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Input-Output Equations and Observability for Polynomial Delay Systems

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Abstract. This paper discusses a result by Fliess about input-output equations for polynomial systems with time delays, and strengthens the result somewhat. The proof given is more detailed and opens the way for constructive methods for determining the input-output behavior. Some such methods based on Gröbner bases are described in detail. Furthermore, some connections with observability are exploited.

Keywords: delay systems, difference-differential equations, commutative algebra, field theory, difference-differential algebra, elimination theory, implicitization

1   Introduction

Let $\tau_1, \ldots, \tau_r$ be an r-tuple of incommensurate time-delays, i.e. $\tau_1, \ldots, \tau_r$ are positive real numbers that are linearly independent over $\mathbb{Q}$. Each time-delay τ_i ($i = 1, \ldots, r$) corresponds to a delay operator δ_i, acting on functions of time in the following way:

$$\delta_i y(t) = y(t - \tau_i).$$

In this paper we consider systems in a kind of generalized state space form, namely systems of coupled differential-difference equations (dde) of differential order one in some internal variables $\bar{x}$:

$$\dot{x}_1(t) = f_1(\bar{x}, u),$$
$$\vdots$$
$$\dot{x}_n(t) = f_n(\bar{x}, u), \tag{1}$$

$$y = h(\bar{x}, u)$$

where $\bar{x} = (x_1, \ldots, x_n)$ and $f_1, \ldots, f_n, h$ are polynomial functions of

$$\bar{x}(t), \delta_1 \bar{x}(t), \ldots, \delta_1^{q_1} \bar{x}(t), \ldots, \delta_r \bar{x}(t), \ldots, \delta_r^{q_r} \bar{x}(t),$$

$$u(t), \delta_1 u(t), \ldots, \delta_1^{q_1} u(t), \ldots, \delta_r u(t), \ldots, \delta_r^{q_r} u(t).$$

for some $q_1, \ldots, q_r \in \mathbb{N}$. If there is only one delay operator involved in the differential-difference equation (1), the time-delays occurring in the system are called *commensurate*.

The problems addressed are the following:

1. Is there an input-output equation, i.e. a dde relating the input u and the output y? In other words: is it always possible to eliminate the latent variables $\bar{x}$?

2. In that case: what is the differential and transformal order of the input-output equation?

3. Can you determine the input-output equation algorithmically?

4. Could there be several, essentially different, input-output equations?

5. Is the order of the input-output equation related to the observability of the system in some sense?

Question no 1 has been given a positive answer by M. Fliess in [4]. The proof given here differs from that given by Fliess.

Question no 2 will be given a complete answer in the case of one delay operator, and a partial answer in the more complex case of several incommensurate delays.

As an answer to question no 3 we give an algorithm relying on Gröbner bases for performing the elimination in question. The algorithm suggested works independently of the number of delay operators involved.

Question no 4 is quite delicate, and here only some partial answers can be given. To the knowledge of the authors, a completely satisfactory answer is still missing in the purely differential or purely transformal case (differential and difference equations, respectively).

Regarding question no 5 we will see that it is possible to define a concept analog to that of algebraic observability of polynomial continuous time systems [3, 9, 12] which is such that the input-output equation is of order n iff all internal variables are algebraically observable..

In the sequel we will use the language of differential and difference algebra without further explanation. The reader is referred to [1, 5, 6, 16] for this terminology.

2 Elimination of the Latent Variables

Let us now describe how the latent variables x_i can be eliminated starting with the one delay case.

2.1 One Delay

Let δ denote a time-shift operator (automorphism), as in [1], so that e.g. $\delta y(t) = y(t-1)$. We use the notation

$$\Delta^q S = \{ \delta^j s; \ j \leq q, \ s \in S \} \tag{2}$$

for a set S of difference indeterminates and $q \in \mathbb{N}$. If S is a singleton $\{s\}$ we write $\Delta^q s$, by abuse of notation. E.g.

$$\Delta^3 x_1 = \{x_1, \delta x_1, \delta^2 x_1, \delta^3 x_1\}$$

We suppose that our original equations are over a differential-difference field k and then form the field K from k in the following manner: adjoin all combinations of shifts and derivatives of the input variable u to the field k. We assume that the input is transformally[1]

---

[1]Recall that the word transformal refers to the difference operators. This is for purely linguistic reasons, since the word differential is already occupied...

differentially transcendental, so that K is a purely transcendental extension of k, with infinite transcendence degree. If ∂ denotes the derivative operator an example of an element in K is

$$\frac{\partial^2 \delta^4 u - 12\delta u(\partial^2 u)^3}{1 - u^2 \delta \partial u}$$

if e.g. $k = \mathbb{Q}$. (δ and ∂ commute, of course.) The construction of K is completely analogous if there are several independent inputs $u_1, \ldots, u_m$, i.e. all results stated here are also valid for multi-input systems.

Furthermore we use the abbreviation

$$\Phi_n^q = K[\Delta^q\{x_1, \ldots, x_n\}] \tag{3}$$

Clearly the ring Φ_n^q has dimension

$$\dim \Phi_n^q = n(q+1) \tag{4}$$

We consider systems in 'pseudo-state space form', i.e. n differential difference equations (dde) of differential order one:

$$\dot{x}_1 = f_1, \ldots, \dot{x}_n = f_n, \qquad \forall i : f_i \in \Phi_n^m \tag{5}$$

and an output map

$$y = h, \qquad h \in \Phi_n^c \tag{6}$$

Now, $h_0 := h$ and we define $h_{i+1} \in \Phi_n^q$ (some q) as the thing we get as we differentiate h_i w.r.t. time and replace every occurrence of $\delta^j \dot{x}_r$ by $\delta^j f_r$.

An example:

$$\dot{x}_1 = 2\delta x_1 \delta^2 x_2^2, \quad \dot{x}_2 = x_2 \delta^3 x_1, \qquad y = x_1 \quad \Rightarrow \tag{7}$$

$$h_1 = 2\delta x_1 \delta^2 x_2^2, \quad h_2 = 4\delta^2 x_2^2 \delta^2 x_1 \delta^3 x_2^2 + 4\delta x_1 \delta^2 x_2^2 \delta^5 x_1 \tag{8}$$

The number $m \in \mathbb{N}$ is such that $f_i \in \Phi_n^m$ for all i, i.e. every f_i is of transformal order $\leq m$. Then

$$h_0 \in \Phi_n^c, \quad h_1 \in \Phi_n^{m+c}, \quad h_2 \in \Phi_n^{2m+c}, \ldots, \quad h_n \in \Phi_n^{nm+c} \tag{9}$$

Theorem 2.1 *For a system of coupled dde of the type (5)–(6) the indeterminate y satisfies a dde of differential order $\leq n$.*

Proof. The key idea is to consider the cardinality of the set

$$H(s) := \{\Delta^{nm+s} h_0, \Delta^{(n-1)m+s} h_1, \ldots, \Delta^s h_n\} \tag{10}$$

Thus all elements of $H(s)$ belong to Φ_n^{nm+c+s}, but not necessarily to $\Phi_n^{nm+c+s-1}$. Now we have that

$$\#H(s) = m\frac{n(n+1)}{2} + (n+1)(s+1) \tag{11}$$

3

denoting cardinality. But according to formula (4)

$$\dim \Phi_n^{nm+c+s} = n(nm + c + s + 1) \tag{12}$$

so, as a function of s, $\#H(s)$ grows faster than $\dim \Phi_n^{nm+s}$, which means that for s large enough the polynomials in $H(s)$ will be algebraically dependent over K. $\square$

To determine an upper bound for the transformal order of the equation for y we solve the equation

$$\#H(s) = \dim \Phi_n^{nm+c+s} + 1 \tag{13}$$

w.r.t. s. This gives

$$s^*(n, m, c) = nc + \frac{n(n-1)}{2} m \tag{14}$$

This implies the existence of an algorithm for determining a dde for y. The algorithm is described in more detail in section 4.

It is worth noticing that equation (13) has an *integer* solution in s for all integers c, n, m, something which is not self evident.

2.2 Several Delays

We will now prove that theorem 2.1 holds for systems with several incommensurate time-delays too.

Suppose that the set of transformations is $\{\delta_1, \ldots, \delta_r\}$, and the free commutative monoid generated by these is denoted Θ_r. Note that the identity operator is in Θ_r. The field K is formed out of k in the obvious way.

The *order* of a transformation $\theta = \delta_1^{\alpha_1} \ldots \delta_r^{\alpha_r} \in \Theta_r$, denoted $|\theta|$, is the number $\sum \alpha_i$. We extend the convention (2) by

$$\Delta^q S = \{\theta s;\ \theta \in \Theta_r,\ |\theta| \le q,\ s \in S\} \tag{15}$$

and Φ_n^q in (3) is changed accordingly. Since

$$\#\{\theta \in \Theta_r;\ |\theta| \le q\} = \binom{r+q}{r} \tag{16}$$

formula (4) becomes

$$\dim \Phi_n^q = n \binom{r+q}{r} \tag{17}$$

From equation (5) and forward we proceed in an obvious analogous way to get the set $H(s)$. Now

$$\dim \Phi_n^{nm+c+s} = n \binom{nm+c+s+r}{r} = \frac{n}{r!} s^r + p_1(s) \tag{18}$$

where p_1 is a polynomial in s of degree $< r$, and

$$\#H(s) = \sum_{i=0}^{n} \binom{im+s+r}{r} = \frac{n+1}{r!} s^r + p_2(s) \tag{19}$$

So again we have that $\#H(s)$ grows faster than $\dim \Phi_n^{nm+s+c}$, i.e. for s large enough the elements of $H(s)$ are algebraically dependent over K. So we have proved that theorem 2.1 holds for systems with r incommensurate time-delay operators, too.

4

Since we have been unable to determine a closed expression for $\#H(s)$ in (19) we are not able to give a bound for the transformal order of the input-output equation. However, for each value of r the sum can be expressed as a polynomial in s, m, n of course. For example we have that

$$
\begin{aligned}
r = 2 \quad \Rightarrow \quad \#H(s) = &\tfrac{1}{2}(n+1)s^2 + \tfrac{1}{2}(n+1)[3 - 2m + 2(n+1)m]\,s+ \\
&+\tfrac{1}{12}(n+1)[m^2 - 3(n+1)\,m^2 - 9m + 12 + 9(n+1)m + 2(n+1)^2 m^2]
\end{aligned}
\tag{20}
$$

2.3 Related Work

In comparison to the aforementioned earlier work by Fliess we note that [4] establishes the existence of an input-output equation, but does not discuss the differential and transformal order of it. The proof given is analogous to the one for differential algebra in [5].

The combinatorial arguments used above are not entirely different from those that can be used to prove that $n + 1$ polynomials in n variables are algebraically dependent, used in e.g. [11]. This idea goes back to Ritt [23], and maybe further.

3 Observability

In this section we discuss an observability concept for delay systems and explain how this type of observability is related to the differential order of the io-equation.

The main tools in this section will be fields; difference fields and differential difference fields. Before we start we should therefore establish that it is mathematically possible to form fields in the variables defined by the system (5)-(6). This is the case iff the difference-differential ideal of the system is prime.

Lemma 3.1 *The difference-differential ideal defined by (5)-(6) is prime.*

Proof. We prove that the corresponding quotient ring is an integral domain. The ideal

$$
\Sigma := [\dot{x}_1 - f_1, \ldots, \dot{x}_n - f_n, \, y - h(x_1, \ldots, x_n)]
$$

is generated as an ordinary polynomial ideal by an infinite set of polynomials of the type

$$
\theta y - p \quad \text{and} \quad \theta x_i - p
\tag{21}
$$

where θ is an arbitrary differential-difference operator and p a polynomial in the difference-differential ring $k\{u, y, x\}$. So the ideal is generated by polynomials that are differences of a variable and a polynomial (such an ideal is sometimes called a *graph-ideal*). The only thing that happens as we take quotients is that some variables θy and θx_i are killed. This means that the quotient ring $k\{u, y, x\}/\Sigma$ is a *free* ring, and in particular a domain. $\square$

We use the abbreviations t.a. = transformally algebraic, t.t. = transformally transcendental and t.a.i. = transformally algebraically independent. Furthermore we use the notation

$$
\text{ttrd}\, K/k
$$

for the transformal transcendence degree of the difference field extension K of k.

The following lemma will prove to be very useful:

5

Lemma 3.2 *If $K \supset L \supset M$ is a tower of difference field extensions, then*

$$\operatorname{ttrd} K/M = \operatorname{ttrd} K/L + \operatorname{ttrd} L/M$$

Proof. The proof is completely analogous to that for ordinary field extensions [25, theorem 12.56]. □

Theorem 3.1 *All x_i are t.a. over the difference-differential field $K\langle y \rangle = k\langle u, y \rangle$ iff $y, \partial y, \ldots, \partial^{n-1} y$ are t.a.i. over K. (Here $\partial = \dfrac{d}{dt}$.)*

Proof. We will make a proof similar to the one given in [9] for differential systems. Consider the difference field extensions

$$K\langle x_1, \ldots, x_n \rangle \supseteq K\langle h_0, \ldots, h_{n-1} \rangle \supseteq K \tag{22}$$

where h_i are obtained as in section 2.1. Since all x_i are t.t. over K we have that

$$\operatorname{ttrd} K\langle x_1, \ldots, x_n \rangle = n \tag{23}$$

Now,

$$\operatorname{ttrd} K\langle h_0, \ldots, h_{n-1} \rangle / K = n \tag{24}$$

iff $y, \partial y, \ldots, \partial^{n-1} y$ are t.a.i. According to lemma 3.2

$$\operatorname{ttrd} K\langle x_1, \ldots, x_n \rangle / K\langle h_0, \ldots, h_{n-1} \rangle = n - \operatorname{ttrd} K\langle h_0, \ldots, h_{n-1} \rangle / K \tag{25}$$

so it follows that $x_1, \ldots, x_n$ are t.a. over $K\langle h_0, \ldots, h_{n-1} \rangle$ iff $y, \partial y, \ldots, \partial^{n-1} y$ are t.a.i.

It remains to prove that $x_1, \ldots, x_n$ are t.a.i. over $K\langle y \rangle$ iff they are t.a.i. over $K\langle h_0, \ldots, h_{n-1} \rangle$. This follows if we can prove that the difference-differential field $K\langle y \rangle$ is a t.a. extension of the difference field $L := K\langle y, \partial y, \ldots, \partial^{n-1} y \rangle$. According to theorem 2.1 $\partial^n y$ is t.a. over L. If we take the derivative of the input-output equation we get something that is linear in $\partial^{n+1} y$, so

$$\partial^{n+1} y \in L\langle \partial^n y \rangle \tag{26}$$

So, in particular, $\partial^{n+1} y$ is t.a. over $L\langle \partial^n y \rangle$ which means that it is t.a. over L (lemma 3.2). Repeating the argument for arbitrary $\partial^j y$ we have proved the last part of the theorem. □

This motivates the following definition:

Definition 3.1 A latent variable x_i is *algebraically observable* if it is t.a. over the difference-differential field $K\langle y \rangle$. □

The system itself is algebraically observable if all latent variables are. A consequence of theorem 3.1 is that the system (5)–(6) is algebraically observable iff the input-output equation is of differential order n.

In words, a variable x_i is algebraically observable iff it satisfies an *iterative functional equation* with inputs defined by u and y. We call such a functional equation an *observer equation* for x_i.

It remains to investigate whether this observability concept is a "natural" one. It would be nice if there is an interpretation of algebraic observability in terms of whether the latent variables can be estimated from measurements of the external variables u and y using the

observer equations. This question is probably rather difficult to answer. The theory for existence and uniqueness of solutions to this kind of equations is very involved. A reference is [17].

A rather interesting case which is not so complicated, though, is that of *parameter identifiability*: we consider some system parameters to be identified to be latent variables satisfying the equation $\dot{x}_i = 0$. Now an observer equation for x_i is just a nonlinear equation, static in x_i depending on $u(t), y(t)$.

Compare this approach to identifiability to the one described in e.g. [19].

Another important question, which we do not address here, is how algebraic observability relates to other observability concepts for this class of systems. For a survey of such concepts, see e.g [18].

4   Algorithms

An advantage with the approach described in section 2 is that it opens the way for constructive methods for determining the input-output-equation. We may see the task of retrieving the input-output equation as a special case of determining the dependency relation of some algebraically dependent polynomials over some field. This is known as *implicitization* in algebraic geometry, and many constructive approaches to this problem have been described in the literature: [7, 10, 20, 22, 24] to mention a few.

Let us here only briefly describe how Gröbner bases can be used to solve the implicitization problem. Gröbner bases (gb) are a well known algorithmic method in elimination theory that has been implemented in all major computer algebra programs, e.g. Maple, Axiom, Reduce and Macsyma. For an introduction to gb we refer to the excellent textbook by Cox et al. [2]. For another application of gb to delay systems, see [15].

Now suppose that we wish to find the dependency relation between some algebraically dependent polynomials $f_1, \ldots, f_N \in k[X_1, \ldots, X_n]$ over k, i.e. we are looking for a nonzero polynomial $p \in k[X_1, \ldots, X_N]$ such that

$$p(f_1, \ldots, f_N) \equiv 0 \tag{27}$$

In our application the f_i are the elements of the set $H(s)$ (s sufficiently large) as defined in formula (10), and p is the io-equation.

The first step is to form the so called *graph-ideal* defined by these polynomials. This is the ideal

$$\mathfrak{g} := \langle Z_1 - f_1, \ldots, Z_N - f_N \rangle \tag{28}$$

in the ring $k[X_1, \ldots, X_n, Z_1, \ldots, Z_N]$ where the $Z_1, \ldots, Z_N$ are new variables, *tag-variables*. It can be showed [2] that the ideal

$$\mathfrak{g}^c := \mathfrak{g} \cap k[Z_1, \ldots, Z_N] \tag{29}$$

i.e. the contraction of $\mathfrak{g}$ to the ring of polynomials in only Z_i, contains all the polynomial relations between $f_1, \ldots, f_N$.

So, if we can find a generating set for $\mathfrak{g}^c$ we have determined an input-output equation. But this is a simple elimination problem in ordinary commutative algebra, which is solved by computing a gb for $\mathfrak{g}$ w.r.t. a lexicographic type term-ordering of the shape

$$\{X_1, \ldots, X_n\} > \{Z_1, \ldots, Z_N\} \tag{30}$$

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The tag-variable technique is described in detail in [24], but it appears that it was known long before Gröbner bases appeared on the stage [21].

Some ideas for lowering the computational complexity of the implicitization problem when using gb are suggested in [7]. These have been used in the Maple-package POLYCON [8].

Let us conclude with a simple example that illustrates the algorithm suggested.

Example 4.1 Consider the system

$$\dot{x}(t) = -x(t)^3, \qquad y(t) = x(t-1) + x(t) \tag{31}$$

With the notation of section 2 we thus have $n = 1, m = 0, r = 1, c = 1$. Now

$$h_0 = \delta x + x, \quad h_1 = -\delta x^3 - x^3 \tag{32}$$

and

$$\delta h_0 = \delta^2 x + \delta x, \quad \delta h_1 = -\delta^2 x^3 - \delta x^3 \tag{33}$$

These four polynomials have to be algebraically dependent according to formula (14). Thus we can eliminate the variables $x, \delta x, \delta^2 x$ in the graph-ideal

$$\langle y_{0,0} - h_0, \ y_{0,1} - \delta h_0, \ y_{1,0} - h_1, \ y_{1,1} - \delta h_1 \rangle \tag{34}$$

As we do this we get the following dde for y:

$$81y_{0,1}^4 y_{1,1}^2 - 18y_{0,1}^8 y_{1,1} - (144y_{0,1}^5 + 81y_{0,1}^4)y_{1,0}y_{1,1} + 36y_{0,1}^2 y_{1,0}^2 y_{1,1} + (16y_{0,1}^9 + 9y_{0,1}^7)y_{1,0} - 6y_{0,1}^{11}$$

$$+3y_{0,1}^{10} + 4y_{1,0}^4 + 4y_{0,1}^{12} + (24y_{0,1}^6 + 126y_{0,1}^5 + 9y_{0,1}^4)y_{1,0}^2 + (16y_{0,1}^3 - 42y_{0,1}^2 + 3y_{0,1})y_{1,0}^3 = 0$$

where

$$y_{i,j} := \frac{d^i}{dt^i} y(t - j)$$

Below is a copy of the Maple session in which this was done:

```
     |\^/|      Maple V Release 2 (University of Linkoping-4)
 ._|\|   |/|_.  Copyright (c) 1981-1993 by the University of Waterloo.
  \  MAPLE  /   All rights reserved. Maple and Maple V are registered
  <____ ____>   trademarks of Waterloo Maple Software.
       |        Type ? for help.

> with(grobner):

> h[0,0] := x[1] + x[0]:

> h[1,0] := -x[1]^3 - x[0]^3:

> h[0,1] := x[2] + x[1]:


> h[1,0] := -x[2]^3 - x[1]^3:


> F := [ y[0,0] - h[0,0],  y[0,1] - h[0,1],  y[1,0] - h[1,0],
>           y[1,1] - h[1,1] ]:

> rankinglist := [ x[2], x[1], x[0], y[1,1], y[1,0] ]:

> G := gbasis( F, rankinglist, plex ):
```

```
> nops(G);
```

<div align="center">6</div>

```
> p := G[6];
```

<div align="right">□</div>

5 Some Open Problems

Some interesting problems that, to the knowledge of the authors, are still open (apart from those already mentioned) are e.g.

- Can it be proved that there is no closed form for the sum corresponding to $\#H(s)$ in formula (19)? Possibly, a solution to this question can be provided by Gosper's theorems on indefinite summation, q.v. e.g. [14, 13].

- Is there a *unique* integer s^* such that

$$\#H(s^*) < \dim \Phi_n^{nm+s^*} \quad \text{and} \quad \#H(s^*+1) \geq \dim \Phi_n^{nm+s^*+1}$$

- How do the results stated here generalize to multi-output systems?

Acknowledgement

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