

# The reference stability of a macro-economic system with a recursive minimum variance control equation

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The Reference Stability of a Macro-  
Economic System with a Recursive Minimum  
Variance Control Equation

by

J.C. Engwerda and P.W. Otter

Eindhoven, the Netherlands

December 1986

# The Reference Stability of a Macro-Economic System with a Recursive Minimum Variance Control Equation

by

Jacob C. Engwerda\* and Pieter W. Otter\*\*

## ABSTRACT

The asymptotic closed loop behaviour of a macro-economic system with time-varying exogenous inputs is studied using an optimal minimum variance control equation. By using the phase- (or echelon) canonical form and by means of a simulation study the dependency of the closed loop system on the weighting matrix in the cost functional is examined.

## I. Introduction

In economics and control engineering a lot of research has been done on the design of optimal controllers and the design of controllers which stabilize the closed loop system. Most of the times an optimal controller is meant to be a controller that minimizes some loss functional over a finite or infinite time horizon. In this sense the word optimal has to be interpreted in this paper too. Moreover, in the sequel it will be assumed that the cost functional is quadratic. For most of the infinite time optimal controllers it has been shown that they have the property that they stabilize the system, provided the weighting matrices occurring in the cost functional are positive definite, and provided that some additional conditions on the system are satisfied (e.g. stabilizability, detectability, stability of reference trajectory), see e.g. Kwakernaak (1972), Chow (1975), Maybeck (1982), Åström (1983,1984). However, the performance of such a controller depends on the particular weighting matrices chosen. Since in many problems it is not clear a priori how these matrices should be chosen, the selection of these matrices is an important issue in the design of optimal stabilizing controllers. When discussing the performance of the LQG-controller, Maybeck stated that "Typically this requires an iteration on choice of cost weighting matrices to provide a benchmark with all desired performance characteristics." (Maybeck 1982 pp.175) and Åström remarks: "In many cases it is difficult to find natural quadratic loss functions." (Åström 1984 pp.267) and "There are several problems when applying LQ-control. One occurs in choosing design parameters, i.e. the weighting in the loss function ... " (Åström 1984 pp.275).

In this paper it will be shown that it is possible to design a stabilizing finite time optimal controller by an appropriate choice of the weighting matrices.

Following Chow (1975), the system that will be considered here is linear, time-invariant and possesses an exogenous input. The cost criterium will be a quadratic time-varying tracking equation. Different from Chow only the one-period ahead cost functional will be used here and the assumption Chow needed to analyze the stabilization properties of his infinite time optimal controller, namely time constancy of the reference trajectory and the exogenous input, will be dropped. Minimizing this one-period ahead cost functional is known as minimum variance (MV)-control in engineering, see e.g. Åström (1984). It will be shown that, by choosing the weighting matrix in this cost functional according to Luenberger's phase canonical form, the resulting feed-back gain matrix is nilpotent of order given by the controllability index. Applying MV-control recursively therefore results in an expected closed loop system which is reference (BIBO) stable for any exogenous input, provided that this input is bounded. The recursive MV-control has the property of being a dead-beat controller in case of no exogenous input.

The choice of this particular cost functional is motivated by the following two arguments:

- (i) Because of the uncertainty in real-life macro-economic situations there is a constant need for short period adaptation of control with respect to new information. A cost functional with a short horizon makes an easy adaptation possible. In engineering this rather simple adaptation is known as self-tuning control (Åström 1983, 1984).
- (ii) The computational ease and relatively simple formulas of the control algorithm.

The paper is organized as follows. In section II the optimal control algorithms of Chow are summarized and specialized for a model with external input other than white noise, a time-varying reference trajectory and a one-period ahead cost functional. Then by transforming the open loop system into its phase canonical form it is shown that the expected closed loop system resulting from a particular choice of the weighting matrix is reference (BIBO) stable. In section III the MV-control will be applied to a two-dimensional macro-economic model with one respectively two control variables. Two control performances will be compared: one with an arbitrarily chosen weighting matrix and one based on the phase canonical form. The paper ends by a conclusion section.

## II. Minimum Variance Control, Reference Stability and Phase Canonical Forms

Consider the following reduced form of a linear econometric model

$$\tilde{y}_t = A_1 \tilde{y}_{t-1} + \dots + A_m \tilde{y}_{t-m} + B_0 u_t + \dots + B_s u_{t-s} + \tilde{c}_t + \tilde{d}_t$$

where  $\tilde{y}_t$  is a  $p_1$ -dimensional target vector;  $u_t$  a  $q$ -dimensional control vector;  $\tilde{c}_t$  a vector of exogenous inputs and  $\tilde{d}_t$  a serially uncorrelated vector with zero mean and covariance  $V$  (white-noise). Following Chow (1975) the reduced form model is rewritten as the first-order system

$$\begin{aligned}
 & \begin{bmatrix} \bar{y}_t \\ \cdot \\ \cdot \\ \cdot \\ \bar{y}_{t-m+1} \\ u_t \\ \cdot \\ \cdot \\ \cdot \\ u_{t-s+1} \end{bmatrix} = \begin{bmatrix} A_1 & \dots & A_m & B_1 & \dots & B_s \\ I & & 0 & 0 & & 0 \\ & \cdot & \cdot & \cdot & & \cdot \\ & & \cdot & \cdot & & \cdot \\ & & & I & 0 & 0 \\ 0 & & 0 & 0 & I & 0 \\ 0 & & 0 & 0 & 0 & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ 0 & & 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} \bar{y}_{t-1} \\ \cdot \\ \cdot \\ \cdot \\ \bar{y}_{t-m} \\ u_{t-1} \\ \cdot \\ \cdot \\ \cdot \\ u_{t-s} \end{bmatrix} + \\
 & + \begin{bmatrix} B_0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ I \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} u_t + \begin{bmatrix} \bar{c}_t \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + \begin{bmatrix} \bar{d}_t \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}
 \end{aligned}$$

$$y_t = A y_{t-1} + B u_t + c_t + \delta_t \quad (1)$$

where  $y_t$  is  $n$ -dimensional with  $n = mp_1 + sq$  with special case  $n = p_1$  if  $m = 1$  and  $s = 0$ . It is assumed that  $\text{rank}(B) = q$  and that the pair  $(A, B)$  is controllable.

Now consider the cost functional

$$J = E \left\{ \sum_{t=1}^N (y_t - y_t^*)^T K (y_t - y_t^*) \right\}$$

where  $y_t^*$  is the reference value for  $y_t$  and  $K_t$  a symmetric positive definite weighting matrix. Then according to Chow the optimal feedback control equation is given by

$$u_t = G_t y_{t-1} + g_t \quad t = 1, \dots, N \quad (2)$$

where

$$G_t = -(B^T H_t B)^{-1} B^T H_t A$$

$$H_t = K_t + (A + B G_{t+1})^T H_{t+1} (A + B G_{t+1}), \text{ with terminal condition } H_N = K_N$$

$$g_t = -(B^T H_t B)^{-1} B^T (H_t c_t - h_t)$$

$$h_{t-1} = K_{t-1} y_{t-1}^* + (A + B G_t) (h_t - H_t c_t), \text{ with terminal condition } h_N = K_N y_N^*.$$

The closed loop system is, by substituting eq.(2) into eq.(1), given by

$$y_t = (A + BG_t) y_{t-1} + B g_t + c_t + \delta_t .$$

Under the conditions that  $K_t = K$ ,  $y_t^* = y^*$  and  $c_t = c$  the feedback gains.  $G_t$  and  $g_t$  tend to the steady-state values  $G$  and  $g$  respectively, provided that  $F = A + BG$  is stable, i.e.  $\lim_{n \rightarrow \infty} F^n = 0$ , see Chow (1975, section 7.8). The steady-state solution is given by  $G = -(B^T H B)^{-1} B^T H A$  where  $H$  is the positive definite solution of the Algebraic Riccati Equation and  $g = -(B^T H B)^{-1} B^T [H c - (I - F)^{-1} (K y^* - H c)]$ . In the sequel we relax the assumptions that  $y_t^*$  and  $c_t$  are time-invariant and study the stability of  $F$  for a special case of the optimal control equation (2), namely an optimal control equation with a one period ahead cost functional i.e.

$$J = E [(y_t - y_t^*)^T K (y_t - y_t^*)] , \quad t = 1, 2, \dots$$

It is assumed that the reference trajectory is a first-order difference equation

$$y_t^* = A^* y_{t-1}^* , \quad t = 1, 2, \dots \quad (3)$$

with  $y_0^*$  and  $A^*$  known.

Subtracting eq.(3) from eq.(1) yields

$$e_t = A e_{t-1} + B u_t + x_t + \delta_t \quad (4)$$

where  $e \triangleq y_t - y_t^*$  and  $x_t \triangleq (A - A^*) y_{t-1}^* + c_t$  is the exogenous vector. Chow's optimal control equation for eq.(4) with  $N = 1$  in the cost functional is given by

$$\begin{aligned} u_t &= -(B^T K B)^{-1} B^T K [A e_{t-1} + x_t] \\ &= -B^+ [A e_{t-1} + x_t] \end{aligned} \quad (5)$$

where  $B^+$  is a pseudo-inverse of  $B$ . The closed loop system is given by

$$\begin{aligned} e_t &= M [A e_{t-1} + x_t] + \delta_t \\ &\triangleq F e_{t-1} + M x_t + \delta_t \end{aligned} \quad (6)$$

where  $M \triangleq (I - B B^+)$  is idempotent. In order to study the asymptotic behaviour of eq.(6) the following definitions will be used.

Def. (1): The closed loop system given by eq.(6) is said to be Lyapunov reference stable if for any  $t$  and  $\epsilon > 0$  there exists a  $\delta(\epsilon, t_0) > 0$  such that  $\|E \{e_{t_0}\}\| \leq \delta$  implies  $\|E \{e_t\}\| \leq \epsilon$  for all  $t \geq t_0$ . The closed loop system is said to be asymptotically reference stable if  $\|E \{e_t\}\| \rightarrow 0$  for  $t \rightarrow \infty$ .

Def. (2): The reference trajectory  $\{y_t^*\}$  is said to be weakly admissible for the minimum variance control sequence  $\{u_t\}$  if there exists a  $\epsilon > 0$  and a  $t_0$  such that  $\|E \{e_t\}\| \leq \epsilon$  for  $t \geq t_0$ . The reference trajectory is strongly admissible for the control sequence  $\{u_t\}$  if  $\|E \{e_t\}\| = 0$  for  $t \geq t_0$ .

Here  $E \{ \cdot \}$  denotes the expectation and  $\| \cdot \|$  a norm.

Theorem 1

The closed-loop system given by eq.(6) is reference stable if the exogenous input sequence  $\{c_t\}$  and the reference trajectory  $\{y_t^*\}$  are bounded for all  $t$  (which implies  $\{x_t\}$  is bounded), and  $F$  is stable i.e.

$$\lim_{n \rightarrow \infty} F^n = 0.$$

Proof:

Equation (6) can be rewritten as

$$\begin{aligned} e_t &= F e_{t-1} + M x_t + \delta_t \\ &= F (F e_{t-2} + M x_{t-1} + \delta_{t-1}) + M x_t + \delta_t \\ &= \dots \\ &= \sum_{i=0}^{\infty} F^i (M x_{t-i} + \delta_{t-i}). \end{aligned}$$

Taking norms we have

$$\| E \{ e_t \} \| = \left\| \sum_{i=0}^{\infty} F^i M x_{t-i} \right\| \leq \sum_{i=0}^{\infty} \| F^i M \| \alpha, \text{ where } \alpha = \sup_t \| x_t \|.$$

Now  $\sum_{i=0}^{\infty} \| F^i M \| \leq \beta < \infty$  if  $F$  is stable. □

In order to study the stability of  $F$  eq.(6) is transformed into its so-called Phase canonical form, see e.g. Luenberger (1967).

Theorem 2 (Phase Canonical Form)

If the pair  $(A, B)$  is controllable and  $\text{rank}(B) = q$  then there exists non-singular transformation matrices  $S$  and  $T$  such that  $\bar{A} = SAS^{-1}$  and  $\bar{B} = SBT$  with

$$\bar{A} \triangleq \begin{pmatrix} k_1 \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ & & \cdot & & & & & & & \\ & & & 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ * & * & * & * & * & \dots & \dots & \dots & * & * & * \end{pmatrix} \\ k_2 \begin{pmatrix} 0 & \dots & \dots & 0 & 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ & & & & \cdot & & & & & & \\ 0 & & & & & & 1 & 0 & \dots & \dots & 0 \\ * & * & * & * & * & \dots & * & * & \dots & * & * & * \end{pmatrix} \\ \vdots \\ k_q \begin{pmatrix} 0 & & & & & & & 0 & 1 & 0 & & \\ 0 & & & & & & & & & & & 1 \\ * & * & * & * & * & \dots & * & * & \dots & * & * & * \end{pmatrix} \end{pmatrix} \quad \bar{B} \triangleq \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & & & \\ 1 & 0 & & \\ \vdots & & & \\ 0 & & & \\ \vdots & & & \\ 0 & 1 & 0 & \\ \vdots & & & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

where  $k_1 \geq k_2 \geq \dots \geq k_q \geq 1$  with  $\sum_{i=1}^q k_i = n$  are the controllability indices, with  $k_1$  as "the" controllability index, and where \* denotes a "free" parameter.

**Theorem 3**

The feedback gain matrix  $F$  in eq.(6) is stable if the weighting matrix in the cost functional is chosen as  $K = S^T S$ .

Proof:

Premultiplying eq.(6) by  $S$  we have  $S e_t = S M A S^{-1} S e_{t-1} + S M x_t + S \delta_t$ . Defining  $\tilde{e}_t$  as  $S e_t$  and choosing  $K = S^T S$  we can rewrite this equation as follows:

$$\tilde{e}_t = [I - \bar{B} (\bar{B}^T \bar{B})^{-1} \bar{B}^T] \bar{A} \tilde{e}_{t-1} + S M x_t + S \delta_t$$

$$\triangleq \bar{F} \tilde{e}_{t-1} + S M x_t + S \delta_t.$$

By simple calculation it can be shown that  $\bar{F} = \text{diag} (D_{1,\dots}, D_q)$  is nilpotent with index  $k_1$ , i.e.  $\bar{F}^{k_1} = 0$ , where

$$D_i = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & & \cdot & & & & \\ \cdot & & & & & & \\ \cdot & & & \cdot & & & \\ \cdot & & & & \cdot & & \\ \cdot & & & & & & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

which is consistent with Wonham (1974, pp.122-126). Now, since  $\bar{F}^n = S F^n S^{-1}$  the result follows.  $\square$

From the proof of Theorem 3 it is seen that the expected closed loop system is given by

$$E \{ \tilde{e}_t \} = \bar{F} E \{ \tilde{e}_{t-1} \} + S M x_t . \tag{7}$$

Evaluating eq.(7) we obtain, for starting value  $\tilde{e}_0 \neq 0$ , after  $k_1$  steps

$$E \{ \tilde{e}_t \} = \bar{F}^{k_1} \tilde{e}_0 + \sum_{i=0}^{k_1-1} \bar{F}^i S M x_{t-i} ,$$

or in general:

$$E \{ \tilde{e}_t \} = \sum_{i=0}^{k_1-1} \bar{F}^i S M x_{t-1} \text{ for } t \geq k_1 . \tag{8}$$

Taking norms in this equation yields

$$\| E \{ \tilde{e}_t \} \| \leq \sum_{i=0}^{k_1-1} \| \bar{F}^i \| \| S M x_{t-1} \| = \epsilon(k_1) . \tag{9}$$



From eq.(8) and eq.(9) the following conclusions can be drawn:

- (i) The closed loop system with a minimum variance control equation with weighting matrix  $K = S^T S$  is reference stable if the sequence  $\{x_t\}$ , and more in particular the reference trajectory  $\{y_t^*\}$  and exogenous input sequence  $\{c_t\}$ , are bounded for all  $t$  (no exponential growth is allowed).
- (ii) Under the condition that the sequence of exogenous inputs  $\{c_t\}$  is bounded for all  $t$ , all reference trajectories such that  $\{y_t^*\}$  is bounded are weakly admissible.
- (iii) If  $x_t = 0$  for all  $t$ , and more in particular  $y_t^* = 0$  and  $c_t = 0$  for all  $t$ , the minimum variance control equation is a deadbeat controller.
- (iv) In case the number of control variables is smaller than the number of target variables, the reference trajectory  $\{y_t^*\}$  is strongly admissible for  $t > k_1$  if the following equation holds for all  $t$

$$x_t = (A - A^*) y_{t-1}^* + c_t = 0 .$$

In a special case, with no exogenous input  $c_t$  and  $y_t^* \neq 0$ , it follows that  $A$  must equal  $A^*$ , that is the reference transition matrix must then equal the process transition matrix. The reader interested in an exact characterization of all obtainable reference trajectories is referred to Engwerda.

- (v) A quantitative measure for the degree of controllability is given by the controllability index  $k_1 \leq n$ . The upper bound on the norm,  $\varepsilon(k_1)$  is a non-decreasing function of  $k$ .

### III. A simulation study

Consider the following reduced-form model:

$$\begin{bmatrix} C(k) \\ I(k) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} C(k-1) \\ I(k-1) \end{bmatrix} + \\ + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} u_1(k-1) \\ u_2(k-1) \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} x(k) + \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix}$$

where

$C(k)$  = Private Consumption;

$I(k)$  = Bruto Private Investment;

$u_1(k)$  = Governmental Expenditures;

$u_2(k)$  = Money Supply;

$x(k)$  = Exogeneous Noise variable;

$V^T(k) = (v_1^T(k) \ v_2^T(k))$  is a white noise vector with  $cov \{V(k) \ V^T(s)\} = \Sigma_V \delta_{k3}$ . All quantities are measured in billions of dollars, in quarter  $k$ .

The simulation experiments are performed on two macro-economic models estimated by Kendrick for the U.S. economy in (1981) and (1982). The parameters he obtained are respectively:

Model I: An estimated macro-economic model with one control ( $m = 1$ ).

$$A = \begin{bmatrix} 1.014 & 0.002 \\ 0.093 & 0.752 \end{bmatrix}; \quad B = \begin{bmatrix} -0.004 \\ -0.100 \end{bmatrix}; \quad C = \begin{bmatrix} -1.312 \\ 0.448 \end{bmatrix}; \quad \Sigma = \begin{bmatrix} 9 & 0 \\ 0 & 10 \end{bmatrix};$$

with initial values:  $C(0) = 460.1$ ;  $I(0) = 113.1$  and  $x(0) = 10$ .

Model II: An estimated macro-economic model with two controls ( $m = 2$ ).

$$A = \begin{bmatrix} 0.914 & -0.016 \\ 0.097 & 0.424 \end{bmatrix}; \quad B = \begin{bmatrix} 0.305 & 0.424 \\ -0.101 & 1.459 \end{bmatrix}; \quad C = \begin{bmatrix} -0.25 \\ -0.777 \end{bmatrix}; \quad \Sigma = \begin{bmatrix} 3.73 & 0 \\ 0 & 8.58 \end{bmatrix};$$

with initial values:  $C(0) = 387.9$ ;  $I(0) = 85.3$  and  $x(0) = 237.75$ .

To show the effect of a different choice of weighting matrix  $Q$  on the controlled system, the results of some experiments with model I are discussed first. We simulated with two  $Q$  matrices, namely:

$$i) \quad Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad ii) \quad Q_2 = S^T S = \begin{bmatrix} 1328847 & -44962 \\ -44962 & 1571 \end{bmatrix}.$$

The choices of these weighting matrices are motivated by the fact that  $Q_1 = I$  will give rise to an unstable closed loop system, while  $Q_2$  makes from the Minimum Variance controller a deadbeat controller. The second weighting matrix is found by taking it equal to  $S^T S$ , where  $S$  is the transformation matrix obtained by transforming equation (1) into the Phase canonical form (see Luenberger [ ]). The stabilization properties of a good weighting matrix are best illustrated by figs. I.1 and I.2, where we assumed that all reference and exogeneous noise variables are in time constant and that the model contains no white noise components. The control error proves to be constant in time for the  $Q_2 = S^T S$  matrix (reference- stable), and unstable for  $Q_1 = I$ . For the case of a constant growth of 2% per year for the reference and exogeneous noise variables the destabilization effects of the applied control are shown in figs. I.3 and I.4. It is seen that the investments behave worse when the stabilizing  $Q_2$  matrix is applied, but that this misconduct is by far compensated by a better behaviour of the consumption. The disadvantage of the stabilizing controller is in this case however that, due to the big components of the  $Q_2$  matrix, it is very sensible to white noise terms. This aspect can be seen in fig. I.3.iv and I.4.iv. For model II the same experiments were carried out for the controls  $u_1(k)$  and  $u_2(k)$  seperately. It proved

that the weighting matrix  $Q = S^T S$  for obtaining a deadbeat controller in both cases was much smaller than for model *I*, namely

$$Q_5 = \begin{bmatrix} 42.5 & 78.7 \\ 78.7 & 185.4 \end{bmatrix} \text{ respectively } Q_6 = \begin{bmatrix} 60.0 & -13.9 \\ -13.9 & 3.5 \end{bmatrix}.$$

As a result the stabilizing control proved to be much less sensitive to exogenous white noise. An example of this is given in fig. II.5. Here the applied control, in case a constant growth of 2% per year of reference and exogenous noise variables is assumed and the model possesses white noise terms, is illustrated when only governmental expenditures is the control variable. All other simulation results with model II appeared to be similar to those of model *I*.

#### IV. (Simulation) conclusions

The choice of the weighting matrices in infinite time optimal control problems is of importance for the performance of the control equations. How to make a "good" choice is not a trivial problem, see e.g. Maybeck (1982) or Åström (1984). In economics the choice is in general motivated by political arguments. Since the behaviour of economic variables in the long run is hard to predict (especially exogenous one), the choices that have to be made about reference trajectories and weighting matrices can not be argued properly. Therefore there is a need for control equations which are optimal for some finite time cost criterium, and which stabilize the system by a recursive application.

In this paper we have shown that if the weighting matrix is based on the phase canonical form, the closed loop system with a minimum variance control equation and with exogenous input (other than white noise) is reference (BIBO) stable.

If no exogenous input is present the minimum variance control is even a dead-beat controller. From the simulations it can be seen that if the weighting matrix is chosen arbitrarily, for this type of controller in general not a stable closed loop system is obtained. So the conclusion can be drawn that for the minimum variance cost criterium the class of weighting matrices among which politicians can make a free choice, should at least be restricted to those matrices which stabilize the closed loop system. The simulations showed moreover that it is not self-evident that always the weighting matrix should be chosen such that the controller becomes a deadbeat controller. The elements of this matrix may namely be so large that the resulting controller becomes too sensitive for small model disturbances. That is, small disturbances in the economy will give rise to heavy fluctuations in the applied control. Concluding one can say that the choice of weighting matrices should be a well considered choice between target preferences, tracking speed and disturbance sensitiveness of the controller.

So the problems that emerged in the infinite time optimal control problems arise here again, extended with the problem that the chosen weighting matrix should be a stabilizing one. But now, since only a one-period ahead optimality criterion is used, there is maybe a more fundamental discussion possible about the choice of weighting matrices and reference trajectories.

## References

- Åström, K.J. and Wittemark, B. "Computer Controlled Systems" Prentice Hall, 1984.
- Åström, K.J. "Theory and Applications of Adaptive Control - a survey". *Automatica*, Vol.19, no.5, pp.471-486, 1983.
- Bertsekas, D.P. "Dynamic Programming and Stochastic Control". Academic Press, 1976.
- Chow, G.C. "Analysis and Control of Dynamic Economic Systems". John Wiley, 1975.
- Engwerda, J.C. "On the set of obtainable reference trajectories using minimum variance control". Submitted for publication.
- Kendrick, D.A. "Stochastic Control for Economic Models". Mc. Graw-Hill, 1981.
- Kendrick, D.A. "Caution and Probing in a Macroeconomic Model". *Journal of Economic Dynamics and Control* 4, 1982.
- Kwakernaak, H. and Sivan, D. "Linear Optimal Control Systems". Wiley-Interscience, 1972.
- Luenberger, D.G. "Canonical Forms for Linear Multivariable Systems". *I.E.E.E. Trans. on Automatic Control*, June 1967.
- Otter, P.W. "Dynamic Feature Space Modelling, Filtering and Self-Tuning Control of Stochastic Systems". Vol.246 in the series "Lecture Notes in Economics and Mathematical Systems". Springer-Verlag, 1985.
- Wonham, W.M. "Linear Multivariable Control: A Geometric Approach". Springer-Verlag, 1974.

Figures Model I

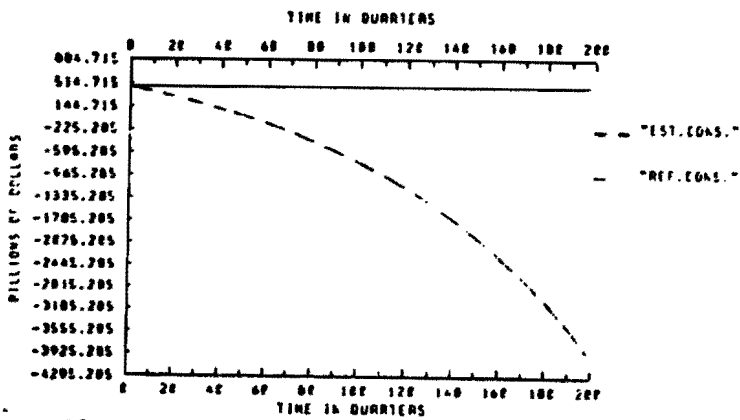


fig. 1-i

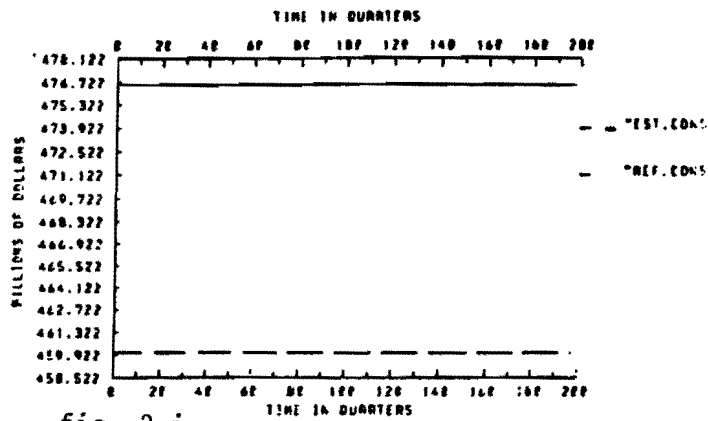


fig. 2-i

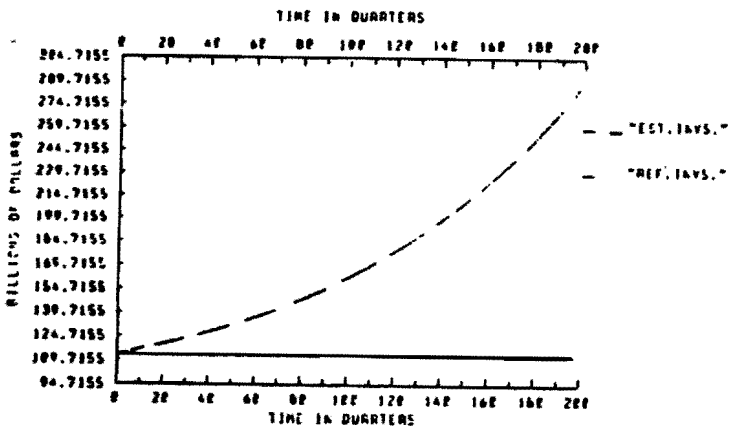


fig. 1-ii

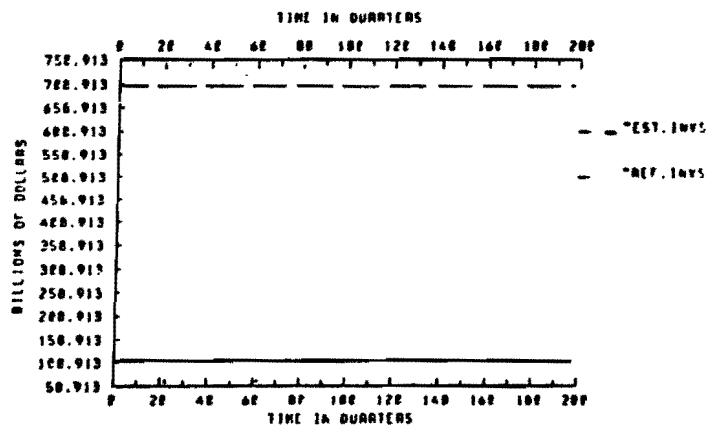


fig. 2-ii

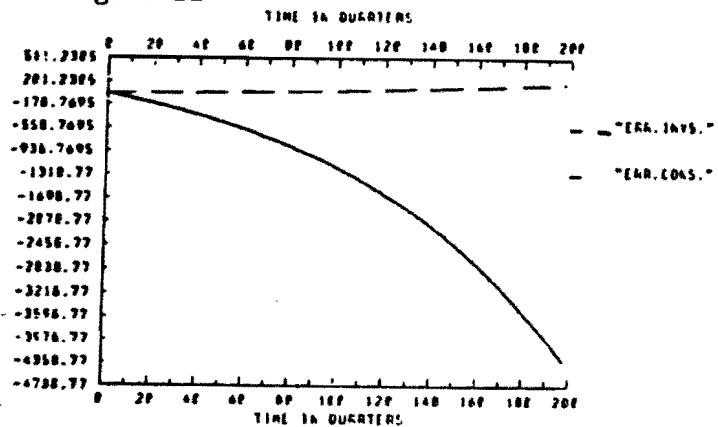


fig. 1-iii

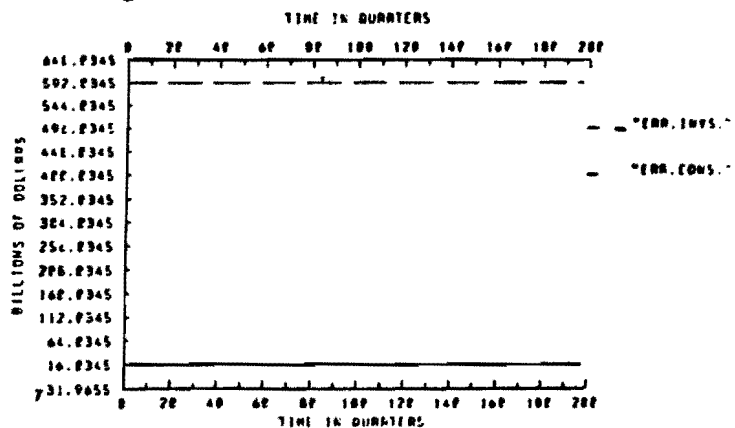


fig. 2-iii

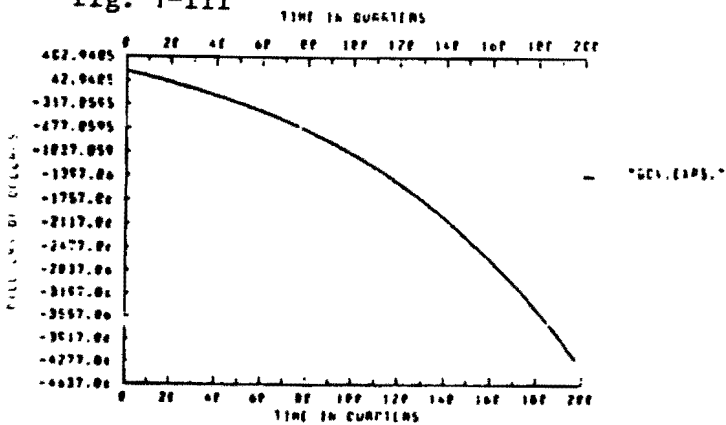


fig. 1-iv

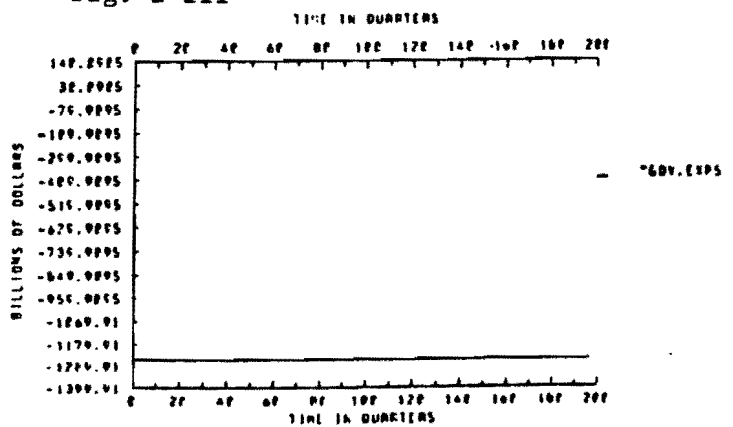


fig. 2-iv

Figures Model I (cont.)

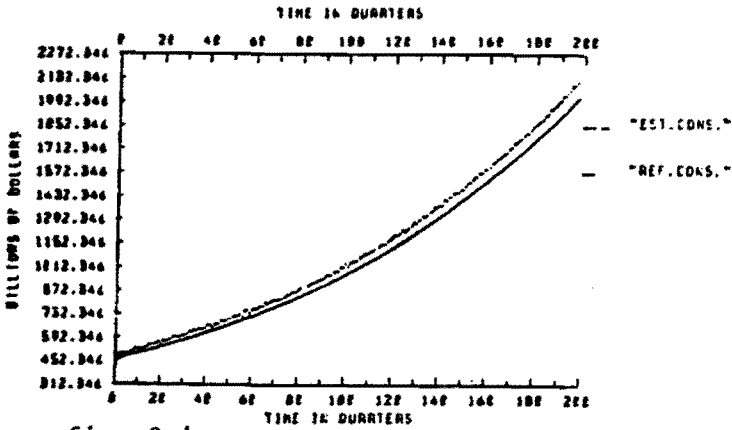


fig. 3-i

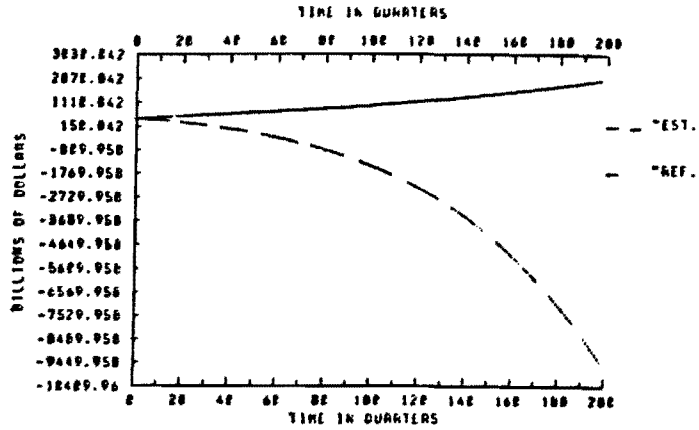


fig. 4-i

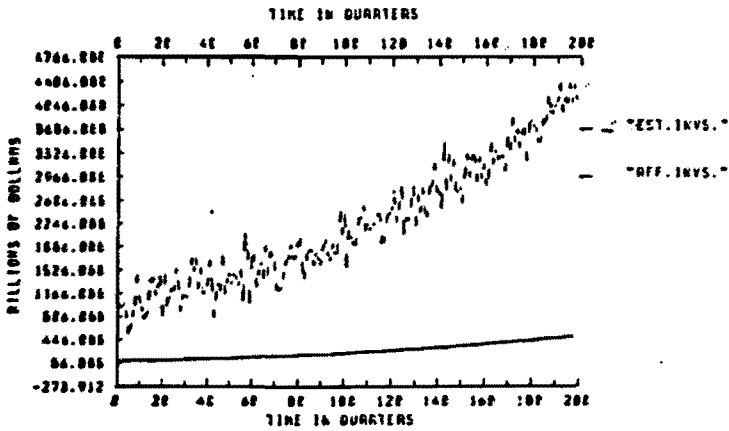


fig. 3-ii

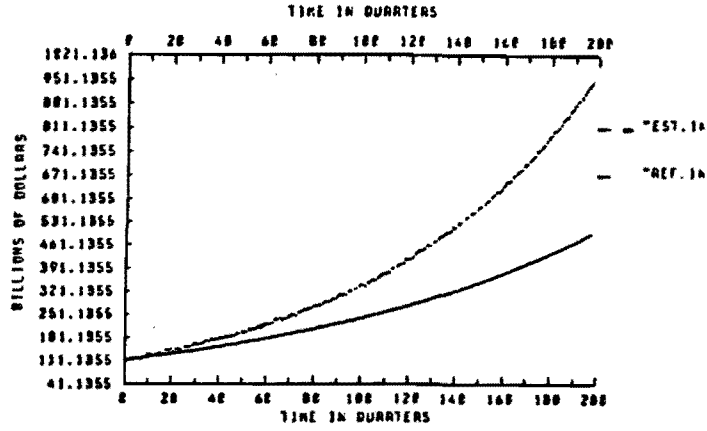


fig. 4-ii

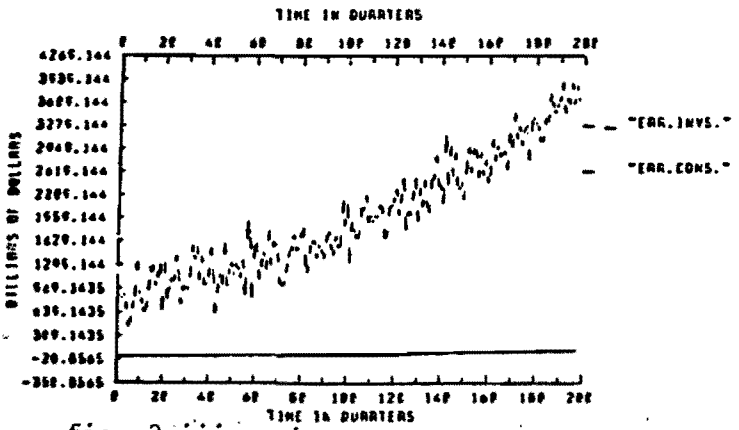


fig. 3-iii

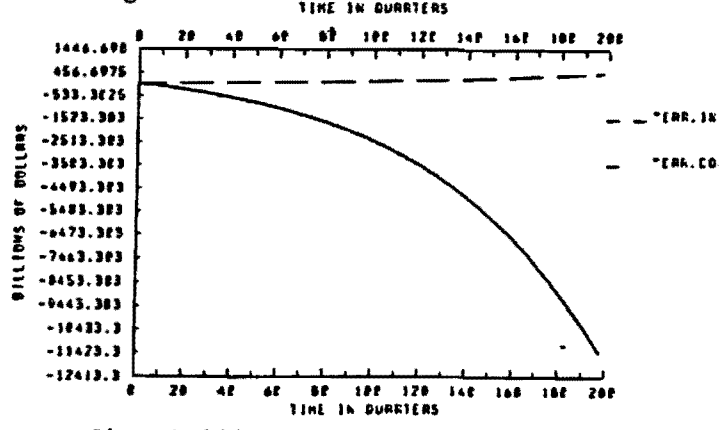


fig. 4-iii

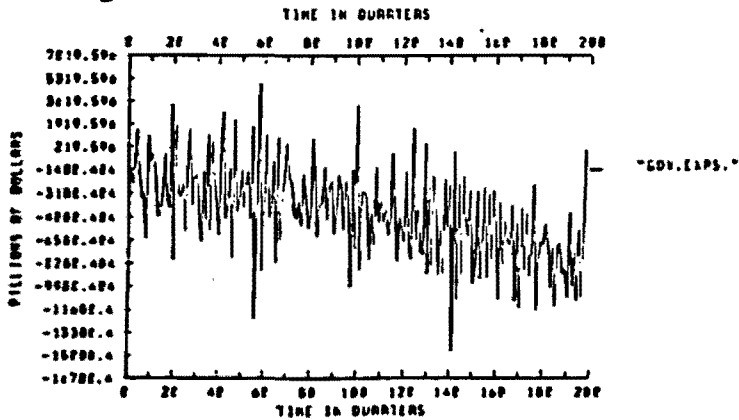


fig. 3-iv

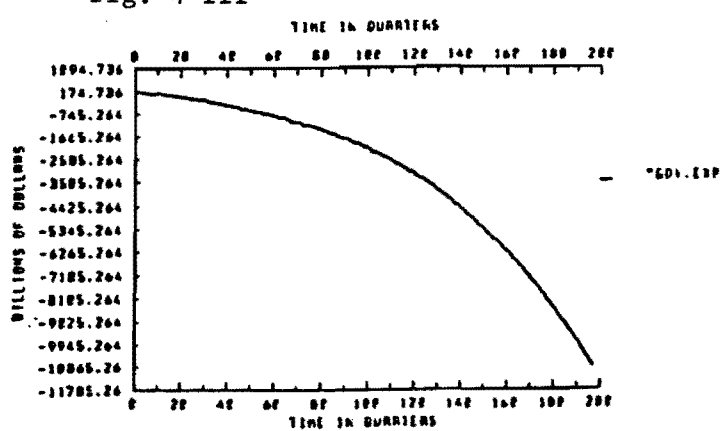


fig. 4-iv

Figures Model II

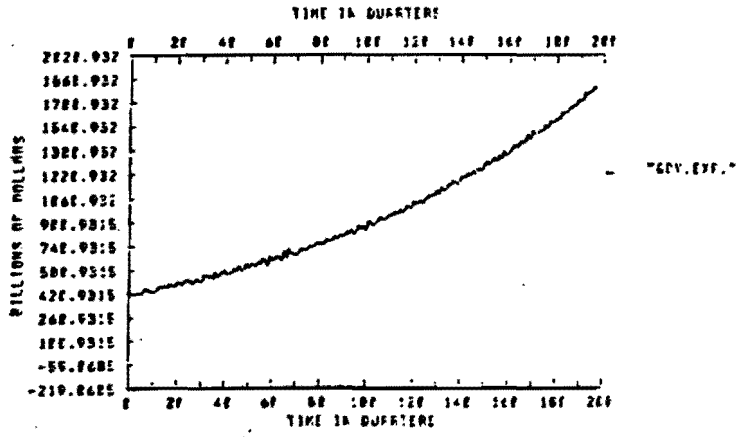


fig. 5