

Solution to Problem 91-19 : A finite part integral

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with respect to c_2 , we obtain

$$\int_0^\pi \frac{\sin^2 \theta}{g^2} d\theta = \frac{\pi}{2c_1 c_2^3}.$$

These integrals, on substitution in the formula for I , confirm that $I = c_1/c_2$ as desired.

Also solved by C. C. GROSJEAN (State University of Ghent, Belgium).

A Finite Part Integral

Problem 91-19, by B. BERTRAM AND O. G. RUEHR (Michigan Technological University).

Evaluate, in terms of elementary functions, the integral

$$I_\nu(x) = \int_0^1 \frac{t^\nu}{(t-x)^2} dt, \quad 0 < x < 1$$

for rational $\nu > -1$. Here, the double bar denotes that the integral is to be taken in the Hadamard or finite part sense, which can be shown to be the derivative of the Cauchy principal value integral, i.e.,

$$\int_0^1 \frac{f(t)}{(t-x)^2} dt = \frac{d}{dx} \int_0^1 \frac{f(t)}{t-x} dt.$$

The problem arose in a numerical analysis of errors encountered in applying product integration to linear finite part integral equations. Such equations occur in fracture mechanics, gas radiation, and fluid flow.

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Solution by J. BOERSMA and P. J. DE DOELDER (Eindhoven University of Technology, Eindhoven, the Netherlands).

Introduce the function

$$F_\nu(x) = \int_0^1 \frac{t^\nu}{t-x} dt;$$

then the derivative $f'_\nu(x) = I_\nu(x)$ is to be evaluated for $0 < x < 1$ and rational $\nu > -1$.

Using [1, form. 6.2(4)], we determine the Mellin transform

$$\int_0^\infty F_\nu(x) x^{s-1} dx = \int_0^1 t^\nu dt \int_0^\infty \frac{x^{s-1}}{t-x} dx = \frac{\pi \cot(\pi s)}{s + \nu},$$

valid for $\max(0, -\nu) < \operatorname{Re} s < 1$. By inversion we obtain the representation

$$F_\nu(x) = \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} \frac{\cot(\pi s)}{s + \nu} x^{-s} ds, \quad x > 0, \quad \max(0, -\nu) < c < 1.$$

By means of [1, form. 7.2(18)] the derivative $F'_\nu(x)$ can be expressed as

$$F'_\nu(x) = -\frac{1}{2i} \int_{c-i\infty}^{c+i\infty} \frac{s}{s+\nu} \cot(\pi s) x^{-s-1} ds$$

$$= -\frac{1}{x(1-x)} + \frac{\nu}{2i} \int_{c-i\infty}^{c+i\infty} \frac{\cot(\pi s)}{s+\nu} x^{-s-1} ds, \quad x > 0, \quad \max(0, -\nu) < c < 1.$$

The latter integral is evaluated by closing the contour to the left, which is permissible if $0 < x < 1$. The integrand has poles at $s = -\nu$ and $s = -k, k = 0, 1, 2, \dots$. Notice that all poles are simple, except for a second-order pole at $s = -\nu$ if ν is integral.

In the case of integral ν , say $\nu = n, n = 0, 1, 2, \dots$, we thus find

$$F'_n(x) = -\frac{1}{x(1-x)} - nx^{n-1} \log x + n \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{x^{k-1}}{n-k}$$

$$= -\frac{1}{x(1-x)} + nx^{n-1} [\log(1-x) - \log x] + n \sum_{k=0}^{n-1} \frac{x^{k-1}}{n-k}, \quad 0 < x < 1.$$

In the case of nonintegral $\nu > -1$ we find

$$(*) \quad F'_\nu(x) = -\frac{1}{x(1-x)} - \pi\nu \cot(\pi\nu) x^{\nu-1} + \nu \sum_{k=0}^{\infty} \frac{x^{k-1}}{\nu-k}, \quad 0 < x < 1.$$

If ν is rational, we may set $\nu = n - p/q$, where $n = [\nu] + 1$, and p and q are mutually prime integers with $0 < p < q$. The final series in (*) is now rewritten as

$$\nu \sum_{k=0}^{\infty} \frac{x^{k-1}}{\nu-k} = \nu \sum_{k=0}^{n-1} \frac{x^{k-1}}{\nu-k} - \nu x^{n-1} q \sum_{k=0}^{\infty} \frac{x^k}{p+qk}.$$

To evaluate the latter series we use a technique adopted from Nielsen [2, p. 21]. Starting from the well-known identity

$$\sum_{m=0}^{q-1} \left(e^{2\pi im/q} \right)^{l-p} = \begin{cases} q & \text{if } l = p \pmod{q}, \\ 0 & \text{if } l \neq p \pmod{q}, \end{cases}$$

we have

$$q \sum_{k=0}^{\infty} \frac{t^{qk}}{p+qk} = \sum_{l=1}^{\infty} \sum_{m=0}^{q-1} \frac{(te^{2\pi im/q})^{l-p}}{l}$$

$$= -t^{-p} \sum_{m=0}^{q-1} e^{-2\pi im p/q} \log(1 - te^{2\pi im/q}), \quad |t| < 1.$$

By combining the previous results we find

$$F'_\nu(x) = -\frac{1}{x(1-x)} + \nu x^{\nu-1} \left[\sum_{m=0}^{q-1} e^{2\pi im\nu} \log(1 - x^{1/q} e^{2\pi im/q}) - \pi \cot(\pi\nu) \right]$$

$$+ \nu \sum_{k=0}^{[\nu]} \frac{x^{k-1}}{\nu-k}, \quad 0 < x < 1,$$

valid for nonintegral rational ν . Here, q is the smallest positive integer such that $q\nu$ is integral.

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 Also solved by C. C. GROSJEAN (University of Ghent, Ghent, Belgium) and W. F. FORD and J. A. PENNLINE (NASA Lewis Research Center) and the proposers. The proposers' solution, with a generalization, has appeared in *Product integration for finite-part singular integral equations; numerical asymptotics and convergence acceleration*, J. Comput. Appl. Math., 41 (1992), pp. 163–173.

Approximation for the Location of the Center of a Disk

*Problem 91-20**, by M. L. GLASSER AND J. KOPLOWITZ (Clarkson University).

A digital disk [1] consists of all points on a square lattice (of lattice spacing 1) that falls within a disk. If the radius of the disk is given, how accurately can one find the location of its center from the knowledge of the digital disk? Is the closest lattice point to a random circle of radius r of $O(1/r)$? If so, what is a good estimate for the coefficient?

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Solution by H. L. ABBOTT (University of Alberta).

It will be shown first that the answer to the second question raised in the proposal is no. Let $f(r)$ denote the distance from the circle $x^2 + y^2 = r^2$ to the nearest exterior lattice point.

THEOREM 1. *There exists a positive constant c such that for arbitrarily large values of r ,*

$$f(r) > \frac{c(\log r)^{1/2}}{r}.$$

Proof. Denote by $h(z)$ the number of integers not exceeding z that may be written in the form $u^2 + v^2$, u , and v integers. It is a classical result of Landau [3] that there exists a positive constant α such that, as $z \rightarrow \infty$,

$$h(z) = (\alpha + o(1)) \frac{z}{(\log z)^{1/2}}.$$

Let $\ell = h(z) - h(z/2) + 1$, and let $m = z/2\ell$. For $j = 1, 2, \dots, \ell$, let $I_j = ((z/2) + (j-1)m, (z/2) + jm]$. Then I_1, I_2, \dots, I_ℓ are nonoverlapping intervals whose union is $I = (z/2, z]$. The number of integers in I of the form $u^2 + v^2$ is $h(z) - h(z/2)$, and hence, by the definition of ℓ , one of the subintervals, say I_t , does not contain any such number. Set $r^2 = z/2 + (t-1)m$ and $w^2 = (z/2) + tm$. Then the nearest exterior lattice point to the circle $x^2 + y^2 = r^2$ must lie outside the circle $x^2 + y^2 = w^2$. Thus

$$f(r) > w - r = \frac{w^2 - r^2}{w + r} = \frac{m}{w + r} > \frac{m}{2w} = \frac{z}{4w\ell}.$$