cases} e^{-\lambda t} + \displaystyle\int_0^t \lambda e^{-\lambda x} M_p(t-x, w-x)dx & (0 \le t < w) \\ \displaystyle\int_0^w \lambda e^{-\lambda x} M_p(t-x, w-x)dx & (t > w) \ , \end{cases}$$

where $M_p(t,z) = 2pC_p(t,z) + (1-p)\displaystyle\int_0^t \mu e^{-\mu y} C_p(t-y, z)$, as is easily verified.

For $t < w$, the value of w is immaterial; we may take $\mu = \infty$. Taking Laplace transforms and some simple calculations lead to

Lemma 5.5 For $t < w$ equation (5.5) is solved by

$$(5.6) \qquad C_p(t,w) = ce^{s_1 t} + (1-c)e^{s_2 t} \ ,$$

where s_1 and s_2 are the zeroes of $s^2 + (\lambda + \mu - 2\lambda p)s - \lambda\mu p$ and $c = (s_1 + \mu)/(s_1 - s_2)$. There is a downward jump at $t = w$ of magnitude $\exp(-\lambda(1-2p)w)$.

The final statment concerns the expected number of electrons reaching the anode unobstructed.

Corollary 5.6 For $p = 0$ and $t < w$ one has

$$C_0(t, w) = \frac{\mu}{\mu + \lambda} + \frac{\lambda}{\mu + \lambda} e^{-(\lambda+\mu)t} .$$

The solution of (5.5) for $t > w$ is more complicated, but it can be reduced to the case $p = 0$, where no ionization takes place. The following result can be proved by taking Laplace transforms in (5.5) with respect to *both* t and w. We omit this proof.

Proposition 5.7

$$C_p(t, w; \lambda, \mu) = e^{s_1 t} C_0(t, w; \lambda', \mu') ,$$

where $\lambda' = -s_2 - \mu > 0$ and $\mu' = \mu + s_1 > \mu$ with s_1 and s_2 as in Lemma 5.5.

So we now restrict attention to the case $p = 0$, and proceed to solve (5.5) for this case. Since values of $t \leq w$ have already been taken care of we, only consider the case $t > w$.

Theorem 5.8 For $t > w$ the current $C_0(t; w)$ is given by

$$C_0(t, w) = \mu \int_0^w e^{-(\lambda+\mu)(w-z)} \{1 - J(\lambda z, \mu(t - w))\} dz ,$$

where J is defined by (5.3).

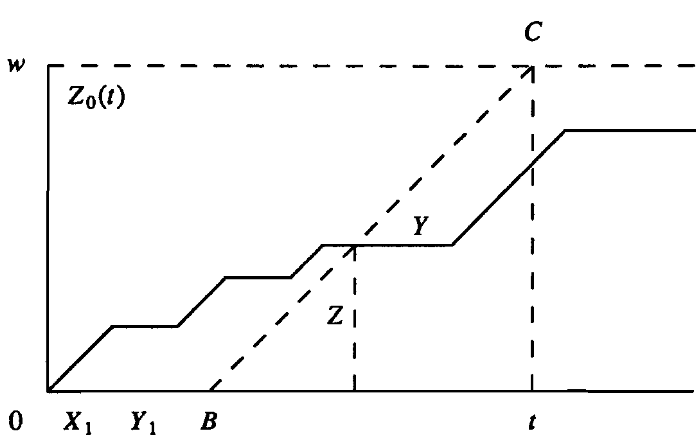

Figure 2:

Proof. The distance $Z_0(t)$ of the electron from the anode is sketched in Figure 2 above.

Clearly, since the number of electrons cannot increase, $C_0(t, w)$ will be decreasing. We condition on the height Z at which $Z_0(t)$ crosses the line BC; this must happen if the electron is not absorbed before t. We use the following three facts. 1. The remaining part of Y after the crossing has the same distribution as Y_1, 2. The process from this point on is identical to the original process for $t < w$, but with X and Y reversed, 3. The distribution of the crossing height Z is given by (cf. Theorem 5.2)

$$F_Z(z) = P(Z \leq z) = P(N_1(\lambda z) > N_2(\mu(t-w)) = 1 - J(\lambda z < \mu(t-w)) .$$

It follows that we have (see Corollary 5.6)

$$C_0(t,w) = \int_0^w \frac{\mu}{\lambda+\mu}(1 - e^{-(\lambda+\mu)(w-z)})dF_Z(z)$$

$$= \mu \int_0^w e^{-(\lambda+\mu)(w-z)}\{1 - J(\lambda z, \mu(t-w))\}dz ,$$

by partial integration.

Proposition 5.7 now yields the values of $C_p(t;w)$. A picture of $C_{2/3}(t,1;5,2\frac{1}{2})$ is given in Figure 3.

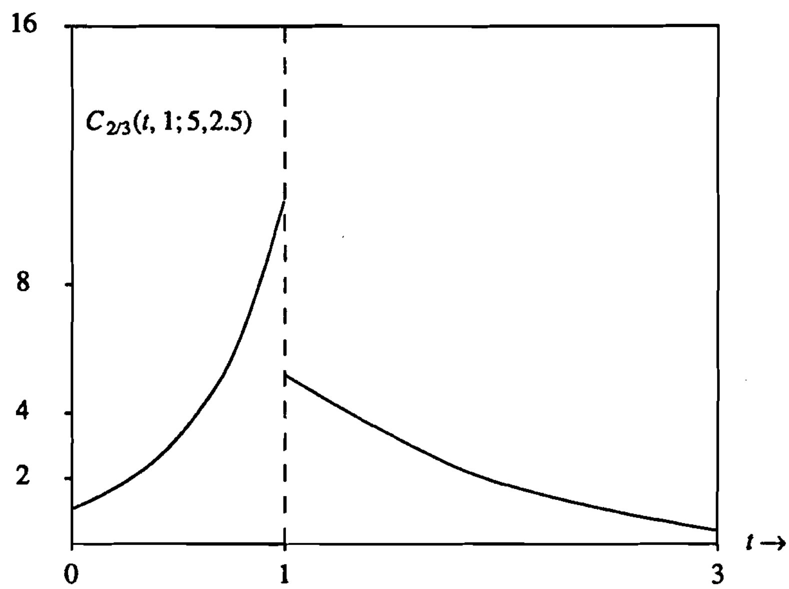

Figure 3:

Since $C_p(t,w)$ is easily computed for $t < w$, approximations are useful only for $t > w$, and these, because of Proposition 5.7, are only needed for $C_0(t,w)$. From Corollary 5.3 we obtain the following, slightly heuristic approximation.

$$C_0(t,1;\lambda,\mu) \approx \frac{\mu}{\lambda+\mu}\Phi\left(\frac{\mu+\lambda-\mu t-1)}{\sqrt{\mu t-\mu+\lambda}}\right) \quad (t > 1) .$$

We remark that $C_0(t,w;\lambda,\mu) = C_0(\frac{t}{w},1;\lambda w,\mu w)$. The approximations turn out to be quite accurate even for values of t not far from w.

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