

## Dynamics of a polymer blend driven by surface tension. Part 2: The zero-order solution

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# Dynamics of a polymer blend driven by surface tension:

## *Part 2: the zero-order solution*

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### **Abstract**

In a preceding report [1], the dynamical behaviour of two infinitely long adjacent parallel polymer threads (dispersed phase) immersed in a different polymeric fluid (continuous phase) is considered. There, the theory and results are presented in concise way. The present report deals with the same subject and contains the full derivation of the theory. Since the evaluation of this theory involves a great number of intricate formulae, the presentation has been split up. In the present report the theory is worked out including only the zero-order terms of the Fourier expansions of all quantities. This implies that it is assumed that the perturbations of the thread have cylinder symmetry. In a forthcoming report [2] the first order contributions are derived.

## **1 Introduction**

The demand for new synthetic materials increases and becomes more specific, nowadays. New synthetic materials may be produced by blending different types of polymers. The material properties of a polymer blend are strongly related to its morphology determined by the blending process in the extruder. Therefore, the eventual material properties can be predicted only if a thorough understanding of this blending process is available.

In the blending process under consideration, relatively big drops of one material are immersed into a shear flow of a second material. Due to dominant shear stresses, long threads are formed. At some moment a thread may become so thin and its radius so small that the inter-facial stress becomes important. The effect of the latter is a tendency to attain the drop-shape. Initiated by perturbations, waves may develop along the thread. Driven by surface tension, these waves may grow in amplitude or attenuate, depending on the stability of the system. In an unstable state, the thread will eventually break up into an array of small spherical droplets.

The problem of breaking up of one isolated thread in a solvent has already been solved long ago. If the fluid is initially at rest, the problem of stability is purely geometrical; that is, instability means that the surface area can be reduced by an infinitesimal surface deformation. The classical example is due to Plateau [3] who studied the instability of an

infinitely long cylindrical fluid thread (or: filament) produced by a jet emanating from a nozzle at high speed. He found that the system is unstable if the wavelength of a perturbation is greater than  $\pi$  times the diameter of the jet. From this it may seem as if disturbances of very long wavelength are the most rapidly growing ones, as they reduce the surface area most. But this leaves out the dynamics of the problem. In fact, the surface tension has to overcome both inertia and viscous dissipation.

The dynamical description of the problem, in terms of linear stability theory, was first introduced by Rayleigh [4, 5]. He considered the stability of a long cylindrical column of an incompressible perfect fluid under the action of capillary forces, neglecting the effect of the surrounding fluid. His result is in accordance with the previous result of Plateau. In his paper, Rayleigh developed the important concept of the mode of maximum instability. The degree of instability, which characterizes the growth rate of the disturbance amplitude and as indicated by the value of  $q$  in the exponential  $e^{qt}$  to which the motion is assumed to be proportional, depends upon the value of wavelength of the disturbances  $\lambda$ . It reaches a maximum when  $\lambda$  equals 4.51 times the diameter of the cylinder. Tomotika [6, 7], generalized Rayleigh's analysis to include viscosity for both the fluid column and the surrounding fluid. He found that if the ratio of viscosities of the two fluids is neither zero nor infinity the maximum instability always occurs at a certain definite value of the wavelength of the assumed initial perturbation. Moreover, he concluded that the formation of droplets of definite size is to be expected. A wide-ranging review of the dynamics of the break-up process of one thread is given by Eggers [8]. The effect of viscosity and surface tension on the stability of the plane interface between two fluids has also been studied by Mikaelian [9, 10]. Using a moment equation approach, he found that perturbations at the interface show damped oscillations when viscosity and surface tension are present. A model describing the nonlinear dynamics of one thread when surface tension drives the motion is discussed by Papageorgiou [11]. Using nonlinear long-wave theory, he is able to describe phenomena such as jet pinching, which are beyond the scope of linear theory. The pinching effect, the scaling behaviour of singularities in nonlinear systems, and drop break-up are studied to some extent in [12]-[18]. In the present paper we restrict ourselves to linear stability theory for a set of two threads, since the literature is still lacking for such systems.

Most blends contain a large volume fraction of the dispersed phase (threads). In the experimental practice, it turns out that the interactions between the threads are of essential importance for the way they break up. In experiments reported in [19] it is observed that neighboring threads may break up in-phase or out-of-phase. A sketch of both possibilities is shown in Figure 1.

In this paper, we study the origin of the phenomena described above and the dependence on the geometrical and rheological parameters. To that end, the dynamical behaviour of two adjacent parallel threads immersed in a polymeric fluid (the matrix) under the action of both surface tension and viscous forces is considered. The threads are initially cylindrical, but due to thermal fluctuations disturbances are always present. The fluids are assumed to be at rest except for small disturbances which are assumed to develop slowly. This implies that the velocities and the shear rates will be small. Under these conditions, it is justified to model both the threads and the matrix as Newtonian fluids and to use the creeping flow approximation, resulting in Stokes equations. These equations are solved by means of

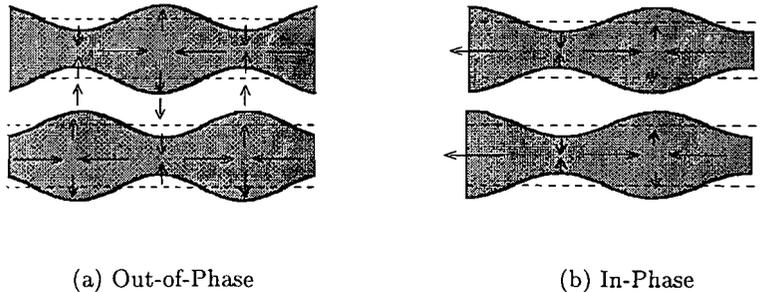


Figure 1: *Two typical examples of break up behaviour observed in the experiment from [19].*

separation of variables in two systems of cylindrical coordinates, each one connected to one of the threads. In this, the dependence on the azimuthal directions is written in the form of Fourier expansion. Substitution of the general solution into the boundary conditions yields an infinite set of linear equations for the unknown coefficients. This set is solved using the method of moments.

Based on the solution found above, the instability (the growth rate) of the initial disturbances is investigated. If the initial disturbances increase in time, the initial state is unstable resulting in breaking up. In the present work, the instability of the threads is examined based on a zero-order Fourier expansion. The zero-order analysis follows the lines presented in [19], but avoids some unreliable assumptions. This refinement leads to expressions that are simpler to evaluate. The model confirms that the break up can indeed occur in-phase as well as out-of-phase. This depends on the viscosity ratio of the two fluids and on the distance between the threads. All this is in correspondence with experimental observations.

The paper is organized as follows. In Section 2, the mathematical model is derived based on the linear theory for two threads. The general solution written in terms of Fourier expansions and a general scheme for solving the unknown coefficients is presented. In this scheme an infinite set of equations is reduced to a finite set of equations. In Section 3, the zero-order solutions are addressed. The general scheme for solving the coefficients derived in the previous section is illustrated for this lowest order of Fourier expansions. The explicit solution for the unknown coefficients is also presented. The core of the paper is Section 4. Based on the solution given in Section 3, the instability of the threads is investigated. Conclusions are given in the last section.

## 2 Linear theory for two threads

Consider two infinitely long parallel threads, both with viscosity  $\mu_d$ , which are surrounded by a viscous fluid with viscosity  $\mu_c$ , where the indices  $c$  and  $d$  refer to the continuous phase (surrounding fluid) and the disperse phase (threads), respectively. We denote the ratio of the viscosities by  $\mu = \mu_d/\mu_c$ . Let  $b$  be the distance between the two centers of the threads. In Figure 2 a radial cross-section is sketched. The indices 1 and 2 refer to thread 1 and

thread 2, respectively. Two cylindrical coordinate systems will be used,  $(r_1, \phi_1, z_1)$  centered

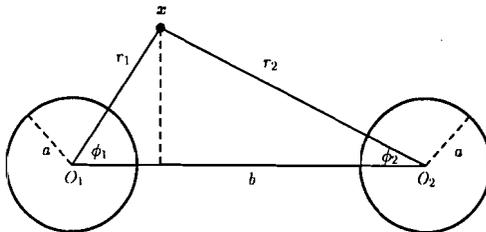


Figure 2: A radial cross-section of the two threads and the corresponding two cylindrical coordinate systems.

along the axis of thread 1, and  $(r_2, \phi_2, z_2)$  centered along the axis of thread 2 (see Figure 2). The disturbed radii are written as a sum of modes, that are periodic in  $z$  with wave number  $k$ . Because the problem is not axisymmetric, the disturbances will also depend on  $\phi$ . This dependence is written as a Fourier expansion. Thus, we consider two disturbed threads with radii  $R_1$  and  $R_2$  given by

$$\begin{aligned} R_1(\phi_1, z, t) &= a + \left( \sum_{n=0}^{\infty} \varepsilon_{1n}(t) \cos n\phi_1 \right) \cos kz, \\ R_2(\phi_2, z, t) &= a + \left( \sum_{n=0}^{\infty} \varepsilon_{2n}(t) \cos n\phi_2 \right) \cos(kz - \alpha). \end{aligned} \quad (1)$$

Since  $z_1$  and  $z_2$  are identical, we drop the index in the  $z$ -coordinates. Here,  $a$  is the mean radius of the threads,  $\varepsilon_{1n}$  and  $\varepsilon_{2n}$  are the time dependent amplitudes of the modes, and  $\alpha$  is a phase difference in the  $z$ -direction between the modes at thread 1 and the modes at thread 2. The parameter  $\alpha$  is unknown in advance. We remark that we only consider the special sub-set of the disturbances for which  $k_1 = k_2 = k$ .

The polymer fluids are assumed to behave Newtonian. This is a good approximation for slow deformation rates. Since the dynamics is only driven by surface tension, the velocities and the shear rates will be small. Also the length scales are very small. The Reynolds number is therefore small and the creeping flow approximation may be used. For this slow and incompressible flow, the system is described by the Stokes equations:

$$\begin{aligned} \nabla \cdot \mathbf{v} &= 0, \\ \nabla p &= \mu \Delta \mathbf{v}, \end{aligned} \quad (2)$$

where  $p$  is the pressure,  $\mu$  is the viscosity, and  $\mathbf{v} = (u, v, w)$ , with  $u, v$  and  $w$  the velocity components in radial, azimuthal and axial directions, respectively. In cylindrical coordi-

nates, the Stokes equations are given by

$$\begin{aligned}
0 &= \frac{1}{r} \frac{\partial [ru]}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z} & (a) \\
\frac{\partial p}{\partial r} &= \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial u}{\partial r} \right] + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} - \frac{2}{r^2} \frac{\partial v}{\partial \phi} - \frac{u}{r^2} \right] & (b) \\
\frac{1}{r} \frac{\partial p}{\partial \phi} &= \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial v}{\partial r} \right] + \frac{1}{r^2} \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial z^2} + \frac{2}{r^2} \frac{\partial u}{\partial \phi} - \frac{v}{r^2} \right] & (c) \\
\frac{\partial p}{\partial z} &= \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial w}{\partial r} \right] + \frac{1}{r^2} \frac{\partial^2 w}{\partial \phi^2} + \frac{\partial^2 w}{\partial z^2} \right]. & (d)
\end{aligned} \tag{3}$$

We first determine the general solution of these equations and then formulate the general solution for two threads. The procedure is as follows. Assuming that separation of variables is applicable and expressing the dependence on  $\phi$  in terms of Fourier modes, we propose as general expressions for the solution:

$$\begin{aligned}
p &= \sum_{n=0}^{\infty} p_n(r, t) \cos n\phi \cos kz & (a) \\
u &= \sum_{n=0}^{\infty} u_n(r, t) \cos n\phi \cos kz & (b) \\
v &= \sum_{n=1}^{\infty} v_n(r, t) \sin n\phi \cos kz & (c) \\
w &= \sum_{n=0}^{\infty} w_n(r, t) \cos n\phi \sin kz. & (d)
\end{aligned} \tag{4}$$

Here, we used that the azimuthal velocity is an odd function of  $\phi$  and the other velocities are even functions of it. We note that there is no contribution of the zeroth order mode in  $v$ , hence without loss of generality we may take  $v_0 = 0$ . By substituting (4) into the Stokes equations (3), we obtain the following equations for the coefficients,  $n \geq 0$ ,

$$\begin{aligned}
0 &= u'_n + \frac{1}{r} u_n + \frac{n}{r} v_n + k w_n & (a) \\
p'_n &= \mu \left[ u''_n + \frac{1}{r} u'_n - \frac{n^2 + (kr)^2 + 1}{r^2} u_n - \frac{2n}{r^2} v_n \right] & (b) \\
-\frac{n}{r} p_n &= \mu \left[ v''_n + \frac{1}{r} v'_n - \frac{n^2 + (kr)^2 + 1}{r^2} v_n - \frac{2n}{r^2} u_n \right] & (c) \\
-k p_n &= \mu \left[ w''_n + \frac{1}{r} w'_n - \frac{n^2 + (kr)^2}{r^2} w_n \right]. & (d)
\end{aligned} \tag{5}$$

The prime indicates derivation with respect to  $r$ . In Stokes flow, the pressure satisfies

$$\nabla^2 p = 0. \tag{6}$$

Substituting (4.a) into this equation leads to

$$p''_n + \frac{1}{r} p'_n - \frac{n^2 + (kr)^2}{r^2} p_n = 0. \tag{7}$$

The general solution of this equation is

$$p_n(r, t) = 2\mu [A_n(t)I_n(kr) + D_n(t)K_n(kr)] \quad (n \geq 0), \quad (8)$$

where  $I_n$  and  $K_n$  are modified Bessel functions of order  $n$ ; the factor  $2\mu$  is added for convenience. Substitution of (8) into (5.d) leads to the general solution for  $w_n(r, t)$ :

$$w_n(r, t) = -A_n(t)rI_{n+1}(kr) + B_n(t)I_n(kr) + D_n(t)rK_{n+1}(kr) + E_n(t)K_n(kr) \quad , \quad n \geq 0. \quad (9)$$

To obtain the solution for the other velocity components, we will follow the same approach. Since there is no zeroth order mode in  $v$  present, one finds from substitution (8) into (5.b), that the solution for  $u_0$  is given by

$$u_0(r, t) = A_0(t)rI_0(kr) + C_0(t)I_1(kr) + D_0(t)rK_0(kr) + F_0(t)K_1(kr). \quad (10)$$

Substituting this expression into (5.a), we find the relations:

$$C_0 = -(B_0 + \frac{2}{k}A_0) \quad \text{and} \quad F_0 = E_0 + \frac{2}{k}D_0. \quad (11)$$

So, for the zeroth order mode in  $u$ , we obtain

$$u_0(r, t) = A_0(t)rI_0(kr) - \left[ B_0(t) + \frac{2}{k}A_0(t) \right] I_1(kr) + D_0(t)rK_0(kr) + \left[ E_0(t) + \frac{2}{k}D_0(t) \right] K_1(kr). \quad (12)$$

For the other modes, we proceed along the same line. We substitute (8) and (9) into (5.a) and (5.b), eliminate  $v_n$  from this two equations, and obtain the equation for  $u_n$ :

$$G(r, t) = \mu \left[ u_n'' + \frac{3}{r}u_n' - \frac{n^2 + (kr)^2 - 1}{r^2}u_n \right] \quad (13)$$

where

$$G(r, t) = 2\mu \left[ 2kA_n(t)I_{n+1}(kr) + \frac{nA_n(t) - kB_n(t)}{r}I_n(kr) - 2kD_n(t)K_{n+1}(kr) + \frac{nD_n(t) - kE_n(t)}{r}K_n(kr) \right].$$

We find for  $n > 0$ :

$$u_n(r, t) = A_n(t)rI_n(kr) - \left[ B_n(t) + \frac{1}{k}(n+2)A_n(t) \right] I_{n+1}(kr) + \frac{C_n(t)}{r}I_n(kr) + D_n(t)rK_n(kr) + \left[ E_n(t) + \frac{1}{k}(n+2)D_n(t) \right] K_{n+1}(kr) + \frac{F_n(t)}{r}K_n(kr). \quad (14)$$

Substitution of (9) and (14) into (5.a) yields that

$$v_n(r, t) = - \left[ (B_n(t) + \frac{1}{k}(n+2)A_n(t) + \frac{k}{n}C_n(t)) I_{n+1}(kr) - \frac{1}{r}C_n(t)I_n(kr) + \left[ E_n(t) + \frac{1}{k}(n+2)D_n(t) + \frac{k}{n}F_n(t) \right] K_{n+1}(kr) - \frac{1}{r}F_n(t)K_n(kr) \right] \quad , \quad n \geq 1. \quad (15)$$

The general solution of the cylindrical Stokes equations (3) is given by (8)-(15), where  $A_n$ ,  $B_n$ ,  $C_n$ ,  $D_n$ ,  $E_n$  and  $F_n$  are still unknown functions of time  $t$ .

We now formulate the solution of the two threads systems. Inside thread 1, the solution is written as

$$\begin{aligned}
p_1(r_1, \phi_1, z, t) &= \sum_{n=0}^{\infty} p_{1n}(r_1, t) \cos n\phi_1 \cos kz, \\
u_1(r_1, \phi_1, z, t) &= \sum_{n=0}^{\infty} u_{1n}(r_1, t) \cos n\phi_1 \cos kz, \\
v_1(r_1, \phi_1, z, t) &= \sum_{n=1}^{\infty} v_{1n}(r_1, t) \sin n\phi_1 \cos kz, \\
w_1(r_1, \phi_1, z, t) &= \sum_{n=0}^{\infty} w_{1n}(r_1, t) \cos n\phi_1 \sin kz.
\end{aligned} \tag{16}$$

The analogue of (16) for thread 2 is obtained by replacing 1 by 2 and shifting  $z$  over  $\alpha$ . For example, the pressure inside thread 2 is given by

$$p_2(r_2, \phi_2, z, t) = \sum_{n=0}^{\infty} p_{2n}(r_2, t) \cos n\phi_2 \cos(kz - \alpha). \tag{17}$$

For the continuous phase, we should note that the local state is influenced by the presence of both threads. As an Ansatz, we represent the solution outside both threads by the sum of an expansion like the one in (16) in terms of cylindrical coordinates centered at thread 1 and a similar expansion in terms of cylindrical coordinates centered at thread 2. The addition of the velocities from both expansions is done vectorially. Then, we obtain

$$\begin{aligned}
p_c &= p_{c1} + p_{c2}, \\
u_c &= u_{c1} - u_{c2} \cos(\phi_1 + \phi_2) + v_{c2} \sin(\phi_1 + \phi_2), \\
v_c &= v_{c1} + u_{c2} \sin(\phi_1 + \phi_2) + v_{c2} \cos(\phi_1 + \phi_2), \\
w_c &= w_{c1} + w_{c2}.
\end{aligned} \tag{18}$$

Note that here  $u_c$  and  $v_c$  are the radial and azimuthal components with respect to the coordinate system of thread 1. The indices  $cj$  ( $j = 1, 2$ ) denote the solution of the continuous phase if only thread  $j$  is taken into account. For example, the radial velocity of the continuous phase due to thread 1 is written as

$$u_{c1} = \sum_{n=0}^{\infty} u_{c1n}(r_1, t) \cos n\phi_1 \cos kz. \tag{19}$$

During the discussion we will take thread 1 as a coordinate of reference. Using this relations,

we find that the solution in the continuous phase is given by

$$\begin{aligned}
p_c(r_1, r_2, \phi_1, \phi_2, z, t) &= \sum_{n=0}^{\infty} p_{c1n}(r_1, t) \cos n\phi_1 \cos kz \\
&\quad + \sum_{n=0}^{\infty} p_{c2n}(r_2, t) \cos n\phi_2 \cos(kz - \alpha), \quad (a) \\
u_c(r_1, r_2, \phi_1, \phi_2, z, t) &= \sum_{n=0}^{\infty} u_{c1n}(r_1, t) \cos n\phi_1 \cos kz \\
&\quad - \sum_{n=0}^{\infty} u_{c2n}(r_2, t) \cos(\phi_1 + \phi_2) \cos n\phi_2 \cos(kz - \alpha) \\
&\quad + \sum_{n=1}^{\infty} v_{c2n}(r_2, t) \sin(\phi_1 + \phi_2) \sin n\phi_2 \cos(kz - \alpha), \quad (b) \\
v_c(r_1, r_2, \phi_1, \phi_2, z, t) &= \sum_{n=1}^{\infty} v_{c1n}(r_1, t) \sin n\phi_1 \cos kz \\
&\quad + \sum_{n=0}^{\infty} u_{c2n}(r_2, t) \sin(\phi_1 + \phi_2) \cos n\phi_2 \cos(kz - \alpha) \\
&\quad + \sum_{n=1}^{\infty} v_{c2n}(r_2, t) \cos(\phi_1 + \phi_2) \sin n\phi_2 \cos(kz - \alpha), \quad (c) \\
w_c(r_1, r_2, \phi_1, \phi_2, z, t) &= \sum_{n=0}^{\infty} w_{c1n}(r_1, t) \cos n\phi_1 \sin kz \\
&\quad + \sum_{n=0}^{\infty} w_{c2n}(r_2, t) \cos n\phi_2 \sin(kz - \alpha). \quad (d)
\end{aligned} \tag{20}$$

Now, considering that the solution inside thread 1 should remain finite for  $r_1 \rightarrow 0$ , we obtain

$$\begin{aligned}
p_{1n}(r_1, t) &= 2\mu_d A_{1n}(t) I_n(kr_1), \quad (a) \\
u_{10}(r_1, t) &= A_{10}(t) r_1 I_0(kr_1) - \left[ B_{10}(t) + \frac{2}{k} A_{10}(t) \right] I_1(kr_1), \quad (b) \\
u_{1n}(r_1, t) &= A_{1n}(t) r_1 I_n(kr_1) - \left[ B_{1n}(t) + \frac{1}{k} (n+2) A_{1n}(t) \right] I_{n+1}(kr_1) \\
&\quad + \frac{C_{1n}(t)}{r_1} I_n(kr_1), \quad (c) \quad (21) \\
v_{1n}(r_1, t) &= - \left[ B_{1n}(t) + \frac{1}{k} (n+2) A_{1n}(t) + \frac{k}{n} C_{1n}(t) \right] I_{n+1}(kr_1) \\
&\quad - \frac{1}{r_1} C_{1n}(t) I_n(kr_1), \quad (d) \\
w_{1n}(r_1, t) &= -A_{1n}(t) r_1 I_{n+1}(kr_1) + B_{1n}(t) I_n(kr_1). \quad (e)
\end{aligned}$$

The analogue of (21) for thread 2 is obtained by replacing 1 by 2. For the continuous phase

one has the requirement that the solution should be bounded at infinity. We obtain

$$p_{c1n}(r_1, t) = 2\mu_c D_{1n}(t) K_n(kr_1), \quad (a)$$

$$p_{c2n}(r_2, t) = 2\mu_c D_{2n}(t) K_n(kr_2), \quad (b)$$

$$u_{c10}(r_1, t) = D_{10}(t)r_1 K_0(kr_1) + \left[ E_{10}(t) + \frac{2}{k} D_{10}(t) \right] K_1(kr_1), \quad (c)$$

$$u_{c20}(r_2, t) = D_{20}(t)r_2 K_0(kr_2) + \left[ E_{20}(t) + \frac{2}{k} D_{20}(t) \right] K_1(kr_2), \quad (d)$$

$$u_{c1n}(r_1, t) = D_{1n}(t)r_1 K_n(kr_1) + \left[ E_{1n}(t) + \frac{1}{k}(n+2)D_{1n}(t) \right] K_{n+1}(kr_1) + \frac{F_{1n}(t)}{r_1} K_n(kr_1), \quad (e)$$

$$u_{c2n}(r_2, t) = D_{2n}(t)r_2 K_n(kr_2) + \left[ E_{2n}(t) + \frac{1}{k}(n+2)D_{2n}(t) \right] K_{n+1}(kr_2) + \frac{F_{2n}(t)}{r_2} K_n(kr_2), \quad (f) \quad (22)$$

$$v_{c1n}(r_1, t) = \left[ E_{1n}(t) + \frac{1}{k}(n+2)D_{1n}(t) + \frac{k}{n}F_{1n}(t) \right] K_{n+1}(kr_1) - \frac{1}{r_1} F_{1n}(t) K_n(kr_1), \quad (g)$$

$$v_{c2n}(r_2, t) = \left[ E_{2n}(t) + \frac{1}{k}(n+2)D_{2n}(t) + \frac{k}{n}F_{2n}(t) \right] K_{n+1}(kr_2) - \frac{1}{r_2} F_{2n}(t) K_n(kr_2), \quad (h)$$

$$w_{c1n}(r_1, t) = D_{1n}(t)r_1 K_{n+1}(kr_1) + E_{1n}(t) K_n(kr_1), \quad (i)$$

$$w_{c2n}(r_2, t) = D_{2n}(t)r_2 K_{n+1}(kr_2) + E_{2n}(t) K_n(kr_2). \quad (j)$$

This completes the general solution for the system with two threads. To find the values for the unknown coefficients  $A_{1n}, A_{2n}, B_{1n}, B_{2n}, \dots, F_{1n}, F_{2n}$ , the expressions (21)-(22) must be substituted into the boundary conditions. These boundary conditions are

- no slip at the interfaces :  $[[u]] = [[v]] = [[w]] = 0$ ; with  $[[u]]$  the jump in  $u$ , etc;
  - continuity of the tangential stresses at the interfaces :  $[[\sigma_{\phi r}]] = [[\sigma_{zr}]] = 0$ ;
  - discontinuity of the normal stresses at the interfaces due to the interfacial tension :  $[[\sigma_{rr}]] = \Delta\sigma_{rr,j}$ .
- (23)

These conditions are evaluated up to linear order, which implies that all evaluations are taken with respect to  $r = a$ , the initial radius of the thread. Here,  $\Delta\sigma_{rr,j}$  is the interfacial tension at thread  $j$ , defined as

$$\Delta\sigma_{rr,j} = \sigma(\kappa_1 + \kappa_2), \quad (24)$$

where  $\sigma$  is the surface tension and  $\kappa_1$  and  $\kappa_2$  are the two main curvatures which in linear approximation are given by

$$\begin{aligned} \kappa_1 &= \frac{\partial^2 R_j / \partial z^2}{(1 + (\partial R_j / \partial z)^2)^{3/2}} \approx \frac{\partial^2 R_j}{\partial z^2}, \\ \kappa_2 &= -\frac{R_j^2 + 2(\partial R_j / \partial \phi)^2 - R_j(\partial^2 R_j / \partial \phi^2)}{(R_j^2 + (\partial R_j / \partial \phi)^2)^{3/2}} \approx \frac{1}{R_j^2} \frac{\partial^2 R_j}{\partial \phi^2} - \frac{1}{R_j}. \end{aligned} \quad (25)$$

Hence, combining (1) and (25) we find for the linearized perturbed interfacial tensions

$$\begin{aligned}\Delta\sigma_{rr,1}(\phi_1, z, t) &= \sigma \left( \sum_{n=0}^{\infty} \frac{1 - (ak)^2 - n^2}{a^2} \varepsilon_{1n}(t) \cos n\phi_1 \right) \cos kz, \\ \Delta\sigma_{rr,2}(\phi_2, z, t) &= \sigma \left( \sum_{n=0}^{\infty} \frac{1 - (ak)^2 - n^2}{a^2} \varepsilon_{2n}(t) \cos n\phi_2 \right) \cos(kz - \alpha).\end{aligned}\quad (26)$$

Substituting (21), (22) and (26) into boundary conditions (23), we obtain an infinite set of equations for the unknown coefficients. We shall obtain approximate solutions by using the method of moments. This method consists of two steps:

- Firstly, we truncate the expansions in (16) at some cut-off value  $N$ , i.e. we take into account

$$2 \times (4 + 6N) = 8 + 12N \quad (27)$$

unknowns and ignore the other ones. Except for  $N = 0$  each order contains 12 unknowns (the expressions for  $N = 0$  do not contain  $C_{j0}$  and  $F_{j0}$ , and therefore for  $N = 0$  we only have 8 unknowns).

- Secondly, we solve the unknowns by means of the boundary conditions at the two interfaces. In linear theory, the boundary conditions (23) have to be evaluated at the interface  $S_1$  given by  $r_1 = a$  and  $S_2$  given by  $r_2 = a$ . In these conditions, we have to evaluate all quantities at  $S_j$  in terms of  $\phi_j$ : At the interface between thread 1 and the continuous phase, the quantities of the continuous phase should be expressed in a Fourier expansion of  $\phi_1$  only, whereas at the interface between thread 2 and the continuous phase it is the other way around. This implies that the product of a Bessel function and a Fourier expansion with respect to thread 2 should be expressed as product of Bessel functions and a Fourier expansion with respect to thread 1 or the other way around. To handle this, the following geometrical relations, derived from Figure 2,

$$\begin{aligned}r_1 \cos \phi_1 &= b - r_2 \cos \phi_2, \\ r_1 \sin \phi_1 &= r_2 \sin \phi_2, \\ z_1 &= z_2,\end{aligned}\quad (28)$$

and the addition theorem of Bessel function [20] are used. The theorem states that for  $r_1 < b$ ,

$$K_n(kr_2) \cos n\phi_2 = \sum_{m=-\infty}^{\infty} K_{n+m}(kb) I_m(kr_1) \cos m\phi_1. \quad (29)$$

This relation also holds with  $\cos$  replaced by  $\sin$ . Applying (28,29), the boundary conditions (23) at  $S_1$  are now functions of  $r_1$  and  $\phi_1$  containing terms with  $\cos m\phi_1$  or  $\sin m\phi_1$ . Thus, the jump  $\llbracket G \rrbracket$ , with  $G$  representing one of the quantities  $u, v, w, \sigma_{\phi r}, \sigma_{zr}, \sigma_{rr}$ , is now described as

$$\llbracket G \rrbracket = \sum_{m=0}^{\infty} \llbracket G \rrbracket_m f_m(\phi_1), \quad (30)$$

where  $f_m(\phi_1)$  is either  $\cos m\phi_1$  or  $\sin m\phi_1$ , depending on whether  $G$  is odd (i.e.  $v, \sigma_{\phi r}$ ) or even in  $\phi_1$ . In principle, boundary conditions (23) must be satisfied *for every value of  $\phi_1$* , which results in an infinite number of equations. To make the problem tractable, we will not require point-wise satisfaction of the boundary conditions, but instead that the  $n$ -th order moments, with  $n \in [0, 1, \dots, N]$ , of the boundary conditions are satisfied. Here, the  $n$ -th order moment of  $\llbracket G \rrbracket$  is defined as

$$\mathcal{M}_n\{\llbracket G \rrbracket\} = \int_0^{2\pi} \llbracket G \rrbracket f_n(\phi_1) d\phi_1. \quad (31)$$

By taking the first  $N + 1$  moments of the boundary conditions (8 conditions for  $N = 0$  and 12 conditions for every  $n$  with  $1 \leq n \leq N$ ), we thus obtain  $(8 + 12N)$  equations for  $(8 + 12N)$  unknowns. Hence, in this way we obtain a finite set of equations for the unknown coefficients.

In this paper, we will restrict ourselves to investigate the solution for the case  $N = 0$ , which is called as the zero-order solution. The case  $N = 1$  is dealt with in [2].

### 3 The zero-order solution ( $N=0$ )

In §3.1 we will evaluate the boundary conditions for  $N = 0$ . We realize from the previous section that this evaluation will lead to an infinite set of equations for the 8 unknown coefficients ( $A_{i0}, B_{i0}, D_{i0}, E_{i0}, i = 1, 2$ ;  $C_{i0}$  and  $F_{i0}$  are already given by relation (11)). Using the method of moments, we will reduce this infinite set to a finite set of 8 linear equations. The solution of this set is discussed in §3.2.

#### 3.1 The evaluation of boundary conditions

In the case  $N = 0$  we have 8 unknowns and we do not need the boundary conditions for  $v$  and  $\sigma_{r\phi}$ . From (16) we see that  $v_1$  is identically zero. This is not so for  $v_c$ , as follows from (18). However,  $v_c$  is an odd function in  $\phi_1$ , and consequently the zero-moment is zero. The same holds for  $\sigma_{r\phi}$ .

The evaluation of boundary conditions at the interface  $S_1$  requires to express all quantities in terms of  $(r_1, \phi_1)$ . As an example, we will work this out for the continuity of the radial velocity. Requiring  $\llbracket u \rrbracket = 0$ , we find using (16)-(22)

$$\llbracket u \rrbracket = X(t) \cos kz + Y(r_2, \phi_1, \phi_2, t) \cos(kz - \alpha) = 0, \quad (32)$$

where

$$\begin{aligned} X(t) = & \left( aI_0(ka) - \frac{2}{k}I_1(ka) \right) A_{10}(t) - I_1(ka)B_{10}(t) \\ & - \left( aK_0(ka) + \frac{2}{k}K_1(ka) \right) D_{10}(t) - K_1(ka)E_{10}(t), \end{aligned} \quad (33)$$

and

$$Y(r_2, \phi_1, \phi_2, t) = \left[ \left( r_2K_0(kr_2) + \frac{2}{k}K_1(kr_2) \right) D_{20}(t) + E_{20}(t)K_1(kr_2) \right] \cos(\phi_1 + \phi_2). \quad (34)$$

We note that  $X$  and  $Y$  are independent of  $z$ . Since (32) should hold for every value of  $z$ , there only exists a non-trivial solution of (32) if either  $\alpha = 0$  or  $\alpha = \pi$ . Hence, we only meet with the two cases

1.  $\alpha = 0$ , which is equivalent to  $\varepsilon_{10}(t) = \varepsilon_{20}(t)$  in (1), implying that the threads will disintegrate in-phase.
2.  $\alpha = \pi$ , which is equivalent to  $\varepsilon_{10}(t) = -\varepsilon_{20}(t)$  in (1), implying that the threads will disintegrate out-of-phase.

Condition (32) then reduces to

$$X(t) \pm Y(r_2, \phi_1, \phi_2, t) = 0, \quad (35)$$

where  $+$  or  $-$  sign corresponds to  $\alpha = 0$  or  $\alpha = \pi$ , respectively. Here,  $Y$  is still written in terms of  $(r_2, \phi_1, \phi_2)$ , but by use of (28) and (29) we can evaluate  $r_2$  and  $\phi_2$  as functions of  $\phi_1$  only. As an example, we show this for the first term of  $Y$  in (34):

$$\begin{aligned} r_2 K_0(kr_2) \cos(\phi_1 + \phi_2) &= K_0(kr_2)[r_2 \cos \phi_1 \cos \phi_2 - r_2 \sin \phi_1 \sin \phi_2] \\ &= K_0(kr_2)[(b - r_1 \cos \phi_1) \cos \phi_1 - r_1 \sin \phi_1 \sin \phi_1] \\ &= \sum_{m=-\infty}^{\infty} K_m(kb) I_m(kr_1) \cos m\phi_1 [b \cos \phi_1 - r_1] \\ &= \sum_{m=-\infty}^{\infty} K_m(kb) I_m(kr_1) \left[ \frac{b}{2} (\cos(m+1)\phi_1 + \cos(m-1)\phi_1) - r_1 \cos m\phi_1 \right]. \end{aligned} \quad (36)$$

The calculation above is a correction on the calculation by Knops [19], who made the approximation  $\cos(\phi_1 + \phi_2) \approx \cos \phi_1$ . Along the same lines, we conclude that (35) can generally be written as

$$[\hat{u}] = [\hat{u}]_0 + \sum_{m=1}^{\infty} [\hat{u}]_m \cos m\phi_1 = 0. \quad (37)$$

Basically, we obtain an infinite set of equations namely  $[\hat{u}]_m = 0$ , for  $m = 0, 1, 2, \dots$ . Since we are interested in the zero-order solution of (37), we approximate this expansion by taking the zero-moment of it, yielding  $[\hat{u}]_0 = 0$ , only. For instance, from expression (35) we find in this way at the interface  $S_1$ :

$$\begin{aligned} &A_{10}[kaI_0(ka) - 2I_1(ka)] - B_{10}kI_1(ka) - D_{10}[kaK_0(ka) + 2K_1(ka)] \\ &- E_{10}kK_1(ka) \pm D_{20}[kbK_1(kb)I_1(ka) - kaK_0(kb)I_0(ka) + 2K_0(kb)I_1(ka)] \\ &\pm E_{20}kK_0(kb)I_1(ka) = 0. \end{aligned} \quad (38)$$

Here, the coefficients  $A_{10}, B_{10}$ , etc. are still unknown. Evaluating the zero-moment of all boundary conditions and choosing an appropriate ordering of the unknowns, we arrive at a matrix equation for the unknown coefficients. In the next section, we will discuss the calculation of these unknown coefficients.

### 3.2 Solving the unknown coefficients

Evaluating the zero-moment of all boundary conditions, we arrive at the following matrix equation for the unknown coefficients

$$\mathbf{MZ} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} = \frac{\sigma}{2\mu_c k a^2} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad (39)$$

where the block matrices  $\mathbf{M}_{ij}$  have size 4 by 4, the vectors  $\mathbf{z}_1$  and  $\mathbf{z}_2$  are defined as

$$\mathbf{z}_i = (A_{i0}, B_{i0}, D_{i0}, E_{i0})^T, \quad \text{for } i = 1, 2, \quad (40)$$

and the right-hand side vectors are given by

$$\mathbf{e}_i = (0, 0, 0, (1 - k^2 a^2)\varepsilon_{i0})^T, \quad \text{for } i = 1, 2. \quad (41)$$

The explicit expressions for the entries of the matrix are given in Appendix B.1. We remark here that  $\mathbf{z}_i$  and  $\mathbf{e}_i$  are time-dependent. To obtain the lower part of the matrix  $\mathbf{M}$ , we simply may replace coordinates referring to thread 1 by coordinates referring to thread 2. From this we see that  $\mathbf{M}_{12} = \mathbf{M}_{21}$ , and moreover,  $\mathbf{M}_{22} = \mathbf{M}_{11}$ . Thus, we can simplify the calculation as follows. For the case  $\varepsilon_{10} = \varepsilon_{20}$  we find

$$\Delta^+ \mathbf{z}_1 = \frac{\sigma}{2\mu_c k a^2} \mathbf{e}_1, \quad \text{and } \mathbf{z}_2 = \mathbf{z}_1, \quad (42)$$

with  $\Delta^+ = \mathbf{M}_{11} + \mathbf{M}_{12}$ , and for the case  $\varepsilon_{10} = -\varepsilon_{20}$  we find

$$\Delta^- \mathbf{z}_1 = \frac{\sigma}{2\mu_c k a^2} \mathbf{e}_1, \quad \text{and } \mathbf{z}_2 = -\mathbf{z}_1, \quad (43)$$

with  $\Delta^- = \mathbf{M}_{11} - \mathbf{M}_{12}$ . This solves the unknown coefficients of the zero-order solution. For instance, from (43) we have

$$A_{10}(t) = -\frac{\sigma}{2\mu_c k a^2} \frac{(1 - k^2 a^2) |\Delta_{41}^-|}{|\Delta^-|} \varepsilon_{10}(t), \quad (44)$$

where  $|\cdot|$  denotes the determinant and  $\Delta_{41}^-$  is the  $3 \times 3$  sub-matrix of  $\Delta^-$  which can be found by omitting the fourth row and the first column of  $\Delta^-$ . We note that this coefficient is proportional to the amplitude  $\varepsilon_{10}(t)$ , and this evidently also holds for the other coefficients (see Appendix B.2), as we are dealing with a linear problem. This observation will be important in the next discussion.

In the next section, the growth of the disturbance amplitude describing the (in)stability of the breaking up process of the threads is analyzed.

## 4 Stability analysis

In [3], as well as in [21], it is claimed that a cylindrical jet of water becomes unstable if the wavelength of the disturbances is greater than the circumference of the cylinder. Plateau

concluded that a cylindrical jet will break up into pieces of length equal to the circumference of the cylinder. This conclusion is incorrect, as Rayleigh showed [4]. Rayleigh found that the *mode of maximum instability* mainly determines the break up of an unstable system. If a system is characterized by a number of unstable modes labeled by  $k_1, k_2, \dots, k_n$ , having amplitudes  $e^{q_1 t}, e^{q_2 t}, \dots, e^{q_n t}$ ,  $q_i = q(k_i)$ , then the mode of maximum instability is the one pertinent to the  $q$  with the largest real part. In view of this, we will focus on finding the mode of maximum instability of the system under consideration. We start with the zero-order case. Again, thread 1 will be taken as the coordinate of reference. In view of (1) and (16), we may write the disturbance amplitudes for every mode  $n$  as

$$\varepsilon_{1n}(t) = \varepsilon_{1n}(0) + \int_0^t u_{1n}(a, \tau) d\tau. \quad (45)$$

Since  $u_{10}$  is proportional to  $\varepsilon_{10}$ , (45) reduces for  $n = 0$  to

$$\varepsilon_{10}(t) = \varepsilon_{10}(0) + Q(b, \mu, k) \int_0^t \varepsilon_{10}(\tau) d\tau, \quad (46)$$

where  $Q$  is given by

$$Q(b, \mu, k) = \frac{1}{\varepsilon_{10}(0)} \left[ \left( aI_0(ka) - \frac{2}{k} I_1(ka) \right) A_{10}(0) - I_1(ka) B_{10}(0) \right]. \quad (47)$$

Taking the derivative of (46) with respect to  $t$ , we obtain the first order scalar differential equation :

$$\frac{d}{dt} \varepsilon_{10} = Q(b, \mu, k) \varepsilon_{10}(t). \quad (48)$$

The behaviour of the solution of (48) depends on the the sign of  $Q(b, \mu, k)$ . This parameter acts as "the degree of instability" mentioned in [6]. We denote its dimensionless version by  $q(b, \mu, k)$ , which is defined as

$$q = \frac{2\mu c a}{\sigma} Q(b, \mu, k). \quad (49)$$

For the case  $\varepsilon_{10} = \varepsilon_{20}$  we use the notation  $q^+$  and for the case  $\varepsilon_{10} = -\varepsilon_{20}$  the notation  $q^-$ . They correspond with the in-phase and out-of-phase modes, respectively.

To get insight in the break up behaviour of the two threads system, we calculated the factor  $q(b, \mu, k)$  for  $N = 0$ . Fixing the geometrical parameter  $b$  and the material parameter  $\mu$ , we determine the values  $q^+$  and  $q^-$  by optimizing over  $k$ , since the (in)stability of the system is determined by the mode corresponding with the highest  $q^\pm$ -value. We denote this highest value of  $q^\pm$  by  $q_{max}^\pm = \max\{q^\pm(k; \mu, b)\}$ . In Figure 3 the values of  $q_{max}^+$  and  $q_{max}^-$ , calculated this way, are given as functions of the relative distance  $\hat{b} = b/a$  of the threads, for two values of  $\mu$ . The case  $\mu = 0.04$  corresponds to a situation in which the threads are less viscous than the surrounding fluid, whereas for  $\mu = 4$  it is the other way around.

For  $\mu = 0.04$ ,  $q_{max}^-$  is always greater than  $q_{max}^+$ . This implies that the threads will break up out-of-phase. However, for  $\mu = 4$ , the curves cross at a critical distance  $b^*$ . If  $\hat{b} < b^*$  the threads will break up in-phase, whereas if  $\hat{b} > b^*$  the threads will disintegrate out-of-phase. The discovery of the existence of this critical parameter  $b^*$  is a new result in this field. In

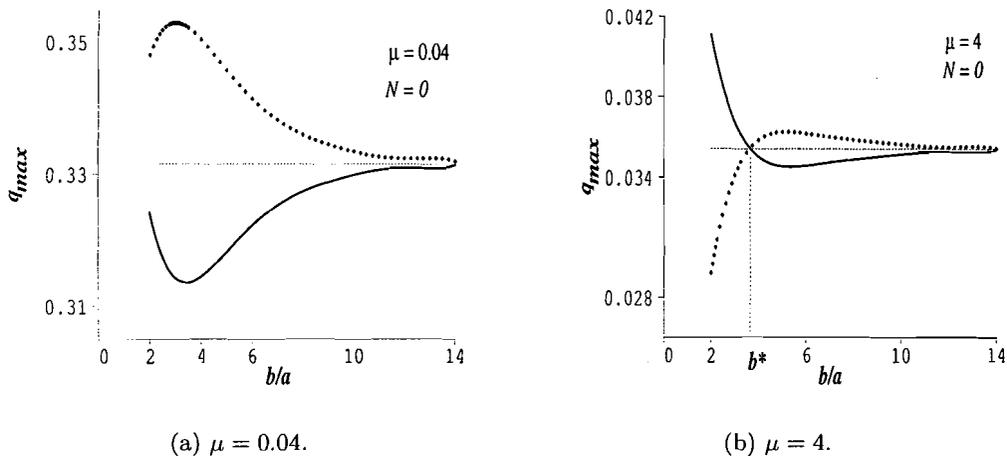


Figure 3: The solid and dotted lines indicate the in-phase and out-of-phase cases, respectively.

the limiting case  $\hat{b} \rightarrow \infty$ , when the problem tends to the simple case of one thread,  $q_{max}^+$  and  $q_{max}^-$  become equal to  $\hat{q}_{max}$ , say. This indicates that in that case the threads will break up independently of each other, and then none of the phase relations are preferable. The limiting value  $\hat{q}_{max}$  for the single thread case is denoted by the horizontal line.

## 5 Conclusions

In this paper, we have described a general method to determine the degree of instability for a set of two parallel viscous threads immersed in a viscous solvent. The general solution of the underlying Stokes problem was decomposed into Fourier modes, and the boundary conditions then led to an infinite set of equations for the unknown coefficients. This set was truncated at finite  $N$ , and the truncated set was solved by use of the method of moments.

The method was evaluated for the zero-order ( $N = 0$ ). The zero-order approximation leads to relatively simple equations for the unknown coefficients. No infinite series are involved in the entries of the matrix of coefficients as was found in [19] due to redundant simplifications. The behaviour of the break-up process of the threads is characterized by the degree of instability  $q$ , which depends on the viscosity ratio  $\mu$ , the wave number of the disturbances  $k$  and the distance between the two threads  $b$ . For large viscosity ratio, when the threads are more viscous than the matrix, we found a critical relative distance  $b_{cr}$ . Below it the threads will break up in-phase, above it out-of-phase. The smaller the viscosity ratio, the more the out-of-phase break up is preferred. This is in accordance with experimental results reported in Knops [19].

From the point of view of blend production the present results may provide important insights for control of the production process. Two aspects deserve mentioning: characteristic drop formation times and spatial distributions of the droplets. As for formation time, the time scale of the dynamical process is governed by the value of the  $q$ -factor, plotted in Figures 3. In most extrusion devices the blend is only a restricted time in the molten state.

As soon as it is cooled down, the spatial droplet distribution at that moment will freeze in. So, from industry there is a need for fast formation processes. From the results summarized in Figures 3 it is clear that high values for  $\mu$  and small values for  $b$  are highly favourable for speeding up the break-up process.

As for the spatial distribution of the droplets, it is highly important to know that either in-phase or out-of-phase break-up may occur. If the present results also apply to system with many threads, it must be possible to produce blends with the droplets either on a cubic grid, resulting from in-phase break-up, or on a body centered cubic grid, resulting from out-of-phase break-up, just by controlling the two parameters  $\mu$  and  $b$  in an appropriate way. So, this implies in practice that important properties of the blends can be adjusted quite easily, since both  $\mu$  and  $b$  are simply controlled in production conditions, since  $\mu$  is a material property and  $b$  is essentially determined by the volume fraction of the big drops in the initial situation.

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## A Some expressions related to the addition theorem for Bessel functions

Below, some relations for Bessel functions used in the evaluation of the boundary conditions are given. The prime denotes derivation with respect to  $x$  where  $x = kr_1$ ,  $[-]_{r_1}$  denotes derivation with respect to  $r_1$ ,  $y = kb$  and  $z = kr_2$ .

$$\begin{aligned}
K_1(z) \cos(\phi_1 + \phi_2) &= \sum_{m=-\infty}^{\infty} K_{m+1}(y) I_m(x) \cos(m+1)\phi_1 \\
r_2 K_1(z) &= \sum_{m=-\infty}^{\infty} K_{m+1}(y) I_m(x) [b \cos m\phi_1 - r_1 \cos(m+1)\phi_1] \\
K_0(z) &= \sum_{m=-\infty}^{\infty} K_m(y) I_m(x) \cos m\phi_1 \\
[K_0(z)]_{r_1} &= \sum_{m=-\infty}^{\infty} k K_m(y) I'_m(x) \cos m\phi_1 \\
[K_1(z) \cos(\phi_1 + \phi_2)]_{r_1} &= \sum_{m=-\infty}^{\infty} k K_{m+1}(y) I'_m(x) \cos(m+1)\phi_1 \\
[r_2 K_1(z)]_{r_1} &= \sum_{m=-\infty}^{\infty} k K_{m+1}(y) I'_m(x) [b \cos m\phi_1 - r_1 \cos(m+1)\phi_1] \\
&\quad - \sum_{m=-\infty}^{\infty} K_{m+1}(y) I_m(x) \cos(m+1)\phi_1 \\
[r_2 K_0(z) \cos(\phi_1 + \phi_2)]_{r_1} &= \sum_{m=-\infty}^{\infty} k K_m(y) I'_m(x) [b (\frac{\cos(m+1)\phi_1 + \cos(m-1)\phi_1}{2}) - r_1 \cos m\phi_1] \\
&\quad - \sum_{m=-\infty}^{\infty} K_m(y) I_m(x) \cos m\phi_1
\end{aligned}$$

## B Appendix

### B.1 Explicit expressions for the block matrices $M_{11}$ and $M_{12}$ in 39

Here  $x = ka$ ,  $y = kb$  and  $\mu = \mu_d/\mu_c$ .

$$M_{11} = \begin{pmatrix} xI_0(x) - 2I_1(x) & -kI_1(x) & -(xK_0(x) + 2K_1(x)) & -kK_1(x) \\ -xI_1(x) & kI_0(x) & -xK_1(x) & -kK_0(x) \\ -2\mu xI_1'(x) & 2\mu kI_1(x) & -2xK_1'(x) & 2kK_1(x) \\ \mu(xI_1(x) - 2I_1'(x))/k & -\mu I_1'(x) & -(2K_1'(x) - xK_1(x))/k & -K_1'(x) \end{pmatrix}$$

$$M_{12} = \begin{pmatrix} 0 & 0 & (yK_1(y)I_1(x) - xK_0(y)I_0(x) + 2K_0(y)I_1(x)) & kK_0(y)I_1(x) \\ 0 & 0 & -(yK_1(y)I_0(x) - xK_0(y)I_1(x)) & -kK_0(y)I_0(x) \\ 0 & 0 & -(2yK_1(y)I_1(x) - 2xK_0(y)I_0(x) + 2K_0(y)I_1(x)) & -2kK_0(y)I_1(x) \\ 0 & 0 & -(xK_0(y)I_1(x) - yK_1(y)I_1'(x) - 2K_0(y)I_1'(x))/k & K_0(y)I_1'(x) \end{pmatrix}$$

where  $x = ka$ ,  $\mu = \mu_d/\mu_c$ ,  $I_1'(x) = I_0(x) - I_1(x)/x$  and  $K_1'(x) = -K_0(x) - K_1(x)/x$ .

### B.2 Explicit solution of (42) and (43)

We denote  $\Delta_{mn}^s$  is a  $3 \times 3$  submatrix of  $\Delta^s$  obtained by omitting the  $m$ -th row and the  $n$ -th column of  $\Delta^s$  and  $|\cdot|$  indicates the determinant of a matrix. In our case,  $s$  can be either  $+$  or  $-$ .

$$A_{10}(t) = -\frac{\sigma}{2\mu_c x a} \frac{(1-x^2)|\Delta_{41}^s|}{|\Delta^s|} \varepsilon_{10}(t)$$

$$B_{10}(t) = \frac{\sigma}{2\mu_c x a} \frac{(1-x^2)|\Delta_{42}^s|}{|\Delta^s|} \varepsilon_{10}(t)$$

$$D_{10}(t) = -\frac{\sigma}{2\mu_c x a} \frac{(1-x^2)|\Delta_{43}^s|}{|\Delta^s|} \varepsilon_{10}(t)$$

$$E_{10}(t) = \frac{\sigma}{2\mu_c x a} \frac{(1-x^2)|\Delta_{44}^s|}{|\Delta^s|} \varepsilon_{10}(t)$$