

A finite basis theorem for packing boxes with bricks

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A FINITE BASIS THEOREM FOR PACKING BOXES WITH BRICKS*)

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Consider a catalogue S which lists one to infinitely many shapes of rectangular bricks with positive integer dimensions. Using as many bricks of each shape as needed, the bricks listed in S may be used to completely fill certain rectangular boxes. We assume the shapes to be oriented, i.e. we are not allowed to turn bricks around when trying to fill a box. Thus, a new catalogue $I(S)$ may be formed which lists the (infinitely many) rectangular boxes which may be completely filled with bricks having their shape listed in S . Some of the bricks listed in S may be shapes of boxes which can be filled up completely with smaller bricks listed in S ; in other words, there may be elements $s \in S$ such that $s \in I(S \setminus \{s\})$. The bricks which may be formed with bricks in S smaller than themselves are *composites*. Bricks in S which are not composites are *primes* in S . If $B_r = B_r(S)$ is the set of primes in S , then B_r is non-empty and every box which can be formed with elements of S can be formed with elements of the subset B_r of S ; in other words, $I(B_r) = I(S)$ (see lemma 4). The subject of this note is the remarkable fact that the set of primes $B_r(S)$ is finite for every set S .

The brief history of this problem is as follows: Results involving the tiling of rectangles and three-dimensional boxes with identical polyominoes and poly-cubes are discussed in ref. 3. One sort of result presented there is typified by the following example. The L -tetromino and two of the smallest rectangles it tiles are shown in fig. 1. An $a \times b$ rectangle can be tiled with copies of the L -tetromino

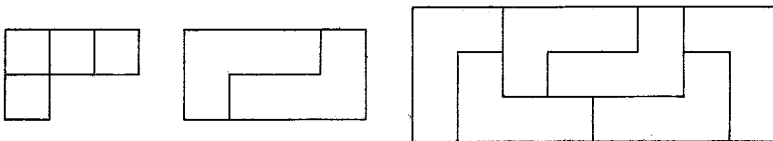


Fig. 1. The L -tetromino and the smallest rectangles it tiles.

*) Dedicated to our mutual friend C.J. Bouwkamp.

if and only if a and b are integers greater than 1 such that 8 divides ab . Also, every $a \times b$ rectangle having integer sides a and b greater than 1 with ab a multiple of 8 can be tiled with 2×4 and 3×8 rectangles.

If S denotes the set of rectangles which can be tiled with L-tetrominoes, one way to characterize S is to list the elements of $B_T(S)$ which are $(2, 4)$, $(3, 8)$, $(4, 2)$, $(8, 3)$; the first two are shown in fig. 1. On reading ref. 3 and doing some investigation of his own, Frits Göbel noted that in every case where one has a characterization of the set S of rectangles which can be tiled with copies of one polyomino, a characterization of S can be given by listing the elements of $B_T(S)$ because this always turned out to be a finite set. Thus, he was led to conjecture that $B_T(S)$ is finite for every set S of rectangular k -dimensional boxes. Bouwkamp (see ref. 1) has done considerable work with a computer to determine the set of prime boxes among the boxes which can be tiled with the Y-pentacube.

Frits Göbel described his conjecture to the second author of the present note who found a proof for the one- and two-dimensional instances of the conjecture. This proof was generalized in collaboration ²⁾, but the proof breaks down in three and higher dimensions (the mistake in the proof is in the bottom third of p. 467 in ref. 2). The proof is illustrated for 2 dimensions, and this correct instance evidently convinced readers of the full generality. The mistake was finally noted and corrected in ref. 4. On reading this correction the first author of the present note thought of a simpler proof. Both proofs are presented here. The first proof is a corrected version of the proof which appears in ref. 2.

Let (d_1, \dots, d_k) denote a k -tuple of positive integers. Such a k -tuple will also be called a *shape*, occasionally written as \vec{d} . Then an (oriented) k -dimensional box with shape (d_1, \dots, d_k) is a set K of $d_1 \dots d_k$ integer points containing one point (the smallest) (c_1, \dots, c_k) such that

$$K = \{(c_1 + x_1, \dots, c_k + x_k) : 0 \leq x_i \leq d_i - 1 \ (i = 1, \dots, k)\}.$$

The set of all k -dimensional boxes with shape (d_1, \dots, d_k) is denoted $\langle d_1, \dots, d_k \rangle$. Boxes with shapes $\vec{d}_1, \vec{d}_2, \dots$ are said to *tile* a box C if there exists a partition $\{C_1, C_2, \dots\}$ of C with $C_1, C_2, \dots \in \langle \vec{d}_1 \rangle \cup \langle \vec{d}_2 \rangle \cup \dots$.

If S is a set of shapes then $T(S)$ is the set of shapes of boxes that can be tiled by S .

We shall also consider a more special kind of tiling (which we might refer to as repeated one-dimensional concatenation). If $\vec{a}, \vec{b}, \vec{c}$ are shapes, and if there is an i ($1 \leq i \leq k$) such that $c_i = a_i + b_i$, and such that $c_j = a_j = b_j$ for all $j \neq i$, then we say that \vec{c} is obtained from \vec{a} and \vec{b} by one-dimensional concatenation.

If S is a set of shapes, we define $\mathcal{A}(S)$ as the smallest set of shapes closed under one-dimensional concatenation, i.e. $\mathcal{A}(S)$ is the smallest set with the properties

- (i) $\Lambda(S) \supseteq S$;
 (ii) if $\bar{a} \in \Lambda(S)$, $\bar{b} \in \Lambda(S)$, and \bar{c} is obtained from \bar{a} and \bar{b} by one-dimensional concatenation, then $\bar{c} \in \Lambda(S)$.

For example, if \bar{a} , \bar{b} , \bar{c} are shapes, $\bar{a} \in S$, $\bar{b} \in S$, if $c_1 = a_1 + b_1$, and if c_j is the least common multiple of a_j and b_j for $2 \leq j \leq k$, then $\bar{c} \in \Lambda(S)$.

It is easy to see that if $\bar{a} \in \Lambda(S)$ then S tiles every box of shape \bar{a} . In other words $\Lambda(S) \subseteq \Gamma(S)$. The following example shows that $\Lambda(S)$ can be smaller than $\Gamma(S)$: If $S = \{(1, 4), (4, 1), (3, 3)\}$ then $(5, 5) \in \Gamma(S)$ but $(5, 5) \notin \Lambda(S)$.

An element \bar{b} of S is called *prime* with respect to Λ and S if $\bar{b} \notin \Lambda(S \setminus \{\bar{b}\})$. The set of all these primes is called $B_\Lambda(S)$ (or B_Λ for short).

The operator Λ has the following properties (for all sets S, T):

- (i) $S \subseteq \Lambda(S)$;
 (ii) if $S \subseteq T$ then $\Lambda(S) \subseteq \Lambda(T)$;
 (iii) $\Lambda(\Lambda(S)) = \Lambda(S)$;
 (iv) if $\bar{a} \in \Lambda(S)$ then there exists a subset U of S with $\bar{a} \in \Lambda(U)$ and such that every shape in U is $\leq \bar{a}$ (we say that $\bar{b} \leq \bar{a}$ if $b_i \leq a_i$ for $i = 1, \dots, k$).

From these properties we derive:

Lemma 1. If $T \subseteq S \subseteq \Lambda(T)$ then $B_\Lambda(S) \subseteq T$.

Proof. Let \bar{b} be any element of $B_\Lambda(S)$. By the definition of $B_\Lambda(S)$ we have $\bar{b} \notin \Lambda(S \setminus \{\bar{b}\})$. Now by (ii): $\bar{b} \notin \Lambda(T \setminus \{\bar{b}\})$. Since $B_\Lambda(S) \subseteq S \subseteq \Lambda(T)$ we have $\bar{b} \in \Lambda(T)$, and it follows that $\Lambda(T \setminus \{\bar{b}\}) \neq \Lambda(T)$, whence $\bar{b} \in T$.

Lemma 2. $\Lambda(B_\Lambda(S)) = \Lambda(S)$.

Proof. According to (i), (ii), (iii) and to $B_\Lambda(S) \subseteq S$ it suffices to show that $S \subseteq \Lambda(B_\Lambda(S))$. We do this by showing that if $S \setminus \Lambda(B_\Lambda(S))$ contains a shape \bar{a} then it contains a shape \bar{b} with $\bar{b} \leq \bar{a}$, $\bar{b} \neq \bar{a}$ (and that cannot go on for ever!). Let \bar{a} be such a shape. Then $\bar{a} \notin B_\Lambda(S)$, whence $\bar{a} \in \Lambda(S \setminus \{\bar{a}\})$. By (iv) we can find U such that $U \subseteq S \setminus \{\bar{a}\}$, $\bar{a} \in \Lambda(U)$ and such that all shapes in U are $\leq \bar{a}$. We cannot have $U \subseteq \Lambda(B_\Lambda(S))$ since that would imply $\bar{a} \in \Lambda(B_\Lambda(S))$. So we can take $\bar{b} \in U$ such that $\bar{b} \notin \Lambda(B_\Lambda(S))$. Since $\bar{b} \in U \subseteq S \setminus \{\bar{a}\}$ we have $\bar{b} \leq \bar{a}$, $\bar{b} \in S$, $\bar{b} \neq \bar{a}$. Hence $\bar{b} \in S \setminus \Lambda(B_\Lambda(S))$.

Lemma 3. If $T \subseteq S \subseteq \Gamma(T)$ then $B_\Gamma(S) \subseteq T$.

Lemma 4. $\Gamma(B_\Gamma(S)) = \Gamma(S)$.

Proofs of lemmas 3 and 4 are obtained from those of lemmas 1 and 2 by replacing all Λ 's by Γ 's.

We now express our main result as a theorem.

Theorem. Let S denote a set of k -dimensional shapes. Then we have

- (i) S has a finite subset T such that $S \subseteq \Gamma(T)$ (whence $\Gamma(S) = \Gamma(T)$);
- (ii) $B_\Gamma(S)$ is finite;
- (iii) S has a finite subset T such that $S \subseteq \Lambda(T)$ (whence $\Lambda(S) = \Lambda(T)$);
- (iv) $B_\Lambda(S)$ is finite.

Remark. It follows from the lemmas that (i) is equivalent to (ii), and that (iii) is equivalent to (iv). Furthermore (iii) implies (i) (since $\Lambda(T) \subseteq \Gamma(T)$), hence it suffices to prove either (iii) or (iv).

First proof. (Corrected version of the proof given in ref. 2; it will prove the theorem in the form (iv).)

The proof is by induction on the dimension k of the boxes. For $k = 1$, S may be regarded as a set of positive integers and $B_\Lambda(S)$ is the smallest subset of S such that every number in S is a non-negative linear combination of elements of $B_\Lambda(S)$. Suppose the greatest common divisor of elements of S is d ; then it is easy to show that some finite subset F of S has greatest common divisor equal to d . Hence, since every large multiple of d is a non-negative linear combination of elements of F , all but a finite subset E of S is contained in $\Lambda(F)$. Hence, $S \subseteq \Lambda(F \cup E)$ and $B_\Lambda(S) \subseteq F \cup E$. This proves $B_\Lambda(S)$ is finite in the 1-dimensional case.

Now suppose $B_\Lambda(S)$ is finite for every set S of k -dimensional shapes for $k = 1, \dots, n-1$ with $n \geq 2$. But, suppose there exists a set T of n -dimensional boxes with $B_\Lambda(T)$ infinite (this leads to a contradiction). Without loss of generality it can be supposed that $B_\Lambda(T) = T$. Also, let the elements of T be put in a sequential order $(\bar{t}_1, \bar{t}_2, \dots)$ with $\bar{t}_i = (t_{i1}, \dots, t_{in})$ for $i = 1, 2, \dots$. Each of the n sequences $(t_{i1} : i = 1, 2, \dots), \dots, (t_{in} : i = 1, 2, \dots)$ must tend to infinity. For example, if the first sequence $(t_{i1} : i = 1, 2, \dots)$ does not tend to infinity, then there exists an infinite constant subsequence $(t_{i1} : i \in I)$ so that the infinite subsequence $(\bar{t}_i : i \in I)$ of $(\bar{t}_i : i = 1, 2, \dots)$ contains boxes all having the same height. It follows from the induction hypothesis that the set $\{\bar{t}_i : i \in I\}$ has finite basis with respect to Λ . A moment's thought may be required here because a detail is being passed over. At bottom, we are using the fact that having boxes all of one height really amounts to dealing with boxes of one lower dimension.

Now we select an infinite subsequence R of T and show that $B_\Lambda(R)$ is finite. This provides the contradiction we seek since this implies there exists an element (in fact, infinitely many elements) of T which is composite. (Supposing $T = B_\Lambda(T)$ means no element of T is composite.)

Given n -tuples $\bar{a} = (a_1, \dots, a_n), \bar{b} = (b_1, \dots, b_n)$ of positive integers, \bar{a} is said to divide \bar{b} if a_i divides b_i for $i = 1, \dots, n$. This notion of division leads to a notion of greatest common divisor of two or more n -tuples of positive integers.

We form the subsequence $R = (\bar{r}_1, \bar{r}_2, \dots)$ as follows: Let $\bar{r}_1 = \bar{t}_1$, and note that \bar{r}_1 has only a finite number of divisors. Hence, \bar{r}_1 has the same greatest common divisor \bar{d}_1 with each one of an infinite set of subsequent terms of T for some divisor \bar{d}_1 of \bar{r}_1 . Let T_2 denote such an infinite subsequence of $(\bar{t}_2, \bar{t}_3, \dots)$, and let \bar{r}_2 denote the first term of T_2 . Now this process is repeated with T_2 in place of T and \bar{r}_2 in place of \bar{r}_1 , thus forming a new subsequence T_3 of T_2 with \bar{r}_2 deleted, and \bar{r}_3 is defined to be the first term of T_3 . Thus, the i th term of R is found by $i - 1$ repetitions of this process. It follows from this construction that R has the peculiar property that, for each i , the greatest common divisor of \bar{r}_i with all subsequent terms of R is the same n -tuple \bar{d}_i for $i = 1, 2, \dots$. Any sequence with this property is said to be *stable*. Note that every subsequence of a stable sequence is again stable. Now we show by construction that $B_A(R)$ is finite.

Let

$$\delta_j = d_{2^{n-1}, j} \quad \text{for } j = 1, \dots, n$$

where

$$\bar{d}_{2^{n-1}} = (d_{2^{n-1}, 1}, \dots, d_{2^{n-1}, n}),$$

then we shall show there exists an integer p such that $A(\{\bar{r}_1, \dots, \bar{r}_{2^n}\})$ contains all boxes having shape $(p_1 \delta_1, \dots, p_n \delta_n)$ with $p_1, \dots, p_n \geq p$. This implies $B_A(R)$ is finite since all but a finite subset of the elements of R have this form. Let $\mu_i(s)$ denote the least common multiple of $r_{1,i}, \dots, r_{2^s,i}$ for $i, s = 1, \dots, n$ where $\bar{r}_i = (r_{i1}, \dots, r_{in})$ for $i = 1, 2, \dots$. Now we show by induction on j :
(A_j). For every stable sequence there exists a number p_j such that every shape

$$(q_1 \delta_1, \dots, q_j \delta_j, \mu_{j+1}(j), \dots, \mu_n(j)) \quad (1)$$

with $q_1, \dots, q_j \geq p_j$ is an element of $A(\{\bar{r}_1, \dots, \bar{r}_{2^j}\})$.

For $j = 1$, $x, y = 1, 2, \dots$, the shape

$$(r_{11}x + r_{21}y, \mu_2(1), \dots, \mu_n(1))$$

may be formed from \bar{r}_1 and \bar{r}_2 by repeated one-dimensional concatenation. But, there exists an integer p_1 such that $q_1 \delta_1$ has the form $r_{11}x + r_{21}y$ with $x, y \geq 1$ for all $q_1 \geq p_1$ because the greatest common divisor of r_{11} and r_{21} divides δ_1 . This proves A_1 .

Now suppose A_j is true for some $j \geq 1$; we shall prove A_{j+1} . Let $\mu_i'(s)$ denote the least common multiple of $r_{2^{s+1}, i}, \dots, r_{2^{s+1}, i}$, and note that A_j also applies to the stable sequence $(\bar{r}_t: t = 2^j + 1, \dots, 2^{j+1})$. Thus, there exists a number p_j' such that every shape

$$(q_1 \delta_1, \dots, q_j \delta_j, \mu_{j+1}'(j), \dots, \mu_n'(j)) \quad (2)$$

where $q_1, \dots, q_j \geq p_j'$ is an element of $A(\{\bar{r}_{2^j+1}, \dots, \bar{r}_{2^{j+1}}\})$. Now we apply

repeated one-dimensional concatenation to shapes given in (1) and (2) to obtain the shape

$$q_1 \delta_1, \dots, q_j \delta_j, x \mu_{j+1}(j) + y \mu_{j+1}'(j), \mu_{j+2}''(j), \dots, \mu_n''(j) \tag{3}$$

for all $q_1, \dots, q_j \geq \max(p_j, p_j')$, $x, y = 1, 2, \dots$ where μ_i'' denotes the least common multiple of μ_i and μ_i' . Now observe that the greatest common divisor of $\mu_{j+1}(j)$ and $\mu_{j+1}'(j)$ divides δ_{j+1} , so there exists an integer

$$p_{j+1} \geq \max(p_j, p_j')$$

such that every number $q_{j+1} \delta_{j+1}$ with $q_{j+1} \geq p_{j+1}$ has the form

$$x \mu_{j+1}(j) + y \mu_{j+1}'(j)$$

with $x, y \geq 1$. Also, note that $\mu_i(j)$ and $\mu_i'(j)$ have the least common multiple $\mu_i(j + 1)$ by definition for $j = 1, \dots, n$. Hence A_j implies A_{j+1} . This completes the proof.

Second proof. We shall prove (iii); the idea is modelled after the following proof (slightly longer than necessary) for the case $k = 1$. Let S be a set of positive integers. If S is empty, nothing has to be proved. If S is not empty, choose an $h \in S$. For any r ($0 \leq r < h$) with the property that $\mathcal{A}(S)$ contains a number $\equiv r \pmod{h}$ we select such a number. The set of selected numbers is denoted by D . Let q be some positive upper bound for the elements of this finite set D . We now have $\mathcal{A}(S) = \mathcal{A}(\{h\} \cup D \cup \{1, \dots, q - 1\})$ since every element of S exceeding $q - 1$ is the sum of an element of D and a non-negative multiple of h .

We next proceed by induction with respect to the dimension k . We take $n > 1$ and assume the theorem correct for $k = n - 1$. One of the dimensions is singled out and referred to as "height". If $\alpha = (a_1, \dots, a_n)$ is an n -dimensional shape we write $(a_1, \dots, a_{n-1}) = \alpha^*$ (the "cross-section") and $a_n = \text{height}(\alpha)$. Furthermore, if t is a positive integer we write $\alpha^{*t} = (a_1, \dots, a_{n-1}, t)$.

As shapes of constant height can be treated as shapes of lower dimension, we note that if $\alpha, \alpha_1, \dots, \alpha_p$ are such that $\alpha^* \in \mathcal{A}(\{\alpha_1^*, \dots, \alpha_p^*\})$ then

$$\alpha^{*t} \in \mathcal{A}(\{\alpha_1^{*t}, \dots, \alpha_p^{*t}\}).$$

Let S be a set of n -dimensional shapes. We put $Y = \{\alpha^* : \alpha \in S\}$. By the induction hypothesis Y has a finite subset Z with $Y \subseteq \mathcal{A}(Z)$. Take $\alpha_1, \dots, \alpha_p \in S$ such that $Z = \{\alpha_1^*, \dots, \alpha_p^*\}$. We put

$$X = \{\alpha_1, \dots, \alpha_p\}, \quad h = \prod_{i=1}^p \text{height}(\alpha_i).$$

Since h is a multiple of $\text{height}(\alpha_i)$, we have $\alpha_i^{*h} \in \mathcal{A}(\{\alpha\})$. If $\alpha \in S$ then $\alpha^* \in Y$, whence $\alpha^* \in \mathcal{A}(\{\alpha_1^*, \dots, \alpha_p^*\})$. Therefore $\alpha^{*h} \in \mathcal{A}(\{\alpha_1^{*h}, \dots, \alpha_p^{*h}\})$. Since h is a multiple of $\text{height}(\alpha_i)$, we have $\alpha_i^{*h} \in \mathcal{A}(X)$. Thus we have proved: if $\alpha \in S$ then $\alpha^{*h} \in \mathcal{A}(X)$.

For $0 \leq r < h$ we consider

$$T_r = \{\alpha^* : \alpha \in S, \text{ height } (\alpha) \equiv r \pmod{h}\}.$$

By the induction hypothesis T_r has a finite subset R_r with $T_r \subseteq \Lambda(R_r)$. We select a finite subset Q_r of S such that $R_r = \{\alpha^* : \alpha \in Q_r\}$. Let q be a positive integer such that $\text{height } (\alpha) \leq q$ for all $\alpha \in R_r$ and all r ($0 \leq r < h$).

Let γ be any element of S whose height m exceeds q ; let r satisfy

$$0 \leq r < h, \quad m - r \equiv 0 \pmod{h}.$$

For every $\alpha \in Q_r$ we have $\alpha^{*h} \in \Lambda(X)$, and since

$$m > q \geq \text{height } (\alpha), \quad m \equiv \text{height } (\alpha) \pmod{h},$$

we conclude $\alpha^{*m} \in \Lambda(X \cup Q_r)$. Furthermore $\gamma^* \in T_r \subseteq \Lambda(R_r)$, whence

$$\gamma = \gamma^{*m} \in \Lambda(\{\alpha^{*m} : \alpha \in Q_r\}) \subseteq \Lambda(X \cup Q_r).$$

Consequently: if $\beta \in S$, $\text{height } (\beta) > q$, then $\beta \in \Lambda(X \cup Q_0 \cup \dots \cup Q_{h-1})$.

If $1 \leq i \leq q$ we consider the sets

$$W_i = \{\alpha : \alpha \in S, \text{ height } (\alpha) = i\}, \quad V_i = \{\alpha^* : \alpha \in W_i\}.$$

By the induction hypothesis V_i has a finite subset U_i with $V_i \subseteq \Lambda(U_i)$. Let P_i be the set of all shapes with height i and cross-section in V_i . Then $W_i \subseteq \Lambda(P_i)$.

Our final conclusion is that

$$\alpha \in \Lambda(X \cup R_0 \cup \dots \cup R_{h-1} \cup V_1 \cup \dots \cup V_q)$$

for every $\alpha \in S$. This proves (iii).

The theorem has the following interesting corollary. Let S be any set of shapes. A box with its shape in $\Gamma(S)$ is called *cleavable* if some hyperplane cuts it into two non-empty boxes with shapes in $\Gamma(S)$. Thus a *non-cleavable* box is a box which can be tiled with boxes having their shape listed in S , but never with a "fault" hyperplane. The corollary is that the set of shapes of non-cleavable boxes is either empty or finite. Note that every shape in B_r is non-cleavable, and that every non-cleavable shape is in B_A .

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