

## Acknowledgement of priority to C. Flye Sainte-Marie on the counting of circular arrangements of $2^n$ zeros and ones that show each $n$ -letter word exactly once

**Citation for published version (APA):**

Bruijn, de, N. G. (1975). *Acknowledgement of priority to C. Flye Sainte-Marie on the counting of circular arrangements of  $2^n$  zeros and ones that show each  $n$ -letter word exactly once*. (EUT report. WSK, Dept. of Mathematics and Computing Science; Vol. 75-WSK-06). Technische Hogeschool Eindhoven.

**Document status and date:**

Published: 01/01/1975

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

**General rights**

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

[www.tue.nl/taverne](http://www.tue.nl/taverne)

**Take down policy**

If you believe that this document breaches copyright please contact us at:

[openaccess@tue.nl](mailto:openaccess@tue.nl)

providing details and we will investigate your claim.

TECHNISCHE HOGESCHOOL EINDHOVEN

NEDERLAND

ONDERAFDELING DER WISKUNDE

TECHNOLOGICAL UNIVERSITY EINDHOVEN

THE NETHERLANDS

DEPARTMENT OF MATHEMATICS

Acknowledgement of priority to C. Flye Sainte-Marie  
on the counting of circular arrangements of  $2^n$   
zeros and ones that show each n-letter word  
exactly once

by

N.G. de Bruijn

The problem of finding the arrangements mentioned in the title became generally known through my paper [2] of 1946, in which the number of solutions was found to be  $2^{2^{n-1}-n}$ .

Quite recently Richard Peter Stanley (Massachusetts Institute of Technology) discovered that the problem had been proposed and solved half a century earlier in the french problem journal "l'Intermédiaire des Mathématiciens" in 1894. The problem was raised by A. de Rivière [14] and solved by C. Flye Sainte-Marie [4].

C. Flye Sainte-Marie found the same number  $2^{2^{n-1}-n}$ , and his method is more or less the same as the one in [2]. His style is of course not the one in which we write today, and it is very condensed. Therefore it is hard to read; nevertheless if we compare [2] to [4] we do not see much difference in content.

As appendices, the items [14], [4], [2] are reproduced at the end of this note.

It seems to be appropriate to end this note with some historical and bibliographical remarks.

After the problem was proposed and solved in 1894, it was entirely forgotten until 1934, when it was reintroduced by M.H. Martin [13]. Independently it was proposed around 1944 by the telecommunication engineer K. Posthumus (at that time at Philips Research Laboratories, Eindhoven). He conjectured the  $2^{2^{n-1}-n}$ , based on his count for  $n = 1, 2, 3, 4, 5$ . This conjecture was proved in [2], (cf. the presentations in [5], [7]). The oriented graphs that played a rôle in this count were produced independently and simultaneously by I.J. Good [6] (who used it for the existence proof, not for the count).

The corresponding counting problem for an alphabet with  $\sigma$  letters seems to have been first raised and solved in [1] (1951); the number is  $(\sigma!) \sigma^{n-1} \sigma^{-n}$ . That paper solved the problem by generalizing the directed graph method of [2] and [4], but also indicated how the count can be related to counting trees in a directed graph, the number of which can be expressed by a determinant. This determinant was not evaluated in [1]; the first paper in which it was achieved is [3] (1957). For this determinant method reference should be made to [8] and to the really very short presentation in [11].

The references in [11] may be used as a point of departure for tracing more literature, in particular papers treating algorithms for finding solutions, rather than counting the total number. In this context one might add a reference to Korobov ([9],[10]) who went beyond what was indicated in [6], using the construction for the production of "normal" real numbers in the sense of Borel. (A real number  $\alpha$  is called normal if, for every  $n$ , in the sequence of words we get by taking  $n$  consecutive digits from the binary representation of  $\alpha$ , each  $n$ -letter word has the same asymptotic frequency).

For the case of the  $\sigma$ -letter alphabet a special construction (although not an algorithm) was presented (as a solution to [14]) by W. Mantel [12] in 1897 with restriction to the case that  $\sigma$  is a prime. Mantel takes an  $n$ -th degree irreducible polynomial  $F(x)$  over the field of  $\sigma$  elements, with  $F(0) = 1$  and develops  $x^{n-1}/F(x)$  in a power series  $a_0 + a_1x + \dots$ , where the coefficients are taken mod  $\sigma$ . He says that Serret has shown that there are  $\frac{1}{2}\phi(\sigma^n-1)$  possibilities for  $F$  such that  $a_0, a_1, a_2, \dots$  has period  $\sigma^n-1$ . If we now take the array  $0, a_0, \dots, a_{\sigma^n-1}$  and turn it into a circular array by pasting head to tail, we get a cycle showing each  $n$ -letter word exactly once.

This construction of Mantel was duplicated 50 years later in [15].

References

1. T. van Aardenne-Ehrenfest and N.G. de Bruijn, Circuits and trees in oriented linear graphs, *Simon Stevin* 28 (1951), 203-217.
2. N.G. de Bruijn, A combinatorial problem, *Nederl. Akad. Wetensch. Proc.* 49 (1946), 758-764 = *Indag. Math.* 8 (1946), 461-467.
3. R. Dawson and I.J. Good, Exact Markov Probabilities from Oriented Linear Graphs, *Ann. Math. Stat.* 28 (1957), 946-956.
4. C. Flye Sainte-Marie, Solution to question nr. 48, *l'Intermédiaire des Mathématiciens* 1 (1894) 107-110.
5. S. Golomb, *Shift Register Sequences*, Holden-Day, Inc., San Francisco (1967).
6. I.J. Good, Normal recurring decimals, *J. London Math. Soc.* 21, 167-169 (1947).
7. M. Hall, *Combinatorial Theory*, Blaisdell Publ. Comp., Waltham Mass. (1967).
8. D.E. Knuth, Oriented Subtrees of an Arc Digraph, *J. Comb. Theory* 3 (1967), 309-314.
9. N.M. Korobov, Concerning some questions of uniform distribution, *Izvestiya Akad. Nauk SSSR. Ser. Mat.* 14, 215-238 (1950). (Russian)
10. N.M. Korobov, Normal periodic systems and a question on the sums of fractional parts, *Uspehi Matem. Nauk (N.S.)* 5, no.3 (37), 135-136 (1950). (Russian).
11. J.H. van Lint, *Combinatorial Theory Seminar Eindhoven University of Technology*, Lecture Notes in Mathematics no. 125, Springer-Verlag 1974.
12. W. Mantel, Resten van wederkerige reeksen, *Nieuw Archief voor Wetkunde*, Serie 2, vol. 1 (1897) 172-184.
13. M.H. Martin, A problem in arrangements, *Bull. Amer. Math. Soc.* 40, (1934) 859-864.
14. A. de Rivière, Question nr. 48, *l'Intermédiaire des Mathématiciens* 1, (1894) 19-20.
15. D. Rees, Note on a paper by I.J. Good, *J. London Math. Soc.* 21 (1947) 169-172.

Appendix 1. The problem (reference [14])

48. [J1a2] Si l'on considère tous les arrangements  $n$  à  $n$  qu'on peut former avec deux objets, il est toujours possible de trouver un arrangement de  $2^n$  termes (formé avec les mêmes deux objets)  $a_1, a_2, a_3, \dots, a_{2^n}$ , tel que les groupes

$$a_1, a_2, \dots, a_n; \quad a_2, a_3, \dots, a_{n+1}; \quad \dots$$

$$a_{2^{n-1}}, a_{2^n}, a_1, \dots, a_{n-2}; \quad a_{2^n}, a_1, a_2, \dots, a_{n-1}$$

représentent tous les arrangements  $n$  à  $n$  dont le nombre est évidemment  $2^n$ . Cette proposition est vérifiée expérimentalement jusqu'à des limites suffisantes pour en présager l'exactitude. Est-elle déjà connue? Pourrait-on en donner une démonstration? Y a-t-il en général plus d'une espèce de solutions et dans ce cas combien?

A. DE RIVIÈRE.

Appendix 2. Solution by C. Flue Sainte-Marie (reference [4])

48. (A. DE RIVIÈRE). — Soient  $a$  et  $b$  les deux objets entrant dans la composition des arrangements considérés. Concevons, disposés dans l'ordre alphabétique, tous les arrangements de  $n - 1$  lettres qu'on peut former avec ces deux caractères et désignons chacun d'eux par le rang qu'il occupe. Tout arrangement  $A$  de  $n$  lettres, composé avec  $a$  et  $b$ , pourra être figuré par un *couple*, ou groupe de deux termes dont le premier (*antécédent*) désigne l'arrangement formé par les  $n - 1$  premières lettres, et le second (*conséquent*), l'arrangement formé par les  $n - 1$  dernières lettres de  $A$ . Les  $2^n$  arrangements de  $n$  lettres seront, ainsi, figurés par les couples

1, 1;	$2^{n-2} + 1, 1;$
1, 2;	$2^{n-2} + 1, 2;$
2, 3;	$2^{n-2} + 2, 3;$
2, 4;	$2^{n-2} + 2, 4;$
⋮	⋮
⋮	⋮
$2^{n-2}, 2^{n-1} - 1;$	$2^{n-1}, 2^{n-1} - 1;$
$2^{n-2}, 2^{n-1};$	$2^{n-1}, 2^{n-1}.$

A l'aide de ce tableau, dans lequel un même terme entre toujours deux fois comme antécédent et deux fois comme conséquent, les arrangements voulus se construisent aussi facilement que les chaînes composées avec les pièces d'un jeu de dominos se succédant suivant la règle de ce jeu.

Supposons, en effet, que tous les couples du Tableau ci-dessus, dont je désignerai l'ensemble par  $T_n$ , soient réunis dans une même chaîne  $C_n$  telle que chaque terme soit, à la fois, conséquent de celui qui précède et antécédent de celui qui suit; cette chaîne formera nécessairement un circuit fermé; pour en déduire un arrangement satisfaisant aux conditions voulues, il suffira de remplacer chaque terme impair par  $a$  et chaque terme pair par  $b$ . Ainsi, le nombre des solutions distinctes du problème est égal au nombre des chaînes  $C_n$  possibles, d'ordres circulaires différents. Je désigne ce nombre par  $\tau(n)$ .

Je dirai, en général, que l'ensemble des couples formés par deux termes consécutifs d'une chaîne constituant un circuit fermé est la *base* de cette chaîne. Ainsi  $T_n$  est la base des chaînes  $C_n$  et les chaînes  $C_n$  *appartiennent* à la base  $T_n$ .

J'appellerai *groupe carré* un ensemble de quatre couples tels que  $\alpha\alpha'$ ,  $\beta\beta'$ ,  $\alpha\beta'$ ,  $\beta\alpha'$ , et je représenterai ce groupe par la nota-

$$\text{tion } \begin{vmatrix} \alpha & \alpha' \\ \beta & \beta' \end{vmatrix}.$$

Cela posé, soit une base quelconque  $B$ , dans laquelle chaque terme entre deux fois seulement comme antécédent, deux fois seulement comme conséquent, et contenant un groupe carré

$$\begin{vmatrix} \alpha & \alpha' \\ \beta & \beta' \end{vmatrix}; \text{ soient } B' \text{ et } B'' \text{ deux nouvelles bases obtenues en sub-}$$

stituant, dans  $B$ , à ce groupe carré, d'une part les couples  $\alpha\alpha'$  et  $\beta\beta'$ , répétés chacun deux fois (base  $B'$ ), d'autre part, les couples  $\alpha\beta'$  et  $\beta\alpha'$  pareillement répétés (base  $B''$ ); soient, enfin,  $N$ ,  $N'$ ,  $N''$  les nombres des chaînes appartenant respectivement aux bases  $B$ ,  $B'$ ,  $B''$ :

On a, en général,  $N = 2(N' + N'')$ , à moins que les bases  $B'$  et  $B''$  ne contiennent plus que des couples répétés chacun deux fois; auquel cas, on a  $N = N' + N''$  (1).

---

(1) Ce théorème est une conséquence de la remarque suivante :  
Chacune des suites de termes qui relie l'un à l'autre les quatre couples

On pourra donc, successivement, éliminer de la base  $T_n$  les  $2^{n-2}$  groupes carrés qui la composent et lui substituer un certain nombre de bases  $\theta$  qui ne contiendront plus, chacune, que des couples répétés deux fois. En désignant par  $\nu$  le nombre total des chaînes  $\Gamma$  appartenant à ces bases, on aura, d'après le théorème précédent,  $\varphi(n) = 2^{2^{n-2}-1}\nu$ .

Je dis enfin que  $\nu = \varphi(n-1)$ . En effet, chaque base  $\theta$  engendre une chaîne unique ou n'en engendre aucune;  $\nu$  est donc le nombre des bases  $\theta$  capables d'engendrer une chaîne  $\Gamma$ . Remarquons que toute chaîne  $\Gamma$  se divise en deux suites  $\sigma$  identiques; or, dans une suite  $\sigma$ , tout couple appartient à un groupe carré de  $T_n$  et l'ordre de succession des couples  $y$  est entièrement défini par l'ordre de succession des groupes auxquels ils appartiennent. Si, en observant l'ordre de succession de ceux-ci, on remplace chaque groupe carré par le plus petit de ses antécédents, on vérifie facilement que la chaîne *corrélative* de  $\sigma$  ainsi formée appartient à la base  $T_{n-1}$ . Inversement, soit  $k$  une des chaînes  $C_{n-1}$  appartenant à  $T_{n-1}$ ; à chaque terme  $x$  de cette chaîne correspond un groupe carré de  $T_n$  ayant pour antécédents  $x$  et  $2^{n-2} + x$ ; si, dans  $K$ , on augmente de  $2^{n-2}$  tout terme ne formant pas, avec celui qui le précède, un couple de  $T_n$ , on

---

du groupe carré étant représentée par une seule lettre, toute chaîne appartenant à  $B$  affectera, selon l'ordre dans lequel ces couples s'y succèdent, un des six dispositifs suivants :

- |     |                     |                     |                    |                 |
|-----|---------------------|---------------------|--------------------|-----------------|
| (1) | $\alpha' g \beta,$  | $\beta' h x,$       | $\beta' k \beta,$  | $\alpha' l x,$  |
| (2) | $\alpha' g \beta,$  | $\beta' k \beta,$   | $\alpha' l x,$     | $\beta' h x,$   |
| (3) | $\alpha' l \alpha,$ | $\beta' k \beta,$   | $\alpha' g \beta,$ | $\beta' h x,$   |
| (4) | $\alpha' g \beta,$  | $\alpha' l x,$      | $\beta' k \beta,$  | $\beta' h x,$   |
| (5) | $\alpha' p \alpha,$ | $\beta' q \beta,$   | $\beta' q' \beta,$ | $\alpha' p' x,$ |
| (6) | $\alpha' r \beta,$  | $\alpha' r' \beta,$ | $\beta' s \alpha,$ | $\beta' s' x.$  |

Chacun de ces dispositifs est composé de quatre suites (séparées par les virgules) qu'on peut écrire dans six ordres circulaires différents; il en résultera ainsi, chaque fois, six chaînes généralement distinctes entre elles, dont quatre appartiennent à la base  $B$  et deux aux bases  $B'$  ou  $B''$ . Si  $B'$  et  $B''$  ne contiennent plus que des couples répétés deux fois, il n'y a de possibles que les dispositifs (5) ou (6), dont les quatre suites, par leurs six arrangements, ne produisent, dans ce cas, qu'une chaîne de base  $B$  et une chaîne de base  $B'$  ou  $B''$ .

Si l'on a  $\alpha = \alpha'$ , ou  $\beta = \beta'$ , les dispositifs sont modifiés, mais le théorème subsiste.

reconnait que la suite ainsi formée est une suite  $\sigma$ , dont les termes appartiendront aux mêmes groupes carrés de  $T_n$  et dont la chaîne corrélatrice est précisément la chaîne  $K$ .

On a donc  $\varphi(n) = 2^{2^n - 2^{n-1}} \varphi(n-1)$ ; d'ailleurs  $\varphi(2) = 1$ , on en déduit, par un calcul facile, la formule générale

$$\varphi(n) = 2^{2^n - 1 - n}.$$

C. FLYE SAINTE-MARIE.

Appendix 3. Reference [2]

**Mathematics.** — *A combinatorial problem.* By N. G. DE BRUIJN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of June 29, 1946.)

1. Some years ago Ir. K. POSTHUMUS stated an interesting conjecture concerning certain cycles of digits 0 or 1, which we shall call  $P_n$ -cycles<sup>1)</sup>. For  $n = 1, 2, 3, \dots$  a  $P_n$ -cycle be an ordered cycle of  $2^n$  digits 0 or 1, such that the  $2^n$  possible ordered sets of  $n$  consecutive digits of that cycle are all different. As a consequence, any ordered set of  $n$  digits 0 or 1 occurs exactly once in that cycle.

For example, a  $P_3$ -cycle is  $\overrightarrow{00010111}$ <sup>2)</sup>, respectively showing the triples 000, 001, 010, 101, 011, 111, 110, 100, which are all possible triples indeed.

For  $n = 1, 2, 3, 4$ , the  $P_n$ -cycles can easily be found.

We have only one  $P_1$ -cycle, viz. 01, and only one  $P_2$ -cycle, viz. 0011. There are two  $P_3$ -cycles, viz.  $\overrightarrow{00010111}$  and  $\overrightarrow{11101000}$ , and sixteen  $P_4$ -cycles, eight of which are

0000110100101111	1111001011010000
0000100110101111	1111011001010000
0000101100111101	1111010011000010
0000110101111001	1111001010000110

the remaining eight being obtained by reversing the order of these, respectively.

Ir. POSTHUMUS found the number of  $P_5$ -cycles to be 2048, and so he had the following number of  $P_n$ -cycles for  $n = 1, 2, 3, 4, 5$ :

$$1, 1, 2, 2^4, 2^{11},$$

$$\text{or } 2^{2^1-1}, 2^{2^2-2}, 2^{2^3-3}, 2^{2^4-4}, 2^{2^5-5}.$$

Thus he was led to the conjecture, that the number of  $P_n$ -cycles be  $2^{2^n-1-n}$  for general  $n$ . In this paper his conjecture is shown to be correct. Its proof is given in section 3, as a consequence of a theorem concerning a special type of networks, stated and proved in section 2. In section 4 another application of that theorem is mentioned.

2. We consider a special type of networks, which we shall call  $T$ -nets.

<sup>1)</sup> These arise from a practical problem in telecommunication.

<sup>2)</sup> With this notation,  $\overrightarrow{00010111}$ ,  $\overrightarrow{00101110}$ , etc., are to be considered as the same cycle. (Properly speaking, the digits must be placed around a circle.) On the other hand we do not identify the cycles  $\overrightarrow{00010111}$  and  $\overrightarrow{11101000}$ , the second of which is obtained by reversing the order of the first one.

Henceforth we simply write  $\overrightarrow{00010111}$  instead of  $\overrightarrow{00010111}$ .

A *T-net* of order  $m$  will be a network of  $m$  junctions and  $2m$  one-way roads (oriented roads), with the property that each junction is the start of two roads and also the finish of two roads. The network need not lie in a plane, or, in other words, viaducts, which are not to be considered as junctions, are allowed. Furthermore we do not exclude roads leading from a junction to that same junction, and we neither exclude pairs of junctions connected by two different roads, either in the same, or in opposite direction. Figs. 1a and 1b show examples of *T-nets*, of orders 3 and 6, respectively.

In a *T-net* we consider closed walks, with the property that any road of the net is used exactly once, in the prescribed direction. Such walks will be called "complete walks" of that *T-net*. Two complete walks are considered to be identical, if, and only if, the sequence of roads<sup>3)</sup> gone through in the first walk is a cyclic permutation of that in the second walk. The nets of figs. 1a and 1b admit 2 and 8 complete walks, respectively.

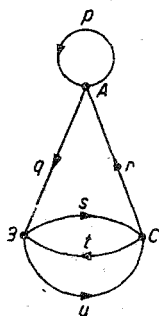


Fig. 1a

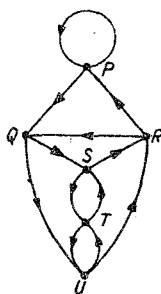


Fig. 1b

The number of complete walks of a *T-net*  $N$  be denoted by  $|N|$ . This number  $|N|$  is zero, if  $N$  is not connected, that is to say, if  $N$  can be divided into two separate *T-nets*<sup>4)</sup>.

We now describe a process, which we call the "doubling" of a *T-net*, and which is illustrated by the relation between the nets of figs. 1a and 1b. Be  $N$  a *T-net* of order  $m$ , with junctions  $A, B, C, \dots$ , and roads  $p, q, r, \dots$ . Then we construct the "doubled" net  $N^*$  by taking  $2m$  junctions  $P, Q, R, \dots$ , corresponding to the roads of  $N$ , respectively. We construct a one-way road from a junction  $P$  to a junction  $Q$ , if the corresponding roads  $p$  and  $q$  of  $N$  have the property, that the finish of  $p$  lies in the same junction of  $N$  as the start of  $q$ . Thus  $4m$  roads are obtained in  $N^*$ , and it is easy to see that  $N^*$  is a *T-net*; its order is  $2m$ .

<sup>3)</sup> If we should replace the word "roads" by "junctions" here, this sentence would get another meaning, since two junctions may be connected by two roads in the same direction.

<sup>4)</sup> The converse is also true: for a connected *T-net* we have  $|N| > 0$ . However, we do not need this result in the proof of our main theorem.

A remarkably simple relation exists between the numbers of complete walks of  $N$  and  $N^*$  <sup>5)</sup>:

**Theorem.** If  $N$  is a  $T$ -net of order  $m$  ( $m = 1, 2, 3, \dots$ ), and  $N^*$  is the doubled net, then we have

$$|N^*| = 2^{m-1} \cdot |N|. \dots \dots \dots (1)$$

**Proof.** We first consider two cases, in which (1) is easily established.

**Case 1.** If  $N$  is not connected, the same holds for  $N^*$ , and hence  $|N| = |N^*| = 0$ .

**Case 2.** We now consider the case, where each junction of  $N$  is connected with itself. For any value of  $m$ , only one connected net of this type exists, consisting of junctions  $A_1, A_2, \dots, A_m$ , connected by roads  $A_1A_2, A_2A_3, \dots, A_{m-1}A_m, A_mA_1$ , and  $A_1A_1, A_2A_2, \dots, A_mA_m$  <sup>6)</sup>. For this net we have  $|N| = 1$ , and some quite trivial considerations show that  $|N^*| = 2^{m-1}$ .

We prove the general case by induction. For  $m = 1$  only one  $T$ -net is possible, consisting of one junction  $A$  and two roads leading from  $A$  to  $A$ . This net belongs to case 2 mentioned above, and we have  $|N| = |N^*| = 1$ .

Now suppose (1) to be valid for all  $T$ -nets of order  $m - 1$  ( $m > 1$ ), and be  $N$  a  $T$ -net of order  $m$ . We may suppose to be able to choose a junction  $A$ , not connected with itself, for otherwise  $N$  belongs to case 2. Hence we have four different roads  $p, q, r, s$ ;  $p$  and  $q$  leading to  $A$ ,  $r$  and  $s$  starting from  $A$ .

A net  $N_1$  arises from  $N$  by omitting  $A, p, q, r, s$ , and constructing two new roads, one from the start of  $p$  to the finish of  $r$ , and one from the start of  $q$  to the finish of  $s$ .

A second net  $N_2$  arises in a similar way, but now by combining  $p$  with  $s$  and  $q$  with  $r$ . This is illustrated by fig. 2; the parts of the nets, which are not drawn, are equal for  $N, N_1$  and  $N_2$ .

A complete walk of  $N$  corresponds to a complete walk either of  $N_1$ , or of  $N_2$ , and so we have

$$|N| = |N_1| + |N_2|. \dots \dots \dots (2)$$

On doubling the nets  $N_1$  and  $N_2$  we obtain nets  $N_1^*$  and  $N_2^*$ , respectively. We shall prove

$$|N^*| = 2|N_1^*| + 2|N_2^*|. \dots \dots \dots (3)$$

<sup>5)</sup> This relation can also be interpreted without introducing the doubling process. Namely, a complete walk of  $N^*$  corresponds to a closed walk through  $N$ , with the property that any road of  $N$  is used exactly twice in that walk, and such that at any junction each of the four possible combinations of a finish and a start is taken exactly once. We can give an even simpler interpretation in terms of  $N^*$ , for a complete walk of  $N$  corresponds to a closed walk through  $N^*$ , visiting any junction of  $N^*$  exactly once. But, since not every  $T$ -net can be considered as a  $N^*$ , this does not lead to an essential simplification of our theorem.

<sup>6)</sup>  $AB$  denotes a one-way road leading from  $A$  to  $B$ .

$N_1^*$  and  $N_2^*$  arise directly from  $N^*$  by simple operations. If  $P, Q, R, S$  are the junctions of  $N^*$  corresponding to the roads  $p, q, r, s$  of  $N$ , we obtain  $N_1^*$  by omitting the roads  $PR, PS, QR, QS$  from  $N^*$ , and identifying the

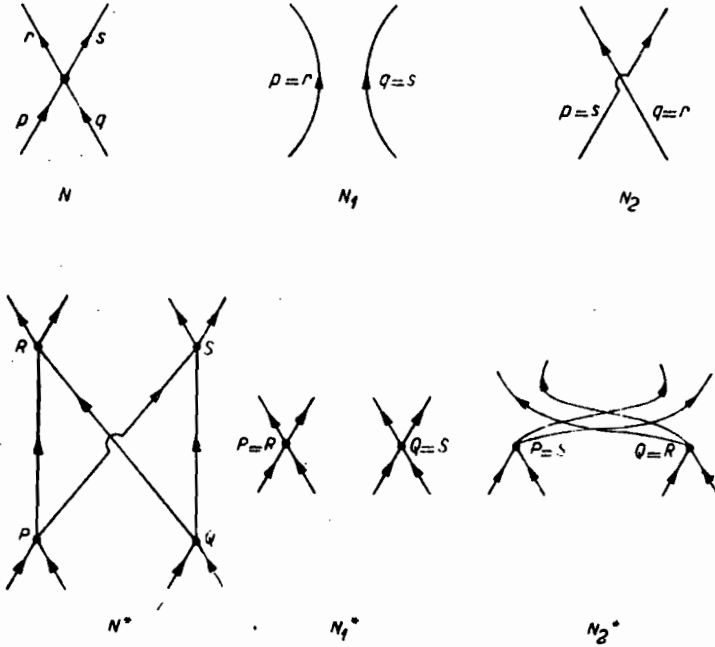


Fig. 2

four junctions two by two:  $P = R$  and  $Q = S$ .  $N_2^*$  is obtained analogously ( $P = S$  and  $Q = R$ ). Again, fig. 2 shows the corresponding details of  $N^*, N_1^*$  and  $N_2^*$ .

Henceforth we deal with  $N^*, N_1^*$  and  $N_2^*$ , and no longer consider  $N, N_1, N_2$ .

We now first introduce the term "path". A path is an ordered sequence of roads, no two of which are identical, such that the finish of each road is the start of the next one. The last one, however, need not lead to the start of the first one.

A complete walk of  $N^*, N_1^*$ , or  $N_2^*$ , contains four special paths, each one leading from one of the junctions  $R, S$  to one of the junctions  $P, Q$ , such that any road of  $N^*$ , except  $PR, PS, QR, QS$ , belongs to just one of those paths. Choosing a definite set of four paths, according to the conditions just mentioned, we consider all (possibly existing) complete walks of  $N, N_1$  and  $N_2$  containing those paths. The numbers of these complete walks be denoted by  $n, n_1, n_2$ , respectively.

The numbers  $n, n_1, n_2$  admit of a simple interpretation. Be  $N^{**}$  the net, arising from  $N^*$  on replacing each of the four paths by one single road, with the same start and finish as the corresponding path. In the same way

nets  $N_1^{**}$  and  $N_2^{**}$  arise from  $N_1^*$  and  $N_2^*$ . Evidently  $n = |N^{**}|$ ,  $n_1 = |N_1^{**}|$ ,  $n_2 = |N_2^{**}|$ . We now show, that

$$n = 2n_1 + 2n_2 \dots \dots \dots (4)$$

for which we have to consider two different cases.

Case A (fig. 3). The paths starting from  $R$  lead to different junctions.

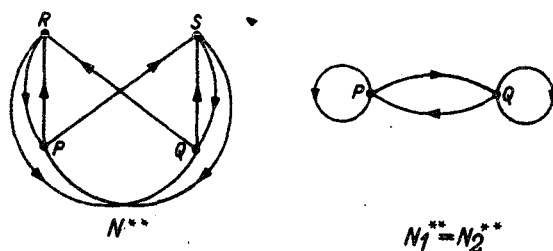


Fig. 3

The four paths thus respectively lead from  $R$  to  $P$ , from  $R$  to  $Q$ , from  $S$  to  $P$ , and from  $S$  to  $Q$ .

Now  $N^{**}$  consists of the junctions  $P, Q, R, S$ , with the roads  $PR, PS, QR, QS, RP, RQ, SP, SQ$ . This net admits four different complete walks.  $N_1^{**}$  consists of only two junctions  $P$  and  $Q$ , with roads  $PP, PQ, QP, QQ$ . This net admits only one complete walk. The net  $N_2^{**}$  is equivalent to  $N_1^{**}$ . Thus we have obtained  $n = 4$ ,  $n_1 = 1$ ,  $n_2 = 1$ , and (4) holds true.

Case B (fig. 4). The paths starting from  $R$  lead to one and the same

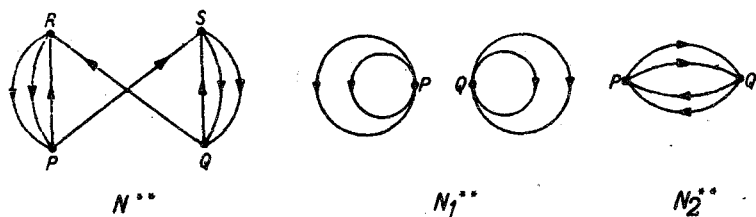


Fig. 4

junction, say  $P$  (the same obtains for  $Q$ ). We now have the four paths  $RP, RP, SQ, SQ$ .

Now  $N^{**}$  consists of the junctions  $P, Q, R, S$ , with roads  $PR, PS, QR, QS, RP, RP, SQ, SQ$ . This net admits four complete walks.  $N_1^{**}$  consists of two junctions  $P$  and  $Q$ , with roads  $PP, PP, QQ, QQ$ , and so it is not connected.  $N_2^{**}$  consists of  $P, Q$ , with roads  $PQ, PQ, QP, QP$ , admitting two complete walks. Now we have  $n = 4$ ,  $n_1 = 0$ ,  $n_2 = 2$ , and hence (4) holds also true in case  $B$ .

Formula (4) being proved for any admissible system of four paths, the truth of (3) is now evident.

Our theorem is an immediate consequence of (3). Namely,  $N_1$  and  $N_2$  being nets of order  $m - 1$ , our assumption of induction yields

$$|N_1^*| = 2^{m-2} |N_1|, \quad |N_2^*| = 2^{m-2} |N_2|,$$

and by (3) and (2) we now have

$$|N^*| = 2 |N_1^*| + 2 |N_2^*| = 2^{m-1} |N_1| + 2^{m-1} |N_2| = 2^{m-1} |N|.$$

3. The theorem of the preceding section provides a proof of POSTHUMUS' conjecture. For  $n \geq 2$ ,  $N_n$  be the following network of order  $2^n$ . As junctions we take the ordered  $n$ -tuples of digits 0 or 1, and we connect two  $n$ -tuples  $A$  and  $B$  by a one-way road  $AB$ , if the last  $n-1$  digits of  $A$  correspond to the first  $n-1$  digits of  $B$ . Fig. 5 shows the nets  $N_2$  and  $N_3$ .

On "doubling" this net  $N_n$  we obtain the net  $N_{n+1}$ . Namely, any road  $AB$  of  $N_n$  (see  $N_2$  in fig. 5) corresponds to an ordered  $(n+1)$ -tuple.

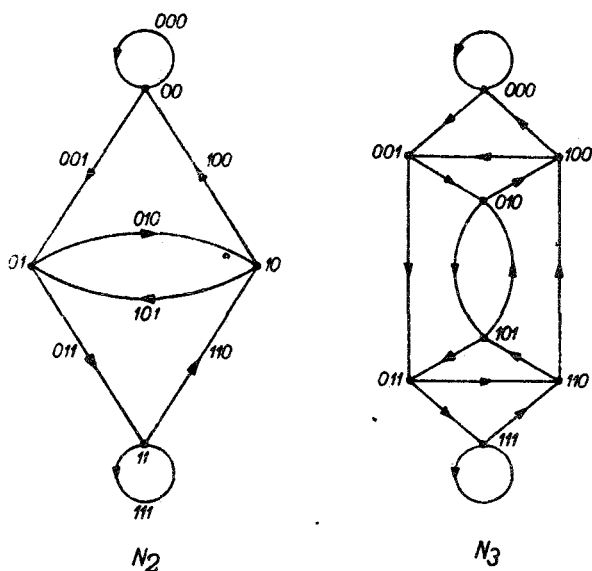


Fig. 5

consisting of the digits of  $A$ , followed by the last digit of  $B$  (or, what is the same, the first digit of  $A$ , followed by the digits of  $B$ ). Two  $(n+1)$ -tuples  $P, Q$ , turn out to be connected in  $N_{n+1}$ , if the last  $n$  digits of the first one correspond to the first  $n$  digits of the second one, since these  $n$  digits characterize the common finish and start of the roads  $p$  and  $q$  of  $N_n$ . Hence  $N_n^* = N_{n+1}$ .

A complete walk of  $N_n$  leads to a  $P_{n+1}$ -cycle in the following way. If such a walk consecutively goes through the roads  $AB, BC, \dots, ZA$ , we write down consecutively, the first digit of  $A$ , the first digit of  $B, \dots$ , the first digit of  $Z$ . This sequence, considered as a cycle, is a  $P_{n+1}$ -cycle. Namely, on taking the first digits of  $n+1$  consecutive junctions  $A, B, C, \dots$  of the walk under consideration, we obtain the  $(n+1)$ -tuple, belonging to the road  $AB$ . The walk in  $N_n$  being complete, it is now clear that any  $(n+1)$ -tuple occurs exactly once in our cycle.

Conversely, any  $P_{n+1}$ -cycle arises from a complete walk in  $N_n$  by the

described process, and different complete walks lead to different  $P_{n+1}$ -cycles. Hence the number of  $P_{n+1}$ -cycles equals  $|N_n|$ .

We now prove POSTHUMUS' conjecture by induction. For  $n = 1, 2, 3$  its truth is already established in section 1. Now take  $n \geq 3$ , and suppose the number of  $P_n$ -cycles to be  $2^{2^{n-1}-n}$ , whence  $|N_{n-1}| = 2^{2^{n-1}-n}$ . The order of  $N_{n-1}$  being  $2^{n-1}$ , the theorem of section 2 yields

$$|N_{n-1}^*| = 2^{2^{n-1}-1} \cdot |N_{n-1}|,$$

and it follows

$$N_n = 2^{2^{n-1}-1} \cdot 2^{2^{n-1}-n} = 2^{2^n-n-1}.$$

The number of  $P_{n+1}$ -cycles equalling  $|N_n|$ , POSTHUMUS' conjecture turns out to be true.

4. Another application of section 2 is the following one. We call a  $n$ -tuple of digits 0, or 2 admissible, if no two consecutive digits are equal; the last digit, however, may be the same as the first one. The number of admissible  $n$ -tuples is easily shown to be  $3 \cdot 2^{n-1}$ . As a  $Q_n$ -cycle we now define an ordered cycle of  $3 \cdot 2^{n-1}$  digits 0, 1 or 2, such that any admissible  $n$ -tuple is represented once by  $n$  consecutive digits of the cycle. For instance twelve  $Q_3$ -cycles exist. Two of them are 012010202121 and 012020102121, whereas the other ten are found by applying permutations of the symbols 0, 1 and 2.

For general  $n > 1$ , the number of  $Q_n$ -cycles amounts to  $3 \cdot 2^{3 \cdot 2^{n-2}-n-1}$ . A proof can be given completely analogous to that in section 3.

*Eindhoven, June 1946.*

*Natuurkundig Laboratorium der  
N.V. Philips' Gloeilampenfabrieken.*