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# A note on local Morse theory in State-Scale space and Gaussian deformations

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## 1 Introduction

In computer vision (see [5, 6]) the main problem is to identify and manipulate objects in a computer screen image. In general the image is given by a "pixel intensity" function. The scale on which the image is considered is changed by applying "blurring". In most literature Gaussian blurring is considered. Starting with an intensity function  $u_0(x)$  on  $\mathbb{R}^n$  Gaussian blurring yields a family of intensity functions  $u(x; \tau)$ , where  $\tau$  can be considered the scale parameter. When considering Gaussian blurring these functions have to satisfy the diffusion equation

$$\frac{\partial u}{\partial \tau} = \Delta u, \tag{1}$$

with  $u(x, 0) = u_0(x)$ ,  $u_0 : \mathbb{R}^n \rightarrow \mathbb{R} \in C^0(\mathbb{R}^n, \mathbb{R})$ . In [4] it is shown that the axioms of state-scale space allow also blurring with respect to the operators  $-(-\Delta)^\alpha$ ,  $0 < \alpha < 1$ . Through blurring a less detailed image is obtained by diffusion of the intensity function which is equivalent to considering a larger scale. This way the image can be blurred in such a way that it only contains the relevant information. At some stages one may however want to reconstruct parts of the image in more detail.

In the following we want to understand the behavior of the intensity functions, especially the qualitative changes the intensity functions will undergo when changing the scale parameter. We will at first restrict to the case of two dimensional images. To get control over the state space we choose to interpret the level lines of the intensity function as integral curves of an Hamiltonian system or equipotential lines of a gradient vector field, therefore we require the intensity functions to be sufficiently differentiable. The scale parameter can now be considered as a deformation parameter of some planar Hamiltonian system or gradient vector field. Consequently, the question reduces to studying bifurcations of vector fields. These vector fields are however parameter dependent in a particular

way. In this paper we introduce the new concept of semigroup deformations and show that this is the proper framework for studying bifurcations under Gaussian blurring. We will illustrate the theory by applying it to Gaussian deformations, that is, work with the semigroup generated by the Laplace operator. For semigroups generated by the operators  $-(-\Delta)^\alpha$ ,  $0 < \alpha < 1$ , the theory should apply as well. Because in this case the computations are much more complicated this will be the subject of a separate paper. Compared to the singularity theoretic approach in [3] we circumvent the problem of defining the right group of transformations by applying geometric arguments. We obtain different results. The origin of these differences lies probably in the fact that the underlying group of transformations will be different from the ones chosen in [3].

## 2 Bifurcations of planar Hamiltonian systems

When we consider a function  $H : \mathbb{R}^2 \rightarrow \mathbb{R}; (x, y) \rightarrow H(x, y)$ ,  $H \in C^2(\mathbb{R}^2, \mathbb{R})$ , then the curves  $H(x, y) = c$  are integral curves of the Hamiltonian differential equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad (2)$$

$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}. \quad (3)$$

The right-hand-side of these equations are usually denoted as the Hamiltonian vector field  $X_H(x, y)$ . Singular points of  $H$  are stationary points of the vector field. The nature of a stationary point can easily be detected by considering the linearized system at the stationary point. The matrix for the linearized system is given by the  $Jd^2H$ , with  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , and  $d^2H$  is the Hessian of  $H$  evaluated at the stationary point. The possible types of stationary points are limited by the following

**Proposition 2.1** *If  $\lambda$  is an eigenvalue of a linear Hamiltonian system then so are  $-\lambda$  and  $\bar{\lambda}$  (the complex conjugate).*

Consequently in the planar case we have either a centre, a saddle or a double zero eigenvalue. So being a saddle or a centre is the generic situation for a stationary point. This is in agreement with the more general local Morse theory in [3]. In the case of a double zero eigenvalue we may apply bifurcation theory. Because in state-scale space we only have one scale parameter this leads to the conclusion that generic bifurcations in state-scale space can be dealt with in the same way as the generic one parameter bifurcations of planar Hamiltonian vector fields. In the next section we will focus on these generic one parameter bifurcations taking into account that the behavior in the scale-parameter direction is governed by the diffusion equation.

Note that the same results could have been obtained by considering the gradient vector field with potential function  $H$ . The gradient vector field approach can easily be generalized to higher dimensions.

### 3 Preliminaries on stability, deformations and unfoldings

Unfoldings or deformations of Hamiltonian vector fields are in general studied through the unfoldings and deformations of the corresponding Hamiltonian function.

We will start with recalling some facts from singularity theory and/or bifurcation theory ([1, 2, 8]) in order to reveal the precise meaning of the terminology used.

Let  $\mathcal{G}$  be a group of transformations acting on the space of functions. (or a local group at 0 acting on the space of germs of functions). For instance  $\mathcal{G} = \mathcal{A} = Diff_n \times Diff_1$ , with  $Diff_n$  the group of  $C^\infty$  diffeomorphisms from  $\mathbb{R}^n$  to itself, acting by  $g \cdot f(x) = \psi \circ f \circ \phi^{-1}(x)$ ,  $g \in \mathcal{G}$ ,  $\phi \in Diff_n$  and  $\psi \in Diff_1$ .

Damon [3] introduces  $\mathcal{G} = \mathcal{H}$ , with  $\mathcal{H}$  the group of pairs  $(\phi, c)$ ,  $\phi \in Diff_{n+1}$  of the form  $\phi(x, t) = (\phi_1(x, t), \phi_2(t))$  with  $\phi_2'(0) > 0$  and  $c \in Diff_1$ , and acting by  $g \cdot f(x, t) = f \circ \phi(x, t) + c(t)$ . In addition  $\mathcal{G} = \mathcal{IS}$  is introduced in [3], with  $\mathcal{IS}$  the group of pairs  $(\phi, \psi)$ ,  $\phi \in Diff_{n+1}$  of the form  $\phi(x, t) = (\phi_1(x, t), \phi_2(t))$  with  $\phi_2'(0) > 0$  and  $\psi \in Diff_2$  of the form  $\psi(y, t) = (\psi_1(y, t), t)$  with  $(\partial\psi_1/\partial y)(0, 0) > 0$  and  $\psi_1(0, t) = 0$ , and acting by  $g \cdot f(x, t) = \psi_1(f \circ \phi(x, t), t) + c$ . The groups  $\mathcal{H}$  and  $\mathcal{IS}$  should be considered as local groups of germs of diffeomorphisms at the origin acting on germs of functions. In the sequel we will consider group actions of groups that do not depend on the scale parameter. Taking the scale parameter  $t$  in  $\mathcal{IS}$  equal to a constant this group reduce to  $\mathcal{A}$  with adding constants, i.e we obtain the group  $\mathcal{IS}_c$  which is the group of triples  $(\phi, \psi, c)$ ,  $\phi \in Diff_n$ ,  $\psi \in Diff_1$  with  $(\partial\psi/\partial y)(0) > 0$  and  $\psi(0) = 0$ ,  $c$  a constant, and acting by  $g \cdot f(x) = \psi(f \circ \phi(x)) + c$ . Note that Damon considers groups of diffeomorphisms on  $\mathbb{R}^{n+1}$ , that is, the parameter is included. In this paper mainly groups of diffeomorphism on  $\mathbb{R}^n$  will be considered. The parameter will be introduced by unfolding.

**Definition 3.1** *Two functions  $f, h$  from  $\mathbb{R}^n$  to  $\mathbb{R}$  are  $\mathcal{G}$ - equivalent if*

$$f = g \cdot h$$

for some  $g \in \mathcal{G}$ .

Note that the notion of equivalence depends on the choice of the group of transformations chosen.

**Definition 3.2** Let  $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ . A  $s$ -parameter unfolding of  $f_0$  is a map  $f : \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R} \times \mathbb{R}^s$  such that

- i.  $f(x, u) = (\tilde{f}(x, u), u)$ , where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^s$ ,  $\tilde{f}(x, u) \in \mathbb{R}$ , i.e.  $\pi \circ f = \pi$ , where  $\pi : \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}^s$  and  $\pi : \mathbb{R} \times \mathbb{R}^s \rightarrow \mathbb{R}^s$  are the canonical projections.
- ii.  $f_0(x) = \tilde{f}(x, 0)$ .

In practice one often calls  $\tilde{f}(x, u)$  an unfolding of  $f_0$ , although, if one wants to be precise  $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}$  is actually a **deformation** of  $f_0$ . The **constant unfolding** is the unfolding  $f$  of  $f_0$  with  $\tilde{f}(x, u) = f_0(x)$ .

When considering unfoldings of functions  $f(x, u)$  we may also consider a group of transformations  $\tilde{\mathcal{G}}$  consisting of unfoldings of  $\mathcal{G}$  acting on these unfolded functions. A transformation  $g(x, u) \in \tilde{\mathcal{G}}$  is an **unfolding of the identity** if  $g(x, 0) = id$ . We denote the unfoldings of the identity by  $\mathcal{G}_{un}$ .

Let  $f$  be a  $s$ -parameter unfolding of  $f_0$ . Consider a map  $\chi$  given by  $\chi : v \rightarrow u = \chi(v)$ , i.e. a transformation acting on the parameter space is considered. The **pull-back** of  $f$  by  $\chi$  is the  $t$ -parameter unfolding

$$\chi^* f : \mathbb{R}^n \times \mathbb{R}^t \rightarrow \mathbb{R} \times \mathbb{R}^t; (x, v) \rightarrow (\tilde{f}(x, \chi(v)), v) .$$

**Definition 3.3** Two  $s$ -parameter unfoldings  $f$  and  $h$  of  $f_0$  are **equivalent** if there exists a  $g \in \mathcal{G}_{un}$  such that

$$h = g \cdot f .$$

An unfolding is **trivial** if it is equivalent to the constant unfolding. An unfolding  $f$  of  $f_0$  is **universal** if every unfolding of  $f_0$  is equivalent to  $\chi^* f$  for some mapping  $\chi$ .

For  $\mathcal{A}_{un}$  equivalence this means that

$$h = \psi \circ f \circ \varphi$$

with

$$\varphi : \mathbb{R}^{n+s} \rightarrow \mathbb{R}^{n+s}; (x, t) \rightarrow (\tilde{\varphi}(x, t), t) ,$$

and

$$\psi : \mathbb{R}^{1+s} \rightarrow \mathbb{R}^{1+s}; (y, t) \rightarrow (\tilde{\psi}(y, t), t) ,$$

where both  $\tilde{\varphi}$  and  $\tilde{\psi}$  are unfoldings of the identity, i.e.  $\tilde{\varphi}(x, 0) = x$  and  $\tilde{\psi}(x, 0) = x$ . Two arbitrary unfoldings  $f$  and  $h$  of  $f_0$  are equivalent if  $h$  is  $\mathcal{G}_{un}$  equivalent to  $\chi^*f$  for some  $C^\infty$  map  $\chi$ .

With abuse of language we will say in the remainder of this paper that unfoldings  $f$  and  $h$  are  $\mathcal{G}$ -equivalent, where the action of the maps is as given above.

Like before a trivial or universal unfolding gives rise to a trivial or universal deformation.

Moreover

**Definition 3.4** *A function  $f_0$  is stable if any unfolding of  $f_0$  is trivial.*

The notion of stability can be rephrased as follows. A function  $f_0$  is stable if all functions that are close to  $f_0$  (in an appropriate topology) are equivalent to  $f_0$ , i.e. are in the  $\mathcal{G}$ -group orbit through  $f_0$ . Stability is usually considered through the equivalent notion of infinitesimal stability, i.e. formulated in terms of the tangent space to the orbit. Let  $f_0$  be in  $\mathcal{E}_0$ , the space of smooth germs of functions at zero. Let  $T_{\mathcal{G}}(f_0)$  denote the tangent space at  $f_0$  to the  $\mathcal{G}$ -orbit through  $f_0$ .

**Proposition 3.5**  *$f_0 \in \mathcal{E}_0$  is stable if and only if  $T_{\mathcal{G}}(f_0) = \mathcal{E}_0$ .*

Thus  $f_0$  is stable if and only if the complement to the tangent space to the  $\mathcal{G}$ -orbit at  $f_0$  is empty. If  $f_0$  is not stable then define

**Definition 3.6** *The  $\mathcal{G}$ -codimension of  $f \in \mathcal{E}_0$  is*

$$d(f, \mathcal{G}) := \dim(\mathcal{E}_0/T_{\mathcal{G}}(f)) .$$

If the co-dimension is non-zero the nontrivial deformations of  $f_0$  can be found by unfolding  $f_0$  in the directions which are in the complement to the tangent space. The tangent space can be given the form of a module of vector fields. Also describing the complement by a basis of vector fields  $X_i$ , we may consider the one-parameter groups  $e^{tX_i}$  acting on the space of functions. Then  $f(x, t) = e^{tX_i}f_0(x)$  gives a deformation of  $f_0$  in the direction of  $X_i$  with initial speed  $X_i f_0$ , where  $X_i f_0$  is the derivative of  $f_0$  along  $X_i$ . A deformation is a universal deformation if the unfolding directions span the complement to the tangent space to the orbit at  $f_0$ . Therefore a universal deformation is stable.

## 4 Deformations by semi-groups

If the unfolding directions are prescribed to follow the orbit of a semigroup, for instance because the deformation is governed by some partial differential equation, we speak of a

semigroup deformation. Let  $S_\tau$ ,  $\tau > 0$  be a semigroup, then the semigroup deformation of  $f_0$  is a function  $f(x, \tau) = S_\tau f_0$ , with  $f : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ . By extending the domain of the parameter  $\tau$  to  $\mathbb{R}$  we obtain  $F(x, \tau)$  with  $F : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ .

**Theorem 4.1**  $F(x, \tau)$  is  $\mathcal{G}$ -stable if and only if either  $f_0$  is stable or  $d(f_0, \mathcal{G})=1$  and  $\frac{\partial S_\tau f_0}{\partial \tau}|_{\tau=0}$  generates the complement of  $T_{\mathcal{G}}(f_0)$  in  $\mathcal{E}_0$ .

**Proof.** If  $f_0$  is stable then any unfolding is trivial and hence stable. If  $f_0$  is of co-dimension 1 then the unfolding  $F(x, \tau)$  is stable if and only if it is universal.  $\square$

In the above the notion of stability is used with respect to the full group action. A semigroup deformation, by definition, only exists for  $\tau > 0$ . That is, we have a **one-sided deformation**. As a consequence the initial speed  $\frac{\partial S_\tau f_0}{\partial \tau}|_{\tau=0}$  of the deformation has to be taken with its direction. Therefore in the  $\mathcal{G}$ -codimension-one case we can at most obtain half of  $\mathcal{E}_0$  because the diffeomorphisms acting on the deformation must respect the sign of  $\tau$ . Consequently the deformation does not cover a full neighborhood of  $f_0$  but only a halfspace. To cover the other half we need an other deformation, which, with respect to the full group, is equivalent to the previous one. The two are not equivalent if we restrict our group action to the proper halfspace and consider stability with respect to the restricted group action. To make this precise, if we have that  $\mathbb{R}\{\frac{\partial S_\tau f_0}{\partial \tau}|_{\tau=0}\} + T_{\mathcal{G}}(f_0) = \mathcal{E}_0$  considered as modules, then  $F(x, \tau)$  is a universal deformation. With  $\mathbb{R}_+ = \{\tau \in \mathbb{R} | \tau > 0\}$  we obtain for a semigroup deformation  $f(x, \tau) = S_\tau f_0$  the halfspace  $\mathbb{R}_+\{\frac{\partial S_\tau f_0}{\partial \tau}|_{\tau=0}\} + T_{\mathcal{G}}(f_0)$  which is half of the tangent space if  $F(x, \tau)$  is a nontrivial universal deformation.

**Definition 4.2** Two non-trivial one-parameter deformations  $h(x, \tau)$  and  $g(x, \tau)$  of  $f(x)$  are one-sided  $\mathcal{G}$ -equivalent if they are  $\mathcal{G}$ -equivalent and

$$\mathbb{R}_+\left\{\frac{\partial h}{\partial \tau}\Big|_{\tau=0}\right\} + T_{\mathcal{G}}(f) = \mathbb{R}_+\left\{\frac{\partial g}{\partial \tau}\Big|_{\tau=0}\right\} + T_{\mathcal{G}}(f).$$

Now if  $f_0$  and  $g_0$  are  $\mathcal{G}$ -equivalent then also universal deformations  $f$  and  $g$  are equivalent. This need not be true for semigroup deformations because  $\mathbb{R}_+\{\frac{\partial S_\tau f_0}{\partial \tau}|_{\tau=0}\} + T_{\mathcal{G}}(f_0)$  and  $\mathbb{R}_+\{\frac{\partial S_\tau g_0}{\partial \tau}|_{\tau=0}\} + T_{\mathcal{G}}(g_0)$  need not be equivalent, i.e. diffeomorphic by a  $\tau$  dependent map which respects the sign of  $\tau$ .

**Definition 4.3** Two non-trivial semigroup deformations  $S_\tau f_1$  and  $S_\tau f_2$  are one-sided  $\mathcal{G}$ -equivalent if  $f_1$  and  $f_2$  are  $\mathcal{G}$ -equivalent, i.e. if there exists a  $g \in \mathcal{G}$  such that  $f_1 = g \cdot f_2$ , and  $S_\tau f_1$  and  $g \cdot S_\tau f_2$  are on-sided  $\mathcal{G}$ -equivalent as one parameter deformations of  $f_1$

Here in  $g \cdot S_\tau f_2$  the action of  $g \in \mathcal{G}$  is on the  $x$  variable only. Consequently  $g \cdot S_\tau f_2$  is an unfolding of  $g \cdot f_2$ .



**Definition 4.4** A non-trivial semigroup deformation  $S_\tau f_0$  is one-sided  $\mathcal{G}$ -stable if any semigroup deformation  $S_\tau f_1$  such that  $f_0 = g \cdot f_1$  for some  $g \in \mathcal{G}$  and such that  $\frac{\partial S_\tau f_0}{\partial \tau}(x, 0)$  and  $\frac{\partial g \cdot S_\tau f_1}{\partial \tau}(x, 0)$  have the same sign as vectors in  $\mathcal{E}_0$  is one-sided  $\mathcal{G}$ -equivalent to  $S_\tau f_0$ .

Note that two one-sided stable semigroup deformations need not be one-sided equivalent. They might lie on different sides of the tangent space to the orbit through  $f_0$ .

For trivial deformations we have to adjust our definition.

**Definition 4.5** A trivial semigroup deformation  $S_\tau f_0$  is two-sided  $\mathcal{G}$ -stable if  $f_0$  is  $\mathcal{G}$ -stable.

Note that in this case

$$\mathbb{R}_+ \left\{ \frac{\partial S_\tau f_0}{\partial \tau} \Big|_{\tau=0} \right\} + T_{\mathcal{G}}(f_0) = T_{\mathcal{G}}(f_0) = \mathcal{E}_0.$$

**Theorem 4.6**

- (i) If for a semigroup deformation  $S_\tau f_0$ ,  $F(x, \tau)$  is a non-trivial  $\mathcal{G}$ -stable deformation then  $S_\tau f_0$  is one-sided  $\mathcal{G}$ -stable.
- (ii) If for a semigroup deformation  $S_\tau f_0$ ,  $F(x, \tau)$  is a trivial  $\mathcal{G}$ -stable deformation then  $S_\tau f_0$  is two-sided  $\mathcal{G}$ -stable.

or phrased differently

**Corollary 4.7** If for a semigroup deformation  $S_\tau f_0$ ,  $F(x, \tau)$  is a non-trivial  $\mathcal{G}$ -universal deformation then  $S_\tau f_0$ ,  $\tau \in \mathbb{R}_+$ , is one-sided  $\mathcal{G}$ -stable.

Note that two one-sided stable semigroup deformations need not be one-sided equivalent. They might lie on different sides of the tangent space to the orbit through  $f_0$ .

## 5 Gaussian deformations

In the case where one wants the deformation of  $f_0$  to be a solution of the diffusion equation the deformation is completely prescribed by the diffusion equation giving  $\frac{\partial u}{\partial \tau}$ , that is, the direction in which one has to unfold is in fact given. The unfolding transformations are given by the semigroup  $\exp(\tau \Delta)$ .

**Definition 5.1** Consider a function  $f_0$ . The one parameter deformations  $f(x; \tau)$  of  $f_0$  with the unfolding direction given by

$$\frac{\partial u}{\partial \tau} = \Delta u ,$$

are

$$f(x, \tau) = \exp(\tau \Delta) f_0 .$$

These deformations are called *Gaussian deformations*.

Note that  $\exp(\tau \Delta)$  is a holomorphic strongly continuous one-parameter semi-group, therefore its action on smooth functions is well defined. The Gaussian deformation can also be obtained by convolution with the Gaussian kernel.

The following theorem allows us to consider the notion of stability for such one-parameter semi-group deformations

**Corollary 5.2**  $f(x, \tau) = \exp(\tau \Delta) f_0$  is  $\mathcal{G}$ -stable if and only if either  $f_0$  is stable or  $d(f_0, \mathcal{G})=1$  and  $\Delta f_0$  generates the complement of  $T_{\mathcal{G}}(f_0)$  in  $\mathcal{E}_0$ .

Thus the possible bifurcations  $f_0$ , or actually its corresponding vector field, can undergo as a consequence of Gaussian blurring are given by the singularities of co-dimension-1 for which  $f(x, \tau) = \exp(\tau \Delta) f_0$  is  $\mathcal{G}$ -stable.

Note that the constant and linear terms in  $f_0$  do not influence the unfolding terms but linear terms can appear as unfolding terms. The linear terms in the deformation do influence the behavior of critical points. Therefore we will work modulo constant terms. Or phrased differently we allow adding constant terms in the group action (compare the groups  $\mathcal{H}$  and  $\mathcal{IS}$  of Damon [3]). If we consider Hamiltonian systems in the neighborhood of a critical point than constant terms in the Hamiltonian  $f_0$  shift the energy .

If we consider  $\mathcal{A}$ -equivalence than the stable functions are the Morse-functions  $a_1 x^2 + a_2 y^2$ . The Gaussian deformations of these Morse-functions are  $a_1 x^2 + a_2 y^2 + 2t(a_1 + a_2)$ . These are trivial  $\mathcal{A}$ -deformations. That is the initial speeds belong to the tangent space to the  $\mathcal{A}$ -orbit through the function. They are stable. Because there is no initial speed versal to the orbit we have two-sided stability.

The standard form of an  $\mathcal{A}$ -co-dimension one function is  $y^3 + x^2$ , with universal  $\mathcal{A}$ -deformation  $y^3 + ty + x^2$ . For  $t > 0$  there are no critical points while for  $t < 0$  there are two critical points, a saddle and a node. We have creation or annihilation of critical points depending on the sign of  $t$ . In terms of vector fields this is known as the saddle-node bifurcation [7]. In catastrophe theory it is the cusp catastrophe [9].

Now this is an  $\mathcal{A}$ -deformation while we have to consider Gaussian deformations. The Gaussian deformation of  $y^3 + x^2$  gives  $y^3 + 6ty + x^2 + 2t$ ,  $t > 0$ . Because it is equivalent to the universal  $\mathcal{A}$ -deformation it is one-sided  $\mathcal{A}$ -stable. The complement to the tangent space is spanned by the vector  $y$ .

In order to cover the codimension one case we need two one-sided  $\mathcal{A}$ -deformations obtained from semi-group deformations. Consider  $y^3 - 6yx^2 + x^2$  which has  $\mathcal{A}$ -codimension one and is actually  $\mathcal{A}$ -equivalent to  $y^3 + x^2$ . Its universal  $\mathcal{A}$ -deformation is  $y^3 - 6yx^2 + ty + x^2$  and its Gaussian deformation is  $y^3 - 6yx^2 - 6ty + x^2 + 2t$ . The complement to the tangent space is spanned by the vector  $-y$ . Thus we obtain the complementary one-sided  $\mathcal{A}$ -stable deformation corresponding to creation of critical points. In terms of catastrophe theory the latter is the cusp embedded in the elliptic umbilic.

This classifies all the one-sided  $\mathcal{A}$ -stable Gaussian deformations.

**Theorem 5.3** *The  $\mathcal{A}$ -stable Gaussian deformations in  $\mathbb{R}^2$  are listed in the following table*

initial function	Gaussian deformation	stability type
$x^2 + y^2$	$x^2 + y^2 + 4t$	two-sided $\mathcal{A}$ -stable
$x^2 - y^2$	$x^2 - y^2$	two-sided $\mathcal{A}$ -stable
$y^3 + x^2$	$y^3 + 6ty + x^2 + 2t$	one-sided $\mathcal{A}$ -stable
$y^3 - 6yx^2 + x^2$	$y^3 - 6yx^2 - 6ty + x^2 + 2t$	one-sided $\mathcal{A}$ -stable

A straightforward generalization to higher dimensions is obtained by adding quadratic Morse functions in the additional variables.

**Theorem 5.4** *The  $\mathcal{A}$ -stable Gaussian deformations in  $\mathbb{R}^n$  are listed in the following table*

$p = 1, \dots, n$	initial function	Gaussian deformation	stability type
(i) <sub>p</sub>	$\sum_{i=1}^p x_i^2 - \sum_{j=1}^{n-p} x_j^2$	$\sum_{i=1}^p x_i^2 - \sum_{j=1}^{n-p} x_j^2 + 2(2p - n)t$	two-sided $\mathcal{A}$ -stable
(ii) <sub>p</sub>	$x_1^3 + Q$	$x_1^3 + 6tx_1 + Q(t)$	one-sided $\mathcal{A}$ -stable
(iii) <sub>p</sub>	$x_1^3 - 6x_1x_i^2 + Q$	$x_1^3 - 6x_1x_i^2 - 6tx_1 + Q(t)$	one-sided $\mathcal{A}$ -stable

Where  $Q$  is a quadratic function as in (i) but with variables  $x_i$ ,  $i > 1$ , with  $Q(t)$  its Gaussian deformation.

## 6 Conclusion

The above framework provides an alternative way to obtain the generic qualitative changes in Gaussian state-scale space. It shows how to generalize to higher dimensions and indicates how to deal with other operators than the Laplace-operator.

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