Solution to Problem 84-17: Patterns in a sequence of symmetric Bernoulli trials

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In the special case where $f'(t)=1/t$, $t_0=1$, and $\lambda_j=j$, then (4) reduces to

$$X(t)=P\begin{bmatrix} t^1 & & & 0 \\ & t^2 & & \\ & & \ddots & \\ 0 & & & t^n \end{bmatrix}P^{-1}X(1).$$

These results can be extended to nonhomogeneous systems of ODE's.

NANCY WALLER *and the proposer* give the following explicit solution by using the Cayley–Hamilton theorem:

$$X(t)=X(1)+\sum_{k=1}^{n}\frac{A(A-I)(A-2I)\cdots(A-(k-1)I)X(1)(t-1)^k}{k!}.$$

A. S. FERNÁNDEZ (E. T. S. Ingenieros Industriales de Madrid) uses the Lagrange interpolating polynomial to obtain

$$X(t)=\sum_{i=1}^{n}\frac{(-1)^{i-1}t^i}{(n-i)!(i-1)!}\prod_{j\neq i}(A-jI)X(1).$$

J. ROPPERT (Wirtschaftsuniversität Wien) also gives a solution if A has multiple eigenvalues by means of Jordan decomposition. Z. J. KABALA and I. P. E. KINNMARK (Princeton University) show how to solve (1) as above as well as when A has multiple eigenvalues. M. LATINA (Community College of Rhode Island) in his solution notes that the method can be extended to more general Euler–Cauchy systems

$$\sum_{k=1}^{m}t^kA_kX^{(k)}(t)=0.$$

Also solved by P. W. BATES (Texas A & M), G. A. BÉCUS (University of Cincinnati), J. BÉLAIR (Université de Montreál), S. COBLE (Student, University of Washington), C. GEORGHIOU (University of Patras, Greece), O. HAJEK (Case Western Reserve University), A. A. JAGERS (Technische Hogeschool Twente, the Netherlands), D. JAMES (San Antonio, Texas), R. A. JOHNS (University of South Carolina-Spartanburg), I. N. KATZ (Washington University, St. Louis), G. LEWIS (Michigan Technological University), H. M. MAHMOUD (George Washington University), H. J. OSER (National Bureau of Standards), D. W. QUINN (Air Force Institute of Technology), P. H. SCHIDT (University of Akron), H. TÜRKE (Universität Tübingen, FRG), G. C. WAKE (Victoria University, New Zealand), two other solutions by NANCY WALLER (Portland State University), J. A. WILSON (Iowa State University), P. T. L. M. VAN WOERKOM (Bussum, the Netherlands) and one other solution by the proposer.

Patterns in a Sequence of Symmetric Bernoulli Trials

Problem 84-17, *by* Y. P. SABHARWAL *and* V. K. MALHOTRA (Delhi University, Delhi, India).

Consider a sequence of $n(>1)$ symmetric Bernoulli trials. Two consecutive trials would yield one of the four patterns SS, SF, FF and FS, where S stands for success and F stands for failure. The study of these patterns is relevant in the context of brand choice processes. Let

$$\begin{aligned} N_1 &= \text{number of occurrences of } SS, \\ N_2 &= \text{number of occurrences of } SF, \\ N_3 &= \text{number of occurrences of } FS, \\ N_4 &= \text{number of occurrences of } FF, \end{aligned}$$

so that

$$N_1 + N_2 + N_3 + N_4 = n - 1.$$

Determine the variance-covariance matrix for (N_1, N_2, N_3, N_4). Here, the (i,j) term is the expectation of $(N_i - \overline{N}_i)(N_j - \overline{N}_j)$.

Solution by O. P. LOSSERS (Eindhoven University of Technology, Eindhoven, the Netherlands).

For $2 \leqq i \leqq n$ define the random variables

$$\tau_{1,i} := \begin{cases} 1 & \text{if } (i-1)\text{th and } i\text{th trials are } S, \\ 0 & \text{otherwise;} \end{cases}$$
$$\tau_{2,i} := \begin{cases} 1 & \text{if } (i-1)\text{th trial is } S \text{ and } i\text{th trial } F, \\ 0 & \text{otherwise;} \end{cases}$$
$$\tau_{3,i} := \begin{cases} 1 & \text{if } (i-1)\text{th trial is } F \text{ and } i\text{th trial } S, \\ 0 & \text{otherwise;} \end{cases}$$
$$\tau_{4,i} := \begin{cases} 1 & \text{if } (i-1)\text{th and } i\text{th trials are } F, \\ 0 & \text{otherwise.} \end{cases}$$

Evidently $N_j = \sum_{i=2}^{n} \tau_{j,i}$, $1 \leqq j \leqq 4$.

Using the obvious joint distribution of τ_{j_1,i_1} and τ_{j_2,i_2} it is an easy verification that

$$\begin{aligned} &EN_j = (n-1)/4 \qquad (1 \leqq j \leqq 4), \\ &EN_1N_2 = EN_1N_3 = EN_4N_2 = EN_4N_3 = (n^2 - 3n + 2)/16, \\ &EN_1N_4 = (n^2 - 5n + 6)/16, \\ &EN_2N_3 = (n^2 - n - 2)/16, \\ &EN_1^2 = EN_4^2 = (n^2 + 3n - 6)/16, \qquad EN_2^2 = EN_3^2 = (n^2 - n + 2)/16. \end{aligned}$$

For the corresponding variance-covariance matrix we obtain

$$\frac{1}{16}\begin{pmatrix} 5n-7 & 1-n & 1-n & 5-3n \\ 1-n & n+1 & n-3 & 1-n \\ 1-n & n-3 & n+1 & 1-n \\ 5-3n & 1-n & 1-n & 5n-7 \end{pmatrix}.$$

Also solved by J. A. GRZESIK (TRW Inc., California), A. A. JAGERS (Technische Hogeschool Twente, the Netherlands), J. A. WILSON (Iowa State University) *and the proposers.*