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Kasner metric in Finsler gravity

van Voorthuizen, Jorn

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Department of Mathematics and Computer Science
Department of Applied Physics

Kasner metric in Finsler gravity

Bachelor Final Project

J. van Voorthuizen
1240581

Supervisors:

Dr. A. Fuster	Department of Mathematics and Computer Science
S.J. Heefer MSc	Department of Mathematics and Computer Science
Dr.ir. L.P.J. Kamp	Department of Applied Physics

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Abstract

Albert Einstein formulated his General Theory of Relativity in the framework of pseudo-Riemannian geometry. In this research, gravity is studied in a more general framework, called pseudo-Finsler geometry. In Finsler geometry, the line element does not need to be quadratic in the infinitesimal components as in Riemannian geometry. For a Finslerian line element, it is a natural generalization to take the multiplicative combination of a Riemannian line element and a certain 1-form. This leads to the so-called Very General Relativity (VGR) line elements. To simplify the field equation in Finsler geometry, we will restrict the pseudo-Finsler geometries to Berwald spacetimes which is a tighter generalization of the Riemannian spacetimes. The goal is to determine the 1-form such that the VGR line element constitutes a Berwald spacetime that solves the field equation in Berwald spacetimes. In this research, we will use the Kasner metric, which is a solution to Einstein's field equations in vacuum, as the Riemannian part in the VGR line element. The first result is that the 1-form must be closed, in order for the VGR line element to describe a Berwald spacetime. With the Kasner metric, we find that there are no suitable 1-forms such that we have a Berwald spacetime. After that, we want to generalize our results for the family of Bianchi type-I metrics, which is a more general class of Riemannian metrics containing the Kasner metric. Here, we find that there are only suitable 1-forms in some very specific cases. However, more research is necessary for the situation with the Bianchi type-I metrics because we are not able to find out whether there exists a suitable 1-form in all specific cases.

Contents

Introduction

In 1916, Albert Einstein published his work on the General Theory of Relativity [1], which is a solid theory to describe gravity very well. Many experiments confirm the General Theory of Relativity, for example, the precession of planetary orbits can be explained by General Relativity [2]. The Kasner metric is a metric in General Relativity that describes a $3 + 1$-dimensional spacetime and solves Einstein's field equations in vacuum. When looking at the corresponding squared line element, we see that the metric tensor is diagonal and that the spatial components are different powers of the time coordinate. Since the exponents are different, the space described by the Kasner metric is anisotropic. This means that space does not behave the same in all directions. For the Kasner metric, there is an expansion in two spatial directions and a contraction in the other spatial direction. Another interesting property of the Kasner metric is that the corresponding volume element is proportional to time. Therefore, the Kasner metric could also describe an expanding universe and in the beginning for small t, even a Big Bang with the singularity at $t = 0$. There are more metrics that could describe a Big Bang such as the FLRW-metric [3], but the Kasner metric is different from the FLRW-metric because the described space is anisotropic. Moreover, the Kasner metric solves the field equations in vacuum while there is matter present for the FLRW-metric.

Although General Relativity performs very well and is self-consistent, there is much research for an improved version. In Einstein's General Relativity, there is local Lorentz symmetry, which means that the physical laws should have the same form locally. There is much research in General Relativity and quantum physics where Lorentz symmetry has been questioned, tracing back to Paul Dirac in [4]. Even in more recent research, Lorentz violations are important in attempts to align General Relativity with quantum physics [5, 6]. Lorentz violations to gravity are not compatible to the framework of Riemannian geometry wherein General Relativity is formulated [7]. Therefore, we will adopt generalization from Riemannian geometry to Finsler geometry. This means that we do not require the squared line elements to be quadratic in the infinitesimal components anymore. Although we work in pseudo-Riemannian and pseudo-Finslerian geometry, we will often drop the prefix "pseudo" for brevity.

Concepts such as the affine connection and the Ricci tensor that have been derived in Riemannian geometry, can be generalized to Finsler geometry. In [8], the field equations developed by Einstein have even been generalized to Finsler geometry. An action that describes gravity is the cornerstone from which the field equation in Finslerian manifolds has been derived. However, the resulting generalized field equation is quite complicated because the Finslerian line elements can be much more general than Riemannian line elements. Therefore, we will not consider all Finslerian manifolds, but only Berwald spacetimes. A Finslerian spacetime is equivalent to a Riemannian spacetime in the sense that there is a temporal component and there are three spatial components and the temporal and spatial part have an opposite contribution to the signature of the metric tensor. Berwald spacetimes form a subclass of all Finslerian spacetimes and are a tighter generalization of Riemannian spacetimes. In these Berwald spacetimes, the generalized field equation simplifies greatly.

For the Finslerian line element, it is natural to take the multiplicative product of a Riemannian line element and a 1-form because it has some weakened Lorentz symmetry [7]. We raise the Riemannian part to the power $1 - b$ and the 1-form to the power $2b$. Multiplying the two parts returns a squared line element that is called the Very General Relativity (VGR) line element. For $b = 0$, we recover the Riemannian metric and therefore, we want the Riemannian metric to solve Einstein's field equations. We want to take the 1-form in such a way that the generalized field equation is also satisfied for $b \neq 0$. Since General Relativity works well in many situations, we could interpret b as some parameter where $b \approx 0$ for these situations where Einstein's theory works well, but b is slightly different from 0 where General Relativity works less well. Therefore, a VGR line element is a slight modification of a General Relativity line element and it can be used to investigate Lorentz violations.

The line element described above has already been investigated for many Riemannian metrics. For example, Finslerian versions of pp-waves are investigated in [9] and the FLRW-metric is studied in [7]. We will investigate the VGR line element with the Kasner metric as the Riemannian part.

Due to the 1-form in the VGR line element, the space described by the VGR line element becomes anisotropic. One of the properties of the Kasner metric is that the space described by the Kasner metric is already anisotropic. This could be an interesting reason we want to know what happens for the Kasner metric. Another interesting feature is to find out what happens in such a generalized description of a Big Bang for small, positive times and close to the singularity of the Kasner metric at $t = 0$. After the Kasner metric, we will also have a look at the more general class of Bianchi type-I metrics. The Kasner metric is one of the Bianchi type-I metrics that solves the vacuum field equations. The other one is the Minkowski metric. The most special properties of the Bianchi type-I metrics are that the metric tensor is diagonal and purely time-dependent and the they can describe anisotropic spaces, as for the Kasner metric.

We will start with the introduction of the Kasner metric in Riemannian geometry in chapter 1. In section 1.1, the Killing vectors will be introduced and computed for the Kasner metric. The underlying framework of Finsler geometry is introduced in chapter 2, while the Berwald spacetimes are introduced in section 2.1. In section 2.2, the generalized field equation in Berwald spacetimes will be formulated. Section 2.3 will introduce Very Special Relativity and in section 2.4, the Berwald spacetimes will eventually be combined with the VGR line element. These first two chapters should contain all necessary theory to finally generalize the Kasner metric to VGR Berwald spacetimes in chapter 3. In chapter 4, we will try to to generalize our results from chapter 3 to the more general Bianchi type-I metrics. Eventually, the results will be concluded and discussed in chapter 5.

In this report, basic knowledge of differential geometry and the General Theory of Relativity will be applied. In particular, the concepts of a pseudo-Riemannian manifold, tangent space, line element, metric tensor, index gymnastics, Einstein summation convention, volume element, 1-form, affine connection/Christoffel symbols, covariant derivative, geodesics, Riemann curvature tensor, Ricci tensor, Ricci scalar and the Einstein field equations in empty space will be used without much explanation. These concepts can be found in any textbook containing the basics of the General Theory of Relativity. For example, (parts of) chapters 2, 3, 4, 7 and 8 in [2] or chapters 1, 3, 4 and 5.2 in [10] should provide a solid basis to understand this research. The definition of Killing vectors will be explained in section 1.1, but can also be found in section 6.6 in [10] for more information.

Also, a few conventions will be adopted:

- $x = (x^0, x^1, x^2, x^3)$ will be reserved for the position in the manifold M;
- $x^0 = t$ denotes the temporal component and x^1, x^2 and x^3 denote the three-dimensional spatial components and the units are chosen in such a way that $c = 1$;
- the tangent space at $x \in M$ is denoted by T_xM and $y \in T_xM$ denotes a vector in this tangent space;
- we will adopt a $(-, +, +, +)$ signature for the metric tensors;
- the partial derivative to x^μ will be shortened by $\frac{\partial}{\partial x^\mu} = \partial_\mu$;
- the Einstein summation convention will be adopted;
- Greek indices $\mu, \nu, \ldots$ are used from 0 to 3 and Latin indices $i, j, \ldots$ only for the spatial components from 1 to 3. In the beginning of chapter 2 till section 2.2, no distinction between the temporal and the spatial components will be made because it is a more general theory for any kind of (pseudo-)Finslerian manifold, that are not necessarily spacetimes.

Chapter 1

Kasner metric

An interesting family of metrics consists of the Bianchi type-I metrics. This is a family of metrics where the line element is of the form

$$ds^2 = -dt^2 + \sum_{i=1}^{3} (A_i(t))^2 \left(dx^i\right)^2, \tag{1.1}$$

for some functions $A_i, i = 1, 2, 3$ that are purely time-dependent. The spatial coordinates x^i are suitably chosen such that the metric is diagonal. The functions A_i can be different in the three spatial directions, and hence the Bianchi type-I metrics can describe an anisotropic space.

The family of Bianchi type-I metrics is mainly used in cosmology. For example, these metrics are used to determine the temperature of cosmic microwave background radiation (CMBR). More recently, the Bianchi type-I metrics are also used for ΛCDM models. Here, Λ represents the cosmological constant related to dark energy and CDM the cold dark matter. This is a simple model in Big Bang theory, taking into account CMBR among others [11, 12]. Nowadays, there is also research in cosmology to find out what happens when quantum effects are incorporated [13, 14]. However, there are also applications in other areas such as in anisotropic hydrodynamics [15]. The metrics are relatively simple because the metric tensor is diagonal and only depends on time. Still, these metrics can describe some interesting phenomena very well [16].

Now, we are interested in metrics of Bianchi type-I that satisfy the Einstein field equations in vacuum from the General Theory of Relativity. This means that the Ricci tensor must vanish. The first solution is quite trivial. If we take the A_i's to be some positive constants, we simply have a Minkowski metric after some rescaling of the spatial coordinates. There is also a more interesting solution and this is the Kasner metric [17].

The Kasner metric is given by

$$ds^2 = -dt^2 + \sum_{i=1}^{3} t^{2p_i} \left(dx^i\right)^2, \tag{1.2}$$

for some Kasner exponents $p_i, i = 1, 2, 3$. The only non-vanishing components of the Christoffel symbols for the Kasner metric are

$$\Gamma^0{}_{ii} = p_i t^{2p_i - 1}, \quad \Gamma^i{}_{0i} = \Gamma^i{}_{i0} = \frac{p_i}{t}, \tag{1.3}$$

for $i = 1, 2, 3$ representing the spatial coordinates. From the Christoffel symbols, the Ricci tensor can be calculated. We adopt the definition in [2] which is

$$R_{\mu\nu} = \partial_\nu \Gamma^\sigma{}_{\mu\sigma} - \partial_\sigma \Gamma^\sigma{}_{\mu\nu} + \Gamma^\rho{}_{\mu\sigma}\Gamma^\sigma{}_{\rho\nu} - \Gamma^\rho{}_{\mu\nu}\Gamma^\sigma{}_{\rho\sigma}, \tag{1.4}$$

and then the only nonzero components are on the diagonal and yield

$$R_{00} = \left(\sum_{j=1}^{3} {p_j}^2 - \sum_{j=1}^{3} p_j\right) t^{-2}, \quad R_{ii} = p_i \left(1 - \sum_{j=1}^{3} p_j\right) t^{2p_j - 2}, \tag{1.5}$$

for $i = 1, 2, 3$. The Einstein field equations in vacuum tells us that the Ricci tensor must vanish. In particular, all components must vanish for all t. Therefore, we get $\sum_{j=1}^{3} {p_j}^2 = \sum_{j=1}^{3} p_j$ from the $(0, 0)$-component and we must have $\sum_{j=1}^{3} p_j = 1$ or $p_i = 0$ for all $i = 1, 2, 3$ from the (i, i)-components. The case with $p_i = 0$ for all $i = 1, 2, 3$ results in the trivial Minkowski metric and the first case results in the Kasner conditions

$$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} p_i^2 = 1. \tag{1.6}$$

A metric of the form (1.2) satisfying the Kasner conditions is a *Kasner metric*. Identically for the general Bianchi type-I metrics, the coordinates are suitably chosen such that the metric is diagonal and there are no constant factors in front of the powers of t in equation (1.2). For the volume element, we know it is proportional to $\sqrt{-g}$, where g is the determinant of the metric tensor and the minus sign comes from the fact that the sign of g is -1. Of course, $g = -t^{\sum_{i=1}^{3} 2p_i} = -t^2$ under the Kasner conditions and therefore the volume element is proportional to t. Hence, the volume starts to increase faster and faster and as t increases, it shows an expansion of the universe in time [18]. Also for very small times, the universe expands and then the Kasner metric describes a singularity such as a Big Bang [17].

The Kasner conditions constitute two conditions on three parameters. Therefore, there is a one-dimensional space wherein the parameters can take values. Graphically, $\sum_{i=1}^{3} p_i = 1$ describes a flat plane through $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ and $\sum_{i=1}^{3} p_i^2 = 1$ describes the unit sphere. Therefore, the parameters must lie on the intersection of the plane with the unit sphere. The parameter space is visualized in figure 1.1. Clearly, not all parameters can be the same while satisfying both conditions at the same time. Therefore it is impossible for the Kasner metric to describe an isotropic space. There are two solutions where two parameters are the same, which we call the degenerate or axisymmetric solutions. The first one is $p_1 = p_2 = 0$ and $p_3 = 1$ and the second one is $p_1 = -\frac{1}{3}$ and $p_2 = p_3 = \frac{2}{3}$. Of course, a permutation of the parameters yields the same result. With the first set of parameters, the Riemann curvature tensor vanishes entirely and the metric is equivalent to the trivial Minkowski metric up to a transformation of coordinates. When we define $\tilde{t} = t \cosh x^3$, $\tilde{x}^3 = t \sinh x^3$ and $\tilde{x}^i = x^i$ for $i = 1, 2$, we see that $d\tilde{t} = \cosh x^3 dt + t \sinh x^3 dx^3$ and $d\tilde{x}^3 = \sinh x^3 dt + t \cosh x^3 dx^3$. Hence,

$$-d\tilde{t}^2 + (d\tilde{x}^3)^2 = -dt^2 + t^2(dx^3)^2. \tag{1.7}$$

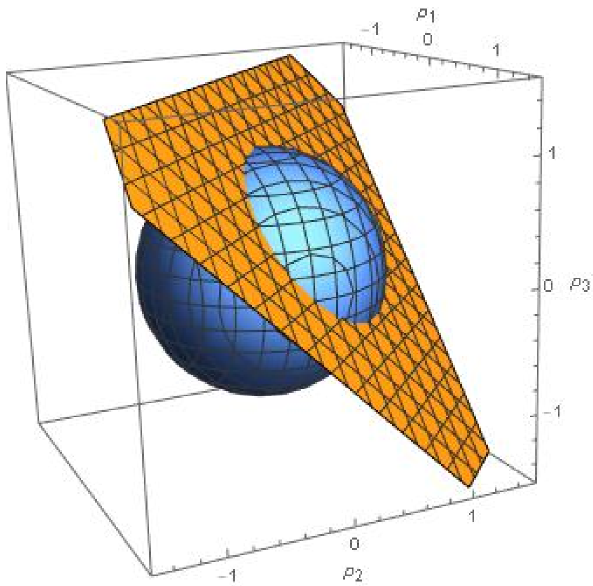

Figure 1.1: A visualization of the Kasner conditions, where the parameter space is the intersection of the blue sphere $\left(\sum_{i=1}^{3} {p_i}^2 = 1\right)$ and the orange plane $\left(\sum_{i=1}^{3} p_i = 1\right)$.

Therefore, the Kasner metric with these transformed components becomes

$$-dt^2 + (dx^1)^2 + (dx^2)^2 + t^2(dx^3)^2 = -d\tilde{t}^2 + (d\tilde{x}^1)^2 + (d\tilde{x}^2)^2 + (d\tilde{x}^3)^2, \tag{1.8}$$

where we recognize the ordinary Minkowski metric in coordinates $\tilde{x}^\mu$. The second degenerate solution is more interesting and does have a nonzero curvature tensor. When we adopt the definition of the Riemann curvature tensor from [2] as for the Ricci tensor, the curvature tensor is defined as

$$R^\rho{}_{\mu\nu\sigma} = \partial_\nu \Gamma^\rho{}_{\mu\sigma} - \partial_\sigma \Gamma^\rho{}_{\mu\nu} + \Gamma^\lambda{}_{\mu\sigma}\Gamma^\rho{}_{\lambda\nu} - \Gamma^\lambda{}_{\mu\nu}\Gamma^\rho{}_{\lambda\sigma}. \tag{1.9}$$

For general Kasner metrics, the nonvanishing components of the Riemann curvature tensor are

$$\begin{aligned} R^0{}_{i0i} = -R^0{}_{ii0} = p_i(p_i-1)t^{2(p_i-1)}, \quad R^i{}_{00i} = -R^i{}_{0i0} = p_i(p_i-1)t^{-2}, \\ R^i{}_{jij} = -R^i{}_{jji} = p_i p_j t^{2(p_j-1)}, \quad -R^j{}_{iij} = R^j{}_{iji} = p_i p_j t^{2(p_i-1)}, \end{aligned} \tag{1.10}$$

for distinct $i, j = 1, 2, 3$. Here, we clearly see the Riemann curvature tensor vanishes if and only if $p_1 = p_2 = 0$ and $p_3 = 1$.

When looking at the constraints, $p_1 = p_2 = 0$ and $p_3 = 1$ turns out to be the only possibility without negative parameters. When we look even more carefully, we can find that there must be a positive, a non-negative and a non-positive parameter. Let us say that p_1 is non-positive, then $-\frac{1}{3} \leq p_1 \leq 0$. When we also suppose $p_2 \leq p_3$, we find $0 \leq p_2 \leq \frac{2}{3}$ and $\frac{2}{3} \leq p_3 \leq 1$. Adopting this order of the Kasner exponents, there is a very useful parametrization of all possible exponents:

$$p_1 = \frac{-u}{1+u+u^2}, \quad p_2 = \frac{1+u}{1+u+u^2}, \quad p_3 = \frac{u(1+u)}{1+u+u^2}, \tag{1.11}$$

where u is a parameter with $1 \leq u \leq \infty$ [19]. Note that $u = 1$ corresponds to the axisymmetric $\left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ case and $u = \infty$ corresponds to the other degenerate $(0, 0, 1)$ case. Above, we already saw that the last case of degenerate parameters results in a flat Minkowski space. Therefore, we can take $1 \leq u < \infty$ when we exclude the trivial case. This parametrization is especially useful when investigating the singularity at $t = 0$ [20].

1.1 Killing vectors

A Killing vector is a vector field X on M such that

$$\mathcal{L}_X(g) = 0, \tag{1.12}$$

where $\mathcal{L}$ denotes the Lie derivative and $g = g_{\mu\nu} dx^\mu dx^\nu$ denotes the Riemannian metric. The Lie derivative also constitutes a tensor. For a vector field $X = X^\mu \partial_\mu$ and a covariant tensor of rank two $T = T_{\mu\nu} dx^\mu dx^\nu$, the Lie derivative $\mathcal{L}_X(T)$ is

$$\mathcal{L}_X(T) = \left(X^\rho \partial_\rho T_{\mu\nu} + \left(\partial_\mu X^\rho\right) T_{\rho\nu} + \left(\partial_\nu X^\rho\right) T_{\mu\rho}\right) dx^\mu dx^\nu. \tag{1.13}$$

Filling in this definition of the Lie derivative into equation (1.12), turns out to be equivalent to the Killing equation [10], which is

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0. \tag{1.14}$$

Here, ∇_μ denotes the covariant derivative with respect to x^μ and we used the covariant components of X instead of the contravariant components. Rewriting equation (1.14) by exploiting the covariant derivative, we find

$$\partial_\mu X_\nu - \Gamma^\rho{}_{\nu\mu} X_\rho + \partial_\nu X_\mu - \Gamma^\rho{}_{\mu\nu} X_\rho = \partial_\mu X_\nu + \partial_\nu X_\mu - 2\Gamma^\rho{}_{\mu\nu} X_\rho = 0, \tag{1.15}$$

where $\Gamma^\rho{}_{\mu\nu}$ denotes the ordinary Christoffel symbol in Riemannian geometry. In the last step we also exploited the symmetry in the Christoffel symbols.

In General Relativity, the Killing vectors are of interest because they express the symmetries in the space induced by the metric. The more symmetry there is, the more Killing vectors there are. For example, if the metric tensor is independent of x^μ for some μ, then ∂_μ is a Killing vector. Similarly, when the metric tensor is symmetric in x^μ and x^ν for some μ and ν, there is a Killing vector that expresses this axisymmetry. The number of Killing vectors can be interpreted as a measure for the symmetry [10]. To generalize a Riemannian metric to a Finslerian line element, it might be interesting to know the symmetries.

Using this theory about Killing vectors and in particular equation (1.15), we want to find the Killing vectors of the Kasner metric. At first, note that the metric tensor is diagonal and only depends on t. Putting this into the definition of the Lie derivative and setting it to zero, tells us that ∂_i for $i = 1, 2, 3$ are the first three Killing vectors. The corresponding covectors are $t^{2p_i} dx^i$ for $i = 1, 2, 3$. For other potential Killing vectors, we will consider equation (1.15).

Then, for the $\mu = \nu = 0$ case, equation (1.15) reduces to

$$2\partial_0 X_0 = 0. \tag{1.16}$$

We set $x^0 = t$ so this equation tells us that X_0 must be independent of time. The case $\mu = 0$ and $\nu = i = 1, 2, 3$ gives

$$\partial_0 X_i + \partial_i X_0 - 2\frac{p_i}{t} X_i = 0, \tag{1.17}$$

and $\nu = 0$ and $\mu = i = 1, 2, 3$ can be found by symmetry. The case $\nu = \mu = i = 1, 2, 3$ yields

$$2\left(\partial_i X_i - p_i t^{2p_i - 1} X_0\right) = 0. \tag{1.18}$$

The only case that is left is $\mu = i = 1, 2, 3$ and $\nu = j = 1, 2, 3$ with $i \neq j$ and then, equation (1.15) gives

$$\partial_i X_j + \partial_j X_i = 0. \tag{1.19}$$

At first, assume $p_i \neq \frac{1}{2}$ for all $i = 1, 2, 3$. Since X_0 is independent of t from equation (1.16), equation (1.17) tells us that

$$X_i = \xi_i(x^j) t^{2p_i} - \frac{t}{1 - 2p_i} \partial_i X_0, \tag{1.20}$$

where $\xi_i, i = 1, 2, 3$ are some functions independent of x^0, but it can depend on all spatial components. Now, substitution in equation (1.18) (divided by two) yields

$$\partial_i \xi_i t^{2p_i} - \frac{t}{1 - 2p_i} \partial_i{}^2 X_0 = p_i t^{2p_i - 1} X_0. \tag{1.21}$$

This equality must hold for all t. Obviously, $t^{2p_i} \neq t^{2p_i - 1}$ for general t. Since we also assumed $p_i \neq \frac{1}{2}$, we also have $t^{2p_i} \neq t$ for general t. Hence, the t^{2p_i} term must vanish independently. Therefore, ξ_i is independent of x^i. If $p_i \neq 1$ for all i, also $t^{2p_i - 1} \neq t$ for general t for all i and hence both other terms must also vanish independently. Then, $X_0 = 0$ since $p_i \neq 0$ for some i. By now, equation (1.20) combined with $X_0 = 0$, the Killing vector field is already reduced to

$$X_i = \xi_i(x^j) t^{2p_i}, \quad X_0 = 0, \tag{1.22}$$

where $\partial_i \xi_i = 0$ for all $i = 1, 2, 3$. Let us now substitute this in equation (1.19):

$$\partial_j \xi_i t^{2p_i} + \partial_i \xi_j t^{2p_j} = 0. \tag{1.23}$$

If $p_i \neq p_j$ for all i, j, the two terms must vanish separately. Therefore, $\partial_j \xi_i = 0$ for $i \neq j$, where we already found it for $i = j$. Therefore, ξ_i is constant and we are left with $X_i = \xi_i t^{2p_i}$. Contracting these coordinates with dx^i, we recognize the same covectors as the ones we found before.

However, we excluded some cases. Let us start with the assumption $p_i \neq p_j$ for all i, j. In the beginning of this chapter, we already saw there are actually two sets of parameters such that the exponents are degenerate. Let us start with the case $p_1 = -\frac{1}{3}$, $p_2 = p_3 = \frac{2}{3}$, for which all other assumptions we made do apply. Since $p_1 \neq p_2, p_3$, we see that ξ_1 is still constant and ξ_2 and ξ_3 are still independent of x^1. However, ξ_2 can now depend on x^3 and ξ_3 can depend on x^2. Equation (1.23) for $i = 2$ and $j = 3$ can be divided by $t^{2p_2} = t^{2p_3}$ which gives

$$\frac{d\xi_2}{dx^3} = -\frac{d\xi_3}{dx^2}, \tag{1.24}$$

where we note that ξ_2 is independent of x^1 and x^2 so we have an ordinary derivative and similarly for ξ_3. Also, the left-hand side now only depends on x^3 while the right hand side only depends on x^2. Therefore, for an equality, both sides must be constant. Hence, $\xi_2 = ax^3 + b$ and $\xi_3 = -ax^2 + c$ for some constants a, b and c. Now, the constants b and c induce Killing vectors that are already implied by the first three constant Killing vectors we found and therefore we disregard those constants by setting $b = c = 0$. We can also set $a = 1$, because the set of Killing vectors is a vector space which means that if we have one Killing vector, we already know that all scalar multiples are also Killing vectors. This gives $X_2 = x^3t^{2p_2}$ and $X_3 = -x^2t^{p_3}$. So we have a new vector field which is

$$x^3\partial_2 - x^2\partial_3, \tag{1.25}$$

after raising the index. The corresponding covector is

$$x^3t^{2p_2}dx^2 - x^2t^{2p_3}dx^3 = t^{4/3}\left(x^3dx^2 - x^2dx^3\right) \tag{1.26}$$

as $p_2 = p_3 = \frac{2}{3}$. We already saw that the Killing vectors correspond to the symmetries, and this Killing vector corresponds to the axisymmetry in x^2 and x^3. Due to this symmetry, rotations around the x^1-axis do not change the metric.

The other axisymmetric case was $p_1 = p_2 = 0$ and $p_3 = 1$. Similarly as for the derivation above, we now find an extra Killing vector $x^2\partial_1 - x^1\partial_2$ due to the degeneracy in the parameters. However, we made another assumption that is not valid anymore as $p_3 = 1$. In equation (1.21) for $i = 3$ we now see $t^{2p_3-1} = t$ always. Therefore, we can find another Killing vector if

$${\partial_3}^2X_0 = X_0. \tag{1.27}$$

This is solved by $X_0 = \xi(x^1,x^2)e^{x^3} + \eta(x^1,x^2)e^{-x^3}$ for some functions ξ and η, independent of x^3 and t. When we now consider equation (1.21) for $i = 1, 2$, we see the right hand side will always vanish as $p_1 = p_2 = 0$. Therefore, we must also have ${\partial_i}^2X_0 = 0$. Therefore, X_0 is of at most first order in x^1 and x^2 separately. Hence,

$$X_0 = ax^1x^2e^{x^3} + bx^1e^{x^3} + cx^2e^{x^3} + de^{x^3} + Ax^1x^2e^{-x^3} + Bx^1e^{-x^3} + Cx^2e^{-x^3} + De^{-x^3}, \tag{1.28}$$

for some constants a, b, c, d, A, B, C, D. Then, for X_i, equation (1.20) tells us that

$$\begin{aligned}
X_1 &= \xi_1(x^2,x^3) - t((ax^2+b)e^{x^3} + (Ax^2+B)e^{-x^3}),\\
X_2 &= \xi_2(x^1,x^3) - t((ax^1+c)e^{x^3} + (Ax^1+C)e^{-x^3}),\\
X_3 &= \xi_3(x^1,x^2)t^2 + t((ax^1x^2+bx^1+cx^2+d)e^{x^3} - (Ax^1x^2+Bx^1+Cx^2+D)e^{-x^3}).
\end{aligned} \tag{1.29}$$

Now, have a look at equation (1.19). For $i = 1$ and $j = 2$, this gives

$$\partial_1\xi_2 - t(ae^{x^3} + Ae^{-x^3}) + \partial_2\xi_1 - t(ae^{x^3} + Ae^{-x^3}) = \partial_1\xi_2 + \partial_2\xi_1 - 2t(ae^{x^3} + Ae^{-x^3}) = 0. \tag{1.30}$$

This equation must vanish for all t and therefore, both $\partial_1\xi_2 + \partial_2\xi_1 = 0$ and $ae^{x^3} + Ae^{-x^3} = 0$. Since e^{x^3} and e^{-x^3} are also linearly independent and the equation must also hold for all x^3, we find $a = A = 0$. For the other term we find $\xi_1(x^2,x^3) = \eta(x^3)x^2 + \eta_1(x^3)$ and $\xi_2(x^1,x^3) = -\eta(x^3)x^1 + \eta_2(x^3)$ for some functions η, η_1, η_2 only dependent on x^3. Now consider equation (1.19) with $i = 1$ and $j = 3$. Then we find, after substituting $a = A = 0$, that

$$\partial_1\xi_3 t^2 + t(be^{x^3} - Be^{-x^3}) + \partial_3\xi_1 - t(be^{x^3} - Be^{-x^3}) = \partial_1\xi_3 t^2 + \partial_3\xi_1 = 0. \tag{1.31}$$

This must hold for all t so both terms vanish separately. Hence, ξ_1 is independent of x^3 so $\xi_1(x^2) = \eta x^2 + \eta_1$ for some constants η and η_1. The other term tells us that ξ_3 is independent of x^1. When we do the same for $i = 2$ and $j = 3$, we find that that $\xi_2 = -\eta x^1 + \eta_2$ for some constant η_2 and ξ_3 is also independent of x^2 and hence constant. Since we already found the constant Killing vectors in the beginning, we can take $\eta_1 = \eta_2 = \xi_3 = 0$. Also, the η-terms are already found by the rotation in the degenerate case. Hence, we can also take $\eta = 0$. Then, we are left with

$$\begin{aligned} X_0 &= bx^1e^{x^3} + cx^2e^{x^3} + de^{x^3} + Bx^1e^{-x^3} + Cx^2e^{-x^3} + De^{-x^3}, \\ X_1 &= -t(be^{x^3} + Be^{-x^3}), \\ X_2 &= -t(ce^{x^3} + Ce^{-x^3}), \\ X_3 &= t((bx + cy + d)e^z - (Bx + Cy + D)e^{-z}). \end{aligned} \tag{1.32}$$

When we now raise the index and use that the set of Killing vectors is a vector space, we find the following six unique Killing vectors, corresponding to the six constants that are left:

$$\begin{aligned} -e^{x^3}\partial_0 + \frac{e^{x^3}}{t}\partial_3, &\quad - e^{-x^3}\partial_0 - \frac{e^{-x^3}}{t}\partial_3, \\ -x^1e^{x^3}\partial_0 - te^{x^3}\partial_1 + \frac{x^1e^{x^3}}{t}\partial_3, &\quad - x^1e^{-x^3}\partial_0 - te^{-x^3}\partial_1 - \frac{x^1e^{-x^3}}{t}\partial_3, \\ -x^2e^{x^3}\partial_0 - te^{x^3}\partial_2 + \frac{x^2e^{x^3}}{t}\partial_3, &\quad - x^2e^{-x^3}\partial_0 - te^{-x^3}\partial_2 - \frac{x^2e^{-x^3}}{t}\partial_3. \end{aligned} \tag{1.33}$$

Now, we have 10 Killing vectors. This is the same as for the Minkowski metric [10], which makes sense because we can transform the Kasner metric with $p_1 = p_2 = 0$ and $p_3 = 1$ into a Minkowski metric. The corresponding covectors are

$$\begin{aligned} e^{x^3}dt + te^{x^3}dx^3, &\quad e^{-x^3}dt - te^{-x^3}dx^3, \\ x^1e^{x^3}dt - te^{x^3}dx^1 + tx^1e^{x^3}dx^3, &\quad x^1e^{-x^3}dt - te^{-x^3}dx^1 - tx^1e^{-x^3}dx^3, \\ x^2e^{x^3}dt - te^{x^3}dx^2 + tx^2e^{x^3}dx^3, &\quad x^2e^{-x^3}dt - te^{-x^3}dx^2 - tx^2e^{-x^3}dx^3. \end{aligned} \tag{1.34}$$

There is only one assumption left we have not considered yet, which is that $p_j \neq \frac{1}{2}$ for all j. There is one unique set of parameters for which this is not the case, and that is $p_1 = \frac{1}{4}(1-\sqrt{5})$, $p_2 = \frac{1}{2}$ and $p_3 = \frac{1}{4}(1 + \sqrt{5})$. Then, equation (1.20) for $j = 2$ is replaced by

$$X_2 = \xi_2(x^1, x^2, x^3)t - t\ln t\partial_2 X_0. \tag{1.35}$$

Because $t\ln t$ and any power of t are linearly independent, we will see the same reasoning still holds and hence we only have the first three Killing vectors.

All Killing vectors are summarized in table 1.1.

Table 1.1: Killing vectors for the Kasner metric

Case	Killing vectors	Covariant form
Always	∂_1	$t^{2p_1}dx^1$
	∂_2	$t^{2p_2}dx^2$
	∂_3	$t^{2p_3}dx^3$
$p_1 = -\frac{1}{3}$, $p_2 = p_3 = \frac{2}{3}$	$x^3\partial_2 - x^2\partial_3$	$x^3t^{2p_2}dx^2 - x^2t^{2p_3}dx^3$
$p_1 = p_2 = 0$, $p_3 = 1$	$x^2\partial_1 - x^1\partial_2$	$x^2t^{2p_1}dx^1 - x^1t^{2p_2}dx^2$
	$-e^{x^3}\partial_0 + \frac{e^{x^3}}{t}\partial_3$	$e^{x^3}dt + te^{x^3}dx^3$
	$-e^{-x^3}\partial_0 - \frac{e^{-x^3}}{t}\partial_3$	$e^{-x^3}dt - te^{-x^3}dx^3$
	$-x^1e^{x^3}\partial_0 - te^{x^3}\partial_1 + \frac{x^1e^{x^3}}{t}\partial_3$	$x^1e^{x^3}dt - te^{x^3}dx^1 + tx^1e^{x^3}dx^3$
	$-x^1e^{-x^3}\partial_0 - te^{-x^3}\partial_1 - \frac{x^1e^{-x^3}}{t}\partial_3$	$x^1e^{-x^3}dt - te^{-x^3}dx^1 - tx^1e^{-x^3}dx^3$
	$-x^2e^{x^3}\partial_0 - te^{x^3}\partial_2 + \frac{x^2e^{x^3}}{t}\partial_3$	$x^2e^{x^3}dt - te^{x^3}dx^2 + tx^2e^{x^3}dx^3$
	$-x^2e^{-x^3}\partial_0 - te^{-x^3}\partial_2 - \frac{x^2e^{-x^3}}{t}\partial_3$	$x^2e^{-x^3}dt - te^{-x^3}dx^2 - tx^2e^{-x^3}dx^3$

Now, it might be interesting to know the (semi-)norms and covariant derivatives of the Killing vectors. Eventually, we want to have a look at the covectors. Therefore, we will consider the corresponding covectors $t^{2p_1}dx^1$, $t^{2p_2}dx^2$ and $t^{2p_3}dx^3$.

$$g\left(t^{2p_1}dx^1, t^{2p_1}dx^1\right) = t^{2p_1}, \quad \nabla_1 t^{2p_1}dx^1 = -p_1t^{2p_1-1}dt, \quad \nabla_0 t^{2p_1}dx^1 = p_1t^{2p_1-1}dx^1, \tag{1.36}$$

$$g\left(t^{2p_2}dx^2, t^{2p_2}dx^2\right) = t^{2p_2}, \quad \nabla_2 t^{2p_2}dx^2 = -p_2t^{2p_2-1}dt, \quad \nabla_0 t^{2p_2}dx^2 = p_2t^{2p_2-1}dx^2, \tag{1.37}$$

$$g\left(t^{2p_3}dx^3, t^{2p_3}dx^3\right) = t^{2p_3}, \quad \nabla_3 t^{2p_3}dx^3 = -p_3t^{2p_3-1}dt, \quad \nabla_0 t^{2p_3}dx^3 = p_3t^{2p_3-1}dx^3. \tag{1.38}$$

The covariant derivatives in the other directions will be zero.

The same can be done for the covector $\hat{v} = x^3t^{2p_2}dx^2 - x^2t^{2p_3}dx^3$ in the case $p_1 = -\frac{1}{3}$ and $p_2 = p_3 = \frac{2}{3}$. Here,

$$\begin{gathered} g\left(\hat{v}, \hat{v}\right) = \left(x^3\right)^2 t^{2p_2} + \left(x^2\right)^2 t^{2p_3}, \quad \nabla_1\hat{v} = 0, \quad \nabla_2\hat{v} = -p_2t^{2p_2-1}x^3dt - t^{2p_3}dx^3, \\ \nabla_3\hat{v} = p_3t^{2p_3-1}x^2dt + t^{2p_3}dx^2, \quad \nabla_0\hat{v} = p_2t^{2p_2-1}x^3dx^2 - p_3t^{2p_3-1}x^2dx^3. \end{gathered} \tag{1.39}$$

We will not consider the case $p_1 = p_2 = 0$ and $p_3 = 1$ because it is a transformation of the Minkowski metric, which already results in a trivial case.

Chapter 2

Finsler geometry

Einstein's Theory of General Relativity has been formulated in the framework of pseudo-Riemannian geometry. We want to generalize pseudo-Riemannian geometry to pseudo-Finsler geometry. Therefore, we first properly introduce Finsler manifolds. We start with an n-dimensional C^∞ manifold M. Define T_xM to be the tangent space at $x \in M$ and $TM := \bigcup_{x\in M} T_xM$ the collection of all tangent spaces. An element of TM consists of a tuple (x, y) where $x \in M$ and $y \in T_xM$.

A *Finsler structure* of M is a mapping

$$F : TM \to [0, \infty) \tag{2.1}$$

with the properties:

(F1) *Regularity*: F is C^∞ on the slit tangent bundle $TM \setminus 0$.

(F2) *Positive homogeneity*: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$ and $(x, y) \in TM$.

(F3) *Strong convexity*: The fundamental tensor

$$g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}, \quad i, j \in \{1, \ldots, n\}$$

is positive definite for all $(x, y) \in TM \setminus 0$.

A pair (M, F), where M is a manifold endowed with a Finsler structure F as defined above, is called a Finsler manifold [21]. When we compare this to a Riemannian manifold, we see that in Riemannian geometry we have that $g_{ij}(x, y) = g_{ij}(x)$. If this condition is satisfied, the Finsler manifold actually is a Riemannian manifold. Therefore, in Riemannian geometry, we also have

$$F^2(x, y) = g_{ij}(x)y^i y^j. \tag{2.2}$$

We recognize this as the squared line element in Riemannian geometry, which becomes even clearer with $y = dx$. The fundamental tensor is also often referred to as the metric tensor, which is a key ingredient in the General Theory of Relativity.

2.1 Pseudo-Finsler and Berwald spacetimes

Actually, Einstein's General Theory of Relativity does not have an underlying Riemannian manifold, but a pseudo-Riemannian manifold. This means that the metric tensor g_{ij} does not need to be positive definite anymore. Nevertheless, it is still impossible for the metric tensor to have zero-eigenvalues, only positive and negative ones are allowed. Therefore, we will also disregard strong convexity for a pseudo-Finslerian manifold. Instead, axiom (F3) is replaced by the requirement of non-degeneracy. Hence, for all $y \in T_xM \setminus 0$ there must exist a $z \in T_xM$ such that $g_{ij}y^iz^j \neq 0$. In other words, a Finsler structure requires g to have only positive eigenvalues while a pseudo-Finsler structure does also allow negative eigenvalues, but zero-eigenvalues are still impossible. Also, the regularity property may not hold when the Finsler structure is 0 [9].

We can be even more precise, and then General Relativity is defined on a Lorentzian manifold, which is a Riemannian manifold where the metric tensor has one eigenvalue with a different sign. We consider a 4-dimensional spacetime with one temporal coordinate and three spatial coordinates. In this situation, we adopt a $(-,+,+,+)$ signature for the metric tensor and then we have one negative and three positive eigenvalues. We want the same in Finsler geometry and call these Finsler spacetimes [22].

To simplify the generalized Einstein field equation, we will not investigate Finsler spacetimes in full generality. They are much more difficult to work with than Riemannian spacetimes due to the much more general line elements. Instead, we will consider Berwald spacetimes. To introduce Berwald spacetimes, a few definitions in general (pseudo-)Finslerian manifolds are needed.

The formal Christoffel symbols (of the second kind) are the first concept and they are defined by

$$\gamma^i{}_{jk} = \frac{1}{2}g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} + \frac{\partial g_{kl}}{\partial x^j}\right). \tag{2.3}$$

As in Riemannian geometry, g^{ij} is the inverse metric tensor. Comparing the definition with the Christoffel symbols in Riemannian geometry, we observe the two definitions are equivalent. However, the Christoffel symbols only depend on x because g_{ij} is independent of y in Riemannian geometry.

The second important concept are the *geodesic spray coefficients*, which are defined by

$$G^i = \gamma^i{}_{jk}y^jy^k. \tag{2.4}$$

In *Berwald spacetimes*, these geodesic spray coefficients are quadratic in y. For Riemannian manifolds, the formal Christoffel symbols will not depend on y and therefore, the geodesic spray coefficients are certainly quadratic in y. However, there are many Berwald spacetimes that are not Riemannian, although the restriction is tighter than general Finsler spacetimes.

There are more ways to express the geodesic spray coefficients. When filling in the definitions of the formal Christoffel symbols and the metric tensor and exploiting Euler's theorem for homogeneous functions, the geodesic spray coefficients become [22]

$$G^i = \frac{1}{2}g^{ik}\left(y^j\frac{\partial^2 F^2}{\partial x^j \partial y^k} - \frac{\partial F^2}{\partial x^k}\right). \tag{2.5}$$

There are still other representations that can be very useful. The first one needs the nonlinear connection. The nonlinear connection can be defined as [22]

$$N^i{}_j = \frac{1}{4}\frac{\partial}{\partial y^j}\left(g^{ik}\left(y^l\frac{\partial^2 F^2}{\partial x^l \partial y^k} - \frac{\partial F^2}{\partial x^k}\right)\right). \tag{2.6}$$

With Euler's theorem for homogeneous functions, we also see that

$$G^i = N^i{}_j y^j \tag{2.7}$$

is an equivalent definition for the geodesic spray coefficients. At the same time, it shows us that

$$N^i{}_j = \frac{1}{2}\frac{\partial G^i}{\partial y^j}. \tag{2.8}$$

In Riemannian geometry, this reduces to $N^i{}_j = \gamma^i{}_{jk}y^k$ as the formal Christoffel symbols are symmetric in j and k and independent of y. Now, the nonlinear connection is used to define the horizontal derivative, which is

$$\frac{\delta}{\delta x^j} = \frac{\partial}{\partial x^j} - N^i{}_j\frac{\partial}{\partial y^i}. \tag{2.9}$$

We define the Christoffel symbols again, but instead of using partial derivatives we replace them by delta derivatives. The result is the Chern connection symbol:

$$\Gamma^i{}_{jk} = \frac{1}{2}g^{is}\left(\frac{\delta g_{sj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} + \frac{\delta g_{ks}}{\delta x^j}\right). \tag{2.10}$$

The Chern connection comes into play with torsion freeness and almost g-compatibility. In the Riemannian limit, g_{ij} is independent of y and therefore the Chern connection symbols reduce to the Christoffel symbols. However, even more interesting is that when the Chern connection symbol and $\gamma^i{}_{jk}$ are both contracted with y^j and y^k, the results are equal. Therefore, the geodesic spray coefficients can alternatively be defined as

$$G^i = \Gamma^i{}_{jk}y^j y^k. \tag{2.11}$$

We defined the Berwald spacetimes by the geodesic spray coefficients being quadratic in y. However, this condition can equivalently be stated as the Chern connection symbols being independent of y. Therefore, in Berwald spacetimes, it is very useful to work with the Chern connection symbols. The usefulness becomes even more apparent because the formal Christoffel symbols do depend on y in general Berwald spacetimes, while equations (2.4) and (2.11) are very similar.

As the name already says, the geodesic spray coefficients have something to do with the geodesics in Finslerian manifolds. In Riemannian geometry, geodesics are curves that connect two points on the manifold by the shortest path. Equivalently, geodesics can be defined by the curve where the tangent vector always points in the same direction (along the curve). From either of these properties can be derived that the geodesics are described by the geodesic equation:

$$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0, \tag{2.12}$$

where $\Gamma^{\mu}{}_{\nu\rho}$ denotes the Christoffel symbol in Riemannian geometry [2] and the position x along the geodesic is parametrized by some parameter u and $\dot{x}$ represents the derivative to this parameter u. In Finsler geometry, the geodesics are defined equivalently and geodesics with constant speed are characterized by

$$\ddot{x}^{\mu} + G^{\mu} = 0, \tag{2.13}$$

where y is replaced by $\dot{x}$ in G^{μ}. When we compare equations (2.12) and (2.13) wherein we substitute the definition from equation (2.4), we see the two equations are equivalent. For non-constant speed geodesics, an additional term appears in equation (2.13), but the form remains the same.

2.2 Field equations in vacuum

A crucial concept in Einstein's Theory of General Relativity is curvature and in particular the Ricci tensor. A quantity that is very important for the curvature in Finsler geometry is

$$R^{i}{}_{k} = \frac{1}{F^2}\left(\left(G^i\right)_{x^k} - \frac{1}{4}\left(G^i\right)_{y^j}\left(G^j\right)_{y^k} - \frac{1}{4}y^j\left(G^i\right)_{y^k x^j} + \frac{1}{2}G^j\left(G^i\right)_{y^k y^j}\right), \tag{2.14}$$

where we define it in terms of the geodesic spray coefficients. The quantity is called the predecessor of the flag curvature. Unfortunately, the predecessor of the flag curvature does not reduce to the Ricci tensor in the Riemannian limit. However, from $R^{i}{}_{k}$ we define the Finsler Ricci scalar as the trace

$$Ric = R^{i}{}_{i}. \tag{2.15}$$

From the Finslerian version of the Ricci scalar, the *Finsler Ricci tensor* is defined by

$$Ric_{ij} = \frac{1}{2}\frac{\partial^2(F^2 Ric)}{\partial y^i \partial y^j}, \tag{2.16}$$

which does converge to the ordinary Ricci tensor in the Riemannian limit.

Using these quantities, it is possible to generalize the Einstein field equations in vacuum to the *generalized field equation*

$$\left(F^2 g^{ij} - 3y^i y^j\right) Ric_{ij} = 0, \tag{2.17}$$

in Berwald spacetimes. In [8], a vacuum field equation for general Finsler spacetimes has been developed using an action describing gravity. In [22], this equation for general Finsler spacetimes has been simplified for Berwald spacetimes, which gives equation (2.17).

In equation (2.17), we see that a vanishing Finsler Ricci tensor in Berwald spacetimes is a sufficient condition for the spacetime to be a solution of the generalized field equation. In Riemannian geometry, Einstein's field equations in vacuum state that the Riemannian Ricci tensor must vanish. In that case, the Finsler Ricci tensor also vanishes in Riemannian manifolds as it generalizes the Riemannian Ricci tensor. Therefore, the generalized field equation is certainly satisfied in a Riemannian manifold that solves Einstein's field equations in vacuum.

2.3 VSR spacetimes

In this section, we consider the line element

$$ds = (\eta_{\mu\nu} dx^\mu dx^\nu)^{\frac{1-b}{2}} (n_\rho dx^\rho)^b , \tag{2.18}$$

where $\eta_{\mu\nu} dx^\mu dx^\nu$ is the Minkowski metric, $n_\rho dx^\rho$ is a 1-form with constant coefficients and b is some parameter. The line element is first investigated by Bogoslovsky and Goenner [23, 24, 25] and therefore it is sometimes also called the Bogoslovsky line element. In the limit $b = 0$, we find that the Minkowski metric is recovered. The corresponding Finsler structure is of course

$$F(x, y) = (\eta_{\mu\nu} y^\mu y^\nu)^{\frac{1-b}{2}} (n_\rho y^\rho)^b , \tag{2.19}$$

where $(x, y) \in TM$. The spacetimes described by this Finsler structure are called Very Special Relativity (VSR) spacetimes.

When we calculate the geodesic spray coefficients using equation (2.5), it turns out to be the case that all coefficients vanish. In other words, $G^\mu = 0$ for all μ. These geodesic spray coefficients are trivially quadratic in y and therefore the space described by the Bogoslovsky line element is a Berwald spacetime. As the geodesic spray coefficients are 0, also the Chern connection vanishes, which corresponds to the vanishing affine connection with the Minkowski metric. Also, the predecessor of the flag curvature vanishes as can be seen from equation (2.14). Hence, the Finsler Ricci scalar and consequently the Finsler Ricci tensor will also vanish using equations (2.15) and (2.16). Therefore, the generalized field equation (2.17) is trivially satisfied. For the Minkowski metric, the Einstein field equations are also trivially satisfied as it describes a flat space and therefore we can see this Finsler structure as a generalization of the Minkowski metric.

For the Kasner metric with Kasner exponents $p_1 = p_2 = 0$ and $p_3 = 1$, we know it is a transformation of the Minkowski metric as we saw in chapter 1. We can also use the transformation as discussed in chapter 1 in the Bogoslovsky line element and we are in very special relativity. After transforming back from the $\tilde{x}^\mu$ coordinates to the ordinary x^μ coordinates, the Kasner metric is recovered. Substituting $d\tilde{t}$ and $d\tilde{x}^3$ as we calculated them in chapter 1, we find

$$n_\rho d\tilde{x}^\rho = -\left(n_0 \cosh x^3 - n_3 \sinh x^3\right) dt + n_1 dx^1 + n_2 dx^2 + t\left(n_3 \cosh x^3 - \sinh x^3\right) dz, \tag{2.20}$$

where some simple 1-forms can be found as dx^i for $i = 1, 2$, $e^{x^3} dt + te^{x^3} dx^3$ and $-e^{-x^3} dt + te^{-x^3} dx^3$ when choosing suitable constants n_ρ. Looking at table 1.1, we see those 1-forms are the covectors corresponding to some of the Killing vectors for $p_1 = p_2 = 0$ and $p_3 = 1$. This gives a Finslerian generalization of the Kasner metric with these Kasner exponents.

2.4 VGR Berwald spacetimes

We will now move on to Very General Relativity (VGR). In VGR, we have the Bogoslovsky line element where the Minkowski metric is replaced by an arbitrary Riemannian metric from General Relativity and the constant 1-form becomes an arbitrary 1-form. Hence, Finsler structures of the form

$$F = (g_{\mu\nu}(x) y^\mu y^\nu)^{\frac{1-b}{2}} (w_\rho(x) y^\rho)^b , \tag{2.21}$$

are investigated. Here, $g_{\mu\nu}$ is a Riemannian metric tensor and $w_\rho(x)y^\rho$ is a 1-form. The powers $\frac{1-b}{2}$ and b are chosen in such a way that F is first-order homogeneous in y. However, with a pseudo-Riemannian metric, this Finsler structure is not necessarily regular anymore on the zero vectors. Using a Lagrangian as in [22, 8], it is possible to tackle such regularity problems. However, this would unnecessarily complicate the discussion here. We could take x in some connected domain excluding the zero-vectors such that the Riemannian part and the 1-form in (2.21) are positive and on this domain, regularity is fine. Alternatively, we can also take other connected domains where zero vectors of the Riemannian metric and the 1-form are excluded and then regularity will also be fine.

To simplify Einstein's field equations, we want equation (2.21) to constitute a Berwald spacetime. Theorem 1 from [22] tells us that a VGR spacetime is of Berwald type if and only if there exists a function $C(x)$ such that the 1-form w satisfies

$$\nabla_\mu w_\nu = C(x)\left(2\left(1+n\right)w_\mu w_\nu - ng(w,w)g_{\mu\nu}\right). \tag{2.22}$$

Here, $g(w,w)$ is the norm of w using the (pseudo-)Riemannian metric, ∇_μ also represents the covariant derivative to x^μ from the underlying Riemannian geometry and $n = \frac{2b}{1-b}$. We will call this equation the *Berwald condition*. Eventually, we also want the spacetime to satisfy the generalized field equation (2.17). To do so, we do not pick a random pseudo-Riemannian geometry, but a solution of the vacuum field equations developed by Einstein in Riemannian geometry.

When we have a closer look at equation (2.22), we notice that the right-hand side is symmetric in μ and ν. Therefore, the left-hand side must also be symmetric, so

$$\nabla_\mu w_\nu = \nabla_\nu w_\mu. \tag{2.23}$$

Comparing this to the Killing equation (1.14), we see the two equations are actually the opposite. Here, we want symmetry in μ and ν while antisymmetry is wanted for the Killing vectors. If both would be satisfied, this can only be the case if all covariant derivatives $\nabla_\mu w_\nu$ vanish.

Exploiting equation (2.23) yields

$$\partial_\mu w_\nu - \Gamma^\rho{}_{\nu\mu} w_\rho = \partial_\nu w_\mu - \Gamma^\rho{}_{\mu\nu} w_\rho, \tag{2.24}$$

where Γ also represents the Christoffel symbols arising from the Riemannian metric. We know the Christoffel symbols are symmetric in the lower indices and therefore these terms cancel against each other in the equation above and we are left with

$$\partial_\mu w_\nu = \partial_\nu w_\mu. \tag{2.25}$$

This equation can be recognized as the condition for a 1-form to be *closed*. The term exact differential is more commonly used in thermodynamics and a 1-form being closed means the same as being an exact differential. The equivalence of closed and exact 1-forms is a result of the Poincaré Lemma [26]. Hence, the 1-form can be written as

$$w = df = \partial_i f(x) dx^i, \tag{2.26}$$

for some smooth function f. This turns out to be a useful property we will exploit in chapters 3 and 4 in an attempt to find suitable 1-forms such that the Finsler function (2.21) describes a Berwald spacetime.

Chapter 3

Kasner metric in VGR Berwald spacetimes

We are interested in Finsler structures from equation (2.21), where the Riemannian part is the Kasner metric. So, $g_{\mu\nu}$ is the same as in equation (1.2). That is, $g_{00} = -1$ and $g_{ii} = t^{2p_i}$ for $i = 1, 2, 3$ are the only nonzero components of the metric tensor. This gives the Finsler function

$$F = \left(-\left(y^0\right)^2 + \sum_{i=1}^{3} t^{2p_i} (y^i)^2 \right)^{\frac{1-b}{2}} \left(w_\rho(x) y^\rho\right)^b , \tag{3.1}$$

for an arbitrary 1-form $w_\rho y^\rho$. At the end of section 2.4, we already saw that this 1-form must be a closed so $w_\rho y^\rho = \partial_\rho f\, y^\rho$ for some sufficiently smooth function f. Since we have covariant derivatives in the Berwald condition from equation (2.22), it is also useful to refresh the Christoffel symbols for the Kasner metric and the only nonzero components are $\Gamma^0{}_{ii} = p_i t^{2p_i - 1}$ and $\Gamma^i{}_{i0} = \Gamma^i{}_{0i} = \frac{p_i}{t}$ for $i = 1, 2, 3$.

Now, we want to have a look at the condition for the Very General Relativity element describing a Berwald spacetime. The $(0, 0)$-component of equation (2.22) is

$$\partial_0{}^2 f = C(x)(2(1+n)\left(\partial_0 f\right)^2 + n g(df, df)), \tag{3.2}$$

the $(i, 0)$- and $(0, i)$-components for $i = 1, 2, 3$ are

$$\partial_0 \partial_i f - \frac{p_i}{t} \partial_i f = C(x) 2(1+n) \partial_0 f \partial_i f, \tag{3.3}$$

the (i, i)-components for $i = 1, 2, 3$ are

$$\partial_i{}^2 f - p_i t^{2p_i - 1} \partial_0 f = C(x)(2(1+n)(\partial_i f)^2 - n g(df, df) t^{2p_i}), \tag{3.4}$$

and, eventually, the (i, j)- and (j, i) components for $i, j = 1, 2, 3$ such that $i \neq j$ are

$$\partial_i \partial_j f = C(x) 2(1+n) \partial_i f \partial_j f. \tag{3.5}$$

In the this chapter, these equations together with the observations from chapters 1 and 2, will be exploited to derive useful conditions for the 1-form in the VGR line element and conclusions will be drawn from this.

3.1 Covariantly constant 1-forms

From equations (3.2-3.5), certain useful properties for the 1-form can be derived. In section 2.4, we already saw that a Killing vector only is a potential suitable 1-form if all the covariant derivatives vanish. That would also mean that we can take $C(x) = 0$ as a suitable function in equation (2.22). In section 1.1, we saw none of the Killing vectors have a vanishing covariant derivative. There is one exception, and that is if one of the p_i's vanishes. However, then we are in the case with $p_1 = p_2 = 0$ and $p_3 = 1$, which is equivalent to the Minkowski metric. If there would be a vector with vanishing covariant derivatives, it would be a Killing vector. Therefore, there are no suitable 1-forms with vanishing covariant derivatives. This also means that the right-hand side of equation (2.22) cannot always be zero which means that $C(x) = 0$ is not possible.

Alternatively, we can also show this by contradiction. If we assume $C(x) = 0$ in equation (2.22), we see that the $(0, i)$-components give

$$\partial_0 w_i - \frac{p_i}{t} w_i = 0. \tag{3.6}$$

This differential equation is solved by

$$w_i = \xi_i(x^j) t^{p_i}, \tag{3.7}$$

where the ξ_i's are functions independent of time. The time-derivative of w_i is

$$\partial_0 w_i = p_i \xi_i(x^j) t^{p_i - 1}. \tag{3.8}$$

The $(0, 0)$-component of equation (2.22) is $\partial_0 w_0 = 0$, so $w_0 = \eta(x^j)$. Now, the 1-form is an exact differential so

$$p_i \xi_i(x^j) t^{p_i - 1} = \partial_0 w_i = \partial_i w_0 = \partial_i \eta(x^j). \tag{3.9}$$

But here, the right-hand side is independent of t, while that is not the case for the left-hand side for $p_i \neq 0, 1$. Therefore, this cannot result in a Berwald spacetime unless both sides would be zero. In that case, also $w_i = 0$ and η would be constant. However, then the (i, i)-components become

$$-p_i t^{2p_i - 1} w_0 = 0, \tag{3.10}$$

which would result in $w_0 = 0$. There is one exception possible, and that is with $p_1 = p_2 = 0$ and $p_3 = 1$ which gives a transformation of the Minkowski metric again. Therefore, this case will be disregarded. So we found, in two ways, that it is impossible to have vanishing covariant derivatives such that equation (2.22) is satisfied with $C(x) = 0$. In particular, the Killing vectors cannot be suitable 1-forms because there are no covariantly constant Killing vectors.

3.2 Vanishing components

It is also useful to know whether all components are needed. For the time-component, it is relatively easy to see what happens. Assume that f is independent of time, then equation (3.3) tells us that

$$-\frac{p_i}{t} \partial_i f = 0, \tag{3.11}$$

where there is no summation over i. Again, we assume that we are not in the trivial $p_1 = p_2 = 0$ and $p_3 = 1$ case and then it follows that $\partial_i f = 0$. Therefore, f must be a constant and then the 1-form would be 0. However, the Finsler structure would be degenerate in that case and cannot describe a proper spacetime. Therefore, a time-component is needed.

Now, assume $\partial_i f = 0$ for all $i \in \{1, 2, 3\}$. Then equation (3.4) becomes

$$-p_i t^{2p_i - 1} \partial_0 f = -C(x) n g(df, df) t^{2p_i}, \tag{3.12}$$

which yields

$$p_i t^{-1} \partial_0 f = C(x) n g(df, df), \tag{3.13}$$

after dividing it by $-t^{2p_i}$. Here, the right-hand sides are the same for all components. However, on the left-hand side, we already know $\partial_0 f \neq 0$ and the p_i's cannot all be the same so there are no proper solutions. Therefore, f cannot be independent of all spatial coordinates.

Now, assume f is independent of x^i, but does depend on x^j for some i and j. Without loss of generality we can take $i = 3$ and $j = 1$ as we can permute the Kasner exponents. Then there is one equation containing $\partial_3 f$ that does not become trivial and that is the $(3, 3)$-component of equation (3.4). This equation gives

$$p_3 t^{-1} \partial_0 f = C(x) n g(df, df), \tag{3.14}$$

after dividing by $-t^{2p_3}$. Using this fact, equation (3.2) becomes

$${\partial_0}^2 f - p_3 t^{-1} \partial_0 f = C(x) 2(1 + n)(\partial_0 f)^2. \tag{3.15}$$

Similarly, the $(1, 1)$-component of equation (3.4) becomes

$${\partial_1}^2 f - (p_1 - p_3) t^{2p_1 - 1} \partial_0 f = C(x) 2(1 + n)(\partial_1 f)^2. \tag{3.16}$$

Next to these equations, we will also exploit the $(1, 0)$-component of equation (3.3) which yields

$$\partial_0 \partial_1 f - p_1 t^{-1} \partial_1 f = C(x) 2(1 + n) \partial_0 f \partial_1 f. \tag{3.17}$$

When we multiply equation (3.15) by $\partial_1 f$ and equation (3.17) by $\partial_0 f$, the right-hand sides are the same. Hence, we get

$${\partial_0}^2 f \partial_1 f - p_3 t^{-1} \partial_0 f \partial_1 f = \partial_0 \partial_1 f \partial_0 f - p_1 t^{-1} \partial_1 f \partial_0 f. \tag{3.18}$$

Rearranging terms and dividing by $\partial_0 f \partial_1 f$ yields

$$\frac{{\partial_0}^2 f}{\partial_0 f} - \frac{\partial_0 \partial_1 f}{\partial_1 f} - (p_3 - p_1) t^{-1} = 0. \tag{3.19}$$

Here, we recognize a derivative with respect to time as

$$\partial_0 \ln \frac{\partial_0 f}{t^{p_3 - p_1} \partial_1 f} = \partial_0 \left(\ln \partial_0 f - \ln \partial_1 f - \ln t^{p_3 - p_1} \right) = 0. \tag{3.20}$$

Therefore,

$$\partial_0 f = \xi(x^1, x^2) t^{p_3 - p_1} \partial_1 f, \tag{3.21}$$

where ξ is some function independent of time and x^3. Taking the derivative to x^1 of this equation also yields

$$\partial_0\partial_1 f = \partial_1\xi t^{p_3-p_1}\partial_1 f + \xi t^{p_3-p_1}{\partial_1}^2 f. \tag{3.22}$$

Alternatively, we can also multiply equation (3.17) by $\partial_1 f$ and equation (3.16) by $\partial_0 f$ which gives the same right-hand sides again. Therefore, we must also have

$$\partial_0\partial_1 f\partial_1 f - p_1 t^{-1}(\partial_1 f)^2 = \partial_0 f{\partial_1}^2 f - (p_1 - p_3)t^{2p_1-1}(\partial_0 f)^2. \tag{3.23}$$

When we rewrite all time derivatives of f in terms of only $\partial_1 f$ derivatives using the identities we found before, we get

$$\partial_1\xi t^{p_3-p_1}(\partial_1 f)^2 + \xi t^{p_3-p_1}{\partial_1}^2 f\partial_1 f - p_1 t^{-1}(\partial_1 f)^2 = \xi t^{p_3-p_1}{\partial_1}^2 f\partial_1 f - (p_1 - p_3)t^{2p_3-1}\xi^2(\partial_1 f)^2. \tag{3.24}$$

Now, the two ${\partial_1}^2 f$ terms cancel against each other. Rearranging terms yields

$$\left(\partial_1\xi t^{p_3-p_1} - p_1 t^{-1} + (p_1 - p_3)t^{2p_3-1}\xi^2\right)(\partial_1 f)^2 = 0. \tag{3.25}$$

Because $\partial_1 f \neq 0$, the first term between the brackets must cancel. In particular, it must vanish for all t and different powers of t are linearly independent. If $p_3 - p_1, 2p_3 - 1 \neq -1$, we must therefore have $p_1 = 0$, which yields the trivial Minkowski space. Therefore, we are interested in cases where $p_3 - p_1 = -1$ or $2p_3 - 1 = -1$. In the second case, we have $p_3 = 0$, which yields the trivial Kasner exponents again. Therefore, we want $p_3 - p_1 = -1$. Combined with the other Kasner conditions, this gives two possibilities for the Kasner exponents, which are the two degenerate cases. We will only consider the nontrivial case and then $p_3 = -\frac{1}{3}$ and $p_1 = \frac{2}{3}$. The relevant part of the equation above becomes

$$\left(\partial_1\xi - \frac{2}{3}\right)t^{-1} + t^{-5/3}\xi^2 = 0. \tag{3.26}$$

However, the $t^{-5/3}$ term must also vanish which gives $\xi = 0$ by linear independence of the powers of t. But then $\partial_0 f$ would be zero by the way ξ is defined. This is a contradiction as we found f must always depend on time. Therefore, we have a contradiction and for nontrivial Kasner exponents, the function f must depend on all coordinates.

3.3 Spatial separability

Another fact can be found when f must depend on all spatial variables. We take the (i, j)-component of equation (3.5) and we multiply it by $\partial_k f$ for distinct $i, j, k \in \{1, 2, 3\}$, which yields

$$\partial_i\partial_j f\partial_k f = C(x)2(1 + n)\partial_i f\partial_j f\partial_k f. \tag{3.27}$$

We see that the right-hand side is symmetric in i, j, k and hence the left-hand side must also be symmetric in i, j, k. Filling in the numbers, we have

$$\partial_1\partial_2 f\partial_3 f = \partial_1\partial_3 f\partial_2 f = \partial_2\partial_3 f\partial_1 f. \tag{3.28}$$

Let us take

$$\partial_i\partial_j f\partial_k f = \partial_i\partial_k f\partial_j f, \tag{3.29}$$

for some distinct $i, j, k \in \{1, 2, 3\}$. We know that f must depend on all spatial variables. Therefore we can divide by $\partial_j f \partial_k f$, which yields

$$\frac{\partial_i \partial_j f}{\partial_j f} = \frac{\partial_i \partial_k f}{\partial_k f}. \tag{3.30}$$

Here, some derivatives to x^i can be recognized as this is

$$\partial_i \ln \partial_j f = \partial_i \ln \partial_k f. \tag{3.31}$$

Bringing everything to the left-hand side and solving the differential equation, we find that $\frac{\partial_j f}{\partial_k f}$ must be independent of x^i. In other words, $\partial_i \frac{\partial_j f}{\partial_k f} = 0$.

A function $h(x^1, \ldots, x^n)$ satisfying $\partial_i \frac{\partial_j h}{\partial_k h} = 0$ for $i \neq j, k$, is called *strongly separable* in econometrics. In econometrics, this identity turns out to be quite interesting. Theorem 1 of [27] tells us that a function is strongly separable if and only if it is of the form $h(x) = \xi\left(\sum_{i=1}^n \phi_i(x^i)\right)$ for some function ξ, when the theorem is applied to the finest, trivial partition $N = \{\{1\}, \ldots, \{n\}\}$. Therefore, f must be of the form $f(x) = \xi\left(\sum_{i=1}^3 \phi_i(x^i, t)\right)$.

Filling in this function into equation (3.5) gives

$$\partial_i \phi_i \partial_j \phi_j \xi'' = C(x) 2(1+n) \partial_i \phi_i \partial_j \phi_i (\xi')^2, \tag{3.32}$$

where the arguments of the functions have been suppressed. Since f must depend on all x^i's, the $\partial_i \phi_i$'s must also be nonzero. Dividing the equation by $\partial_i \phi_i \partial_j \phi_i$ yields

$$\xi'' = C(x) 2(1+n)(\xi')^2, \tag{3.33}$$

which will be substituted in the other components of the condition.

Equation (3.3) becomes

$$\partial_0 \partial_i \phi_i \xi' + \partial_i \phi_i \left(\sum_{j=1}^3 \partial_0 \phi_j \right) \xi'' - \frac{p_i}{t} \partial_i \phi_i \xi' = \partial_i \phi_i \left(\sum_{j=1}^3 \partial_0 \phi_j \right) \xi'', \tag{3.34}$$

where we immediately substituted equation (3.33). In this equation, also the ξ'' terms cancel and we can also divide by ξ' as f cannot be constant. Executing this division yields

$$\partial_0 \partial_i \phi_i - \frac{p_i}{t} \partial_i \phi_i = 0. \tag{3.35}$$

When we divide by $\partial_i \phi_i$ we recognize some derivatives with respect to time as

$$\partial_0 \ln \frac{\partial_i \phi_i}{t^{p_i}} = \frac{\partial_0 \partial_i f}{\partial_i f} - \frac{p_i}{t} = 0. \tag{3.36}$$

Therefore,

$$\partial_i \phi_i = \psi_i(x^i) t^{p_i}, \tag{3.37}$$

where the ψ_i's are functions independent of time. Integrating again but now to x^i gives

$$\phi_i = \eta_i(t) + \psi_i(x^i) t^{p_i}, \tag{3.38}$$

for some functions $\eta_i, i = 1,2,3$ that only depend on t and $\psi_i, i = 1,2,3$ that are the primitive of the old ψ_i's. All those η_i's can be combined, which yields

$$f(x) = \xi\left(\eta(t) + \sum_{i=1}^{3}\psi_i(x^i)t^{p_i}\right), \tag{3.39}$$

for the general form where $\eta = \sum_{i=1}^{3}\eta_i$.

Under condition (3.33), equation (3.39) is the solution to the non-diagonal components of the Berwald condition. However, the diagonal components of the condition must also be satisfied. The (i,i)-components from equation (3.4) become

$$\partial_i{}^2\psi_i t^{p_i}\xi' + (\partial_i\psi_i)^2 t^{2p_i}\xi'' - p_i t^{2p_i-1}\left(\eta' + \sum_{j=1}^{3}p_j\psi_j t^{p_j-1}\right)\xi' = (\partial_i\psi_i)^2 t^{2p_i}\xi'' - ht^{2p_i}, \tag{3.40}$$

where we already substituted equation (3.33) and $h = C(x)ng(df, df)$, which is the same for all (i,i)-components, $i = 1,2,3$. Again, the ξ'' terms cancel which yields

$$\partial_i{}^2\psi_i t^{-p_i}\xi' - p_i t^{-1}\left(\eta' + \sum_{j=1}^{3}p_j\psi_j t^{p_j-1}\right)\xi' = -h, \tag{3.41}$$

after dividing by t^{2p_i}. Here, the right-hand side is the same for all (i,i)-components, so

$$\partial_i{}^2\psi_i t^{-p_i}\xi' - p_i t^{-1}\left(\eta' + \sum_{k=1}^{3}p_k\psi_k t^{p_k-1}\right)\xi' = \partial_j{}^2\psi_j t^{-p_j}\xi' - p_j t^{-1}\left(\eta' + \sum_{k=1}^{3}p_k\psi_k t^{p_k-1}\right)\xi', \tag{3.42}$$

for some $i,j \in \{1,2,3\}$ such that $i \neq j$. Here, we can divide by ξ' and rearranging terms yields

$$\partial_i{}^2\psi_i t^{-p_i} - \partial_j{}^2\psi_j t^{-p_j} = (p_i - p_j)t^{-1}\left(\eta' + \sum_{k=1}^{3}p_k\psi_k t^{p_k-1}\right). \tag{3.43}$$

Now, let $k \in \{1,2,3\}$ such that $k \neq i,j$, then the left-hand side is independent of x^k. To make the right-hand side also independent of x^k, there are three possibilities. The first one is that ψ_k must be independent of x^k, but then we have a contradiction as f must depend on x^k. The second possiblity is that $p_k = 0$, but then we have the trivial Minkowski metric which we will not consider here. The last possibility is that $p_i = p_j$, which has a nontrivial solution by $p_i = p_j = \frac{2}{3}$ and $p_k = -\frac{1}{3}$. However, if we interchange j and k, we still have that ψ_j must be independent of x^j which still gives a contradiction.

Therefore, there are no suitable 1-forms such that equation (2.21) applied to the Kasner metric describes a Berwald spacetime for nontrivial Kasner exponents.

Chapter 4

Bianchi type-I metrics in VGR Berwald spacetimes

In chapter 3, we saw that it is impossible to incorporate the Kasner metric into the VGR Finsler structure from equation (2.21) such that it describes a non-trivial Berwald spacetime. In this chapter, we will investigate Finsler structures of the form

$$F = \left(- \left(y^0\right)^2 + \sum_{i=1}^{3} \left(A_i(t)\right)^2 \left(y^i\right)^2 \right)^{\frac{1-b}{2}} \left(w_\rho(x) y^\rho\right)^b , \tag{4.1}$$

for some purely time-dependent, nonzero functions $A_i, i = 1, 2, 3$ and some arbitrary 1-form $w_\rho y^\rho$. We want to know whether we can generalize our results to the entire family of Bianchi type-I metrics.

Let us first have a brief look at the Bianchi type-I metrics in Riemannian geometry. We already saw in chapter 1 that the line elements are of the form

$$ds^2 = -dt^2 + \sum_{i=1}^{3} \left(A_i(t)\right)^2 \left(dx^i\right)^2 , \tag{4.2}$$

for some suitably chosen coordinates x^i such that the metric tensor becomes diagonal and the A_i's are some purely time-dependent functions. The A_i's cannot be zero because that would give a degenerate metric tensor as we have a zero eigenvalue in the concerned direction. Analogous to the Kasner metric, the only non-vanishing components of the affine connection are

$$\Gamma^0{}_{ii} = A_i \dot{A}_i, \quad \Gamma^i{}_{0i} = \Gamma^i{}_{i0} = \frac{\dot{A}_i}{A_i}, \tag{4.3}$$

for $i = 1, 2, 3$ and where the dot represents the (time) derivative. The Ricci tensor becomes

$$R_{00} = \sum_{j=1}^{3} \frac{\ddot{A}_j}{A_j}, \quad R_{ii} = \dot{A}_i^{\,2} - A_i \ddot{A}_i - A_i \dot{A}_i \sum_{j=1}^{3} \frac{\dot{A}_j}{A_j}, \tag{4.4}$$

for $i = 1, 2, 3$ and all non-diagonal components are zero. Therefore, these components must also be zero to satisfy the Einstein field equations in vacuum, which gives the Kasner metric and the Minkowski metric up to some rescaling of the coordinates [17].

Now we know the Christoffel symbols, we are able to evaluate the covariant derivatives and we can explore the condition for the VGR spacetime to be a Berwald spacetime. Again, we will use that $w_\rho y^\rho = \partial_\rho f\, y^\rho$. Then, the $(0,0)$-component of equation (2.22) is

$$\partial_0{}^2 f = C(x)(2(1+n)\,(\partial_0 f)^2 + ng(df, df)), \tag{4.5}$$

the $(i,0)$- and $(0,i)$-components of the Berwald condition for $i = 1,2,3$ become

$$\partial_0\partial_i f - \frac{\dot{A}_i}{A_i}\partial_i f = C(x)2(1+n)\partial_0 f \partial_i f, \tag{4.6}$$

the (i,i)-components for $i = 1,2,3$ become

$$\partial_i{}^2 f - A_i\dot{A}_i\partial_0 f = C(x)(2(1+n)(\partial_i f)^2 - ng(df, df)A_i{}^2), \tag{4.7}$$

and, eventually, the (i,j)- and (j,i)-components for $i,j = 1,2,3$ such that $i \neq j$ are

$$\partial_i\partial_j f = C(x)2(1+n)\partial_i f \partial_j f. \tag{4.8}$$

Similarly to chapter 3, these equations will be used to derive some restrictions for the 1-form such that the VGR line element describes a Berwald spacetime.

4.1 Vanishing time component

In section 3.2, we investigated the cases where some components of the 1-form vanish. The same will be done with the Bianchi type-I metrics. Similarly to the Kasner metric, we first have a look at the situation where we have independence of time, i.e. f is independent of time. Then, equation (4.6) becomes

$$\frac{\dot{A}_i}{A_i}\partial_i f = 0. \tag{4.9}$$

Therefore, $\dot{A}_i = 0$ or $\partial_i f = 0$ for all $i = 1,2,3$. The first possibility would be that $\partial_i f = 0$ for all $i = 1,2,3$, but then f is constant and the 1-form df would be zero. Then the Finsler structure is degenerate as it becomes 0 for $b \neq 0$. Therefore, this does not describe a suitable VGR spacetime. Secondly, we can also have $\dot{A}_i = 0$ for all $i = 1,2,3$ and then all A_i's are constant. However, when we introduce coordinates $\tilde{x}^i = A_i x^i$, we recover the Minkowski metric and we already investigated that situation in section 2.3. The third situation is where we take $\dot{A}_i = 0$ for some fixed $i \in \{1,2,3\}$ and $\partial_j f = 0$ for $j \neq i$. Then, f only depends on x^i and not on x^j for $j \neq i$. Using a similar transformation as above, namely $\tilde{x}^i = A_i x^i$, we can also take $A_i = 1$ and drop the tilde. The (j,j)-component of equation (4.7) for $j \neq i$ becomes

$$0 = C(x)ng(df, df)A_j{}^2. \tag{4.10}$$

Now, $n = \frac{2b}{1-b}$ is a parameter that is 0 if and only if $b = 0$ which means that we have a Riemannian metric. Of course, we do not want this and therefore one of the other terms must be 0. The function A_j cannot be zero as that would constitute a degenerate metric tensor in Riemannian geometry. Therefore, $g(df, df)$ or $C(x)$ must be zero. In this situation, $g(df, df) = (\partial_i f)^2$ which can only be zero if f is independent of x^i, which cannot be the case as we concluded in the first situation. Therefore, $C(x) = 0$. The only condition that is not satisfied yet is the (i,i)-component which becomes

$$\partial_i{}^2 f = 0. \tag{4.11}$$

Therefore, f is linear in x^i for our fixed i and $f = ax^i + b$ for some constants a, b. The 1-form is $df = a\,dx^i$ for some constant a and this constitutes a proper 1-form for the VGR line element describing a Berwald spacetime. The last situation we have not considered yet is $\partial_j f = 0$ for some $j \in \{1,2,3\}$ and $\dot{A}_i = 0$ for $i \neq j$. Here, the same reasoning as above holds and therefore we must have f at most linear in x^i for $i \neq j$. However, there is another condition that is not satisfied yet and that is the ik-component for $i \neq k$ such that $i, k \neq j$ and this component becomes

$$\partial_i \partial_k f = 0. \tag{4.12}$$

Hence, the combined $x^i x^k$ term must also vanish and $f = ax^i + bx^k + c$ for some constants a, b, c. In the end, $df = a\,dx^i + b\,dx^k$ for constants a, b is a suitable 1-form such that we end up with a Berwald spacetime. These are the only cases in which the VGR line element with a time-independent 1-form constitutes a Berwald spacetime.

In conclusion, we can only have a time-independent 1-form if we have at least one A_i that is constant and if A_i is constant for all $i \in I$ for some $I \subseteq \{1,2,3\}$, then a suitable 1-form is $\sum_{i \in I} c_i dx^i$ for some constants $c_i, i \in I$. We only need one A_i to be constant and we can have non-trivial Berwald spacetimes this way. However, if we want the Riemannian metric to satisfy the field equations, it must be the Minkowski metric of a transformation of the Minkowski metric.

4.2 Vanishing spatial components

Now, we know all cases in which f can be independent of time, we assume f only depends on time for the second step. Then, the (i,i)-components from equation (4.7) become

$$A_i \dot{A}_i \partial_0 f = C(x) n g(df, df) A_i{}^2. \tag{4.13}$$

We can divide this by $A_i{}^2$, which gives

$$\frac{\dot{A}_i}{A_i} \partial_0 f = C(x) n g(df, df). \tag{4.14}$$

Here, the right-hand side is the same for all $i = 1, 2, 3$ and hence the left-hand side must be as well. As f depends on t, $\partial_0 f \neq 0$ and we can divide by $\partial_0 f$. Therefore,

$$\frac{\dot{A}_1}{A_1} = \frac{\dot{A}_2}{A_2} = \frac{\dot{A}_3}{A_3}. \tag{4.15}$$

Here, we recognize the time derivatives of the $\ln A_i$'s and integration tells us that A_1, A_2 and A_3 are scalar multiples of each other. But then, we can choose an appropriate rescaling of the spatial coordinates such that $A_1 = A_2 = A_3 = A$ for some function $A = A(t)$. The (semi-)norm of the 1-form is $g(df, df) = -(\partial_0 f)^2$ as we only have a time-component and substituting this in equation (4.14) yields

$$\frac{\dot{A}}{A} \partial_0 f = -C(x) n (\partial_0 f)^2. \tag{4.16}$$

The other nontrivial component is the $(0,0)$-component from equation (4.5), which gives

$$\partial_0{}^2 f = C(x)(2(1+n)-n)(\partial_0 f)^2 = C(x)(2+n)(\partial_0 f)^2. \tag{4.17}$$

Combining the two equations above, we find

$$\partial_0{}^2 f = -\frac{2+n}{n}\frac{\dot{A}}{A}\partial_0 f. \tag{4.18}$$

Dividing by $\partial_0 f$, we recognize some time derivatives as

$$\partial_0 \ln \partial_0 f = \frac{\partial_0{}^2 f}{\partial_0 f} = -\frac{2+n}{n}\frac{\dot{A}}{A} = \partial_0 \ln A^{-\frac{2+n}{n}}. \tag{4.19}$$

Integration gives

$$\partial_0 f = cA^{-\frac{2+n}{n}}, \tag{4.20}$$

for some constant c. Therefore, we have the 1-form $df = cA^{-\frac{2+n}{n}}dt$. Substituting this 1-form in equation (4.16), returns a specific function $C(x)$ and then all components of the Berwald condition are satisfied. Therefore, it is a suitable 1-form that is not covariantly constant as we cannot take $C(x) = 0$. This result is in agreement with [22] where the FLRW-metric has been considered and our metric is nothing different than the FLRW-metric with zero spatial curvature. Remarkably, the Finsler Ricci tensor also vanishes for the VGR element, and therefore the generalized field equation is satisfied. However, the Einstein equations in Riemannian geometry are only satisfied if A is constant which results in (a rescaling of) the Minkowski metric. This has to do with our 1-form. The way we defined n dependent on b, we have $\frac{2+n}{n} = \frac{1}{b}$. Therefore, the VGR line element becomes

$$\begin{aligned} ds &= \left(-(dt)^2 + A^2\sum_{i=1}^{3}\left(dx^i\right)^2\right)^{\frac{1-b}{2}}\left(cA^{-\frac{1}{b}}dt\right)^b \\ &= \left(-A^{-2}(dt)^2 + \sum_{i=1}^{3}\left(dx^i\right)^2\right)^{\frac{1-b}{2}}\left(cA^{-1}dt\right)^b. \end{aligned} \tag{4.21}$$

But here, we recognize the Minkowski metric when we substitute $\tilde{t} = \ln A(t)$ and then the 1-form is $cd\tilde{t}$, which is nothing different than the 1-form in VSR spacetimes where all spatial components vanish. This is the reason our VGR line element describes a Berwald spacetime and satisfies the field equations. Although we did not start with (a transformation of) the Minkowski metric, we eventually end up with a VSR spacetime from section 2.3.

In conclusion, the only case where a purely time-dependent 1-form gives a Berwald spacetime is when all A_i's are the same (or scalar multiples of each other). When we take $A_i = A$ for all $i = 1, 2, 3$, the 1-form $cA^{-\frac{2+n}{n}}dt$ for a constant c constitutes a suitable VGR line element, but we eventually have a transformation of a VSR spacetime, while we did not start with a transformation of the Minkowski metric.

4.3 A vanishing spatial component

The third situation we investigated in section 3.2 is the situation where f does depend on t and x^i for some i, but not on x^j for some j. Since we can permute the spatial components, we can take f dependent on x^1 and independent of x^3 without loss of generality. The only component of the Berwald condition containing $\partial_3 f$ that does not become trivial is the $(3,3)$-component of equation (4.7), which becomes

$$-A_3\dot{A}_3\partial_0 f = -C(x)ng(df,df){A_3}^2. \tag{4.22}$$

We can divide this by A_3, which yields

$$C(x)ng(df,df) = \frac{\dot{A}_3}{A_3}\partial_0 f. \tag{4.23}$$

When we substitute this in the $(0,0)$-component from equation (4.5), we find

$${\partial_0}^2 f - \frac{\dot{A}_3}{A_3}\partial_0 f = C(x)2(1+n)(\partial_0 f)^2. \tag{4.24}$$

Similarly, the $(1,1)$-component from equation (4.7) becomes

$${\partial_1}^2 f + {A_1}^2\left(\frac{\dot{A}_3}{A_3} - \frac{\dot{A}_1}{A_1}\right)\partial_0 f = C(x)2(1+n)(\partial_1 f)^2. \tag{4.25}$$

We complete this set of equations by repeating the $(0,1)$-component from equation (4.6):

$$\partial_0\partial_1 f - \frac{\dot{A}_1}{A_1}\partial_1 f = C(x)2(1+n)\partial_0 f\partial_1 f. \tag{4.26}$$

We multiply equation (4.24) by $\partial_1 f$ and we multiply equation (4.26) by $\partial_0 f$ such that the right-hand sides of these equations are equal. Equating the left-hand sides yields

$${\partial_0}^2 f\partial_1 f - \frac{\dot{A}_3}{A_3}\partial_0 f\partial_1 f = \partial_0\partial_1 f\partial_0 f - \frac{\dot{A}_1}{A_1}\partial_0 f\partial_1 f. \tag{4.27}$$

Dividing this equation by $\partial_0 f\partial_1 f$ and rearranging terms gives

$$\frac{{\partial_0}^2 f}{\partial_0 f} - \frac{\partial_0\partial_1 f}{\partial_1 f} - \frac{\dot{A}_3}{A_3} + \frac{\dot{A}_1}{A_1} = 0. \tag{4.28}$$

Here, we recognize some time derivatives and eventually

$$\partial_0 \ln\frac{A_1\partial_0 f}{A_3\partial_1} = 0. \tag{4.29}$$

Therefore,

$$\partial_0 f = \xi(x^1,x^2)\frac{A_3}{A_1}\partial_1 f, \tag{4.30}$$

where ξ is a function independent of time. As we want f to be independent of x^3, we also know that ξ is independent of x^3. We will use this identity to replace time derivatives of f by derivatives to x^1. The x^1-derivative of equation (4.30) is

$$\partial_0\partial_1 f = \partial_1\xi\frac{A_3}{A_1}\partial_1 f + \xi\frac{A_3}{A_1}{\partial_1}^2 f. \tag{4.31}$$

Now, we multiply equation (4.26) by $\partial_1 f$ and we multiply equation (4.25) by $\partial_0 f$. Again, the right-hand sides are equal and therefore we set the left-hand sides to be equal. The result is

$$\partial_0\partial_1 f\partial_1 f - \frac{\dot{A}_1}{A_1}(\partial_1 f)^2 = {\partial_1}^2 f\partial_0 f + {A_1}^2\left(\frac{\dot{A}_3}{A_3} - \frac{\dot{A}_1}{A_1}\right)(\partial_0 f)^2. \tag{4.32}$$

Now, we replace our time-derivatives by derivatives to x^1, by applying equations (4.30) and (4.31):

$$\partial_1\xi\frac{A_3}{A_1}(\partial_1 f)^2 + \xi\frac{A_3}{A_1}{\partial_1}^2 f\partial_1 f - \frac{\dot{A}_1}{A_1}(\partial_1 f)^2 = \xi\frac{A_3}{A_1}\partial_1 f{\partial_1}^2 f + {A_3}^2\left(\frac{\dot{A}_3}{A_3} - \frac{\dot{A}_1}{A_1}\right)\xi^2(\partial_1 f)^2. \tag{4.33}$$

The terms containing ${\partial_1}^2 f$ obviously cancel against each other and rearranging terms yields

$$\left(\partial_1\xi\frac{A_3}{A_1} - \frac{\dot{A}_1}{A_1} - {A_3}^2\left(\frac{\dot{A}_3}{A_3} - \frac{\dot{A}_1}{A_1}\right)\xi^2\right)(\partial_1 f)^2 = 0. \tag{4.34}$$

The first term between the brackets must be zero because we assumed f depends on x^1. For the situation with the Kasner metric in section 3.2, we applied linear independence between the powers of t. However, in equation (4.34), the time-dependent terms are $\frac{A_3}{A_1}$, $\frac{\dot{A}_1}{A_1}$ and ${A_3}^2\left(\frac{\dot{A}_3}{A_3} - \frac{\dot{A}_1}{A_1}\right)$ and the linear dependence between these terms is much more difficult to investigate because A_1 and A_3 are arbitrary functions. Therefore, there is no final result for the situation discussed in this section, but this identity could contribute to derive results in future research.

4.4 No vanishing components

Now we considered all situations in the sections above, we can move on to the situation where f really depends on all x^μ's. Similarly as in section 3.3, we have spatial separability. The (i,j)-component from equation (4.8) multiplied by $\partial_k f$ for distinct $i,j,k \in \{1,2,3\}$ is

$$\partial_i\partial_j f\partial_k f = C(x)2(1+n)\partial_i f\partial_j f\partial_k f. \tag{4.35}$$

Because the right-hand side is symmetric in i,j,k, the left-hand side must be as well. Therefore,

$$\partial_i\partial_j f\partial_k f = \partial_i\partial_k f\partial_j f, \tag{4.36}$$

and dividing by $\partial_j f\partial_k f$ yields

$$\partial_i \ln\partial_j f = \frac{\partial_i\partial_j f}{\partial_j f} = \frac{\partial_i\partial_k f}{\partial_k f} = \partial_i \ln\partial_k f. \tag{4.37}$$

Therefore, $\frac{\partial_j f}{\partial_k f}$ is independent of x^i for $i \neq j,k$ and we have strong separability from [27] as for the Kasner metric in section 3.3. Therefore,

$$f(x) = \xi\left(\sum_{i=1}^{3}\phi_i(x^i,t)\right), \tag{4.38}$$

for some functions ξ and ϕ_i for $i = 1, 2, 3$. Substituting this in the (i, j)-components of the condition yields

$$\partial_i\phi_i\partial_j\phi_j\xi'' = C(x)2(1+n)\partial_i\phi_i\partial_j\phi_j(\xi')^2. \tag{4.39}$$

Therefore,

$$\xi'' = C(x)2(1+n)(\xi')^2 \tag{4.40}$$

as we assumed dependence on all components and hence the $\partial_i\phi_i$ terms cannot be zero. When we substitute equation (4.40) in the $(i, 0)$-component of equation (4.6) for some $i \in \{1, 2, 3\}$, we find

$$\partial_0\partial_i\phi_i\xi' + \partial_i\phi_i\left(\sum_{j=1}^{3}\partial_0\phi_i\right)\xi'' - \frac{\dot{A}_i}{A_i}\partial_i\phi_i\xi' = \partial_i\phi_i\left(\sum_{j=1}^{3}\partial_0\phi_i\right)\xi''. \tag{4.41}$$

Obviously, the ξ'' terms cancel and dividing by $\partial_i\phi_i\xi'$ yields

$$\partial_0 \ln\frac{\partial_i\phi_i}{A_i} = \partial_0\left(\ln\partial_i\phi_i - \ln A_i\right) = 0. \tag{4.42}$$

Integrating this equation gives

$$\partial_i\phi_i = \psi_i(x^i)A_i(t), \tag{4.43}$$

for some function ψ_i, independent of time. Integrating this again yields

$$\phi_i = \eta_i(t) + \psi_i(x^i)A_i(t), \tag{4.44}$$

for some function η_i independent of x^i and some function ψ_i, which is a primitive of our old ψ_i. This procedure can be repeated for all $i = 1, 2, 3$ and then all η_i's can be combined, which gives

$$f(x) = \xi\left(\eta(t) + \sum_{i=1}^{3}\psi_i(x^i)A_i(t)\right), \tag{4.45}$$

for some function η. Let us now consider the (i, i)-component while keeping all of this in mind, then we find

$$\partial_i{}^2\psi_iA_i\xi' + (\partial_i\psi_i)^2A_i{}^2\xi'' - A_i\dot{A}_i\left(\dot{\phi} + \sum_{j=1}^{3}\psi_j\dot{A}_j\right)\xi' = (\partial_i\psi_i)^2A_i{}^2\xi'' - hA_i{}^2, \tag{4.46}$$

where $h = C(x)ng(df, df)$. Again, the ξ'' terms cancel and we are left with

$$\partial_i{}^2\psi_i\frac{1}{A_i} - \frac{\dot{A}_i}{A_i}\left(\dot{\phi} + \sum_{j=1}^{3}\psi_j\dot{A}_j\right) = -\frac{h}{\xi'}. \tag{4.47}$$

Here, the right-hand side is independent of the component so must the left-hand side be. Therefore,

$$\partial_i{}^2\psi_i\frac{1}{A_i} - \frac{\dot{A}_i}{A_i}\left(\dot{\phi} + \sum_{k=1}^{3}\psi_k\dot{A}_k\right) = \partial_j{}^2\psi_j\frac{1}{A_j} - \frac{\dot{A}_j}{A_j}\left(\dot{\phi} + \sum_{k=1}^{3}\psi_k\dot{A}_k\right), \tag{4.48}$$

for $i, j \in \{1, 2, 3\}$. Rearranging some terms gives

$$\left(\frac{\dot{A}_i}{A_i} - \frac{\dot{A}_j}{A_j}\right)\left(\dot{\phi} + \sum_{k=1}^{3} \psi_k \dot{A}_k\right) = \frac{\partial_i{}^2\psi_i}{A_i} - \frac{\partial_j{}^2\psi_j}{A_j}. \tag{4.49}$$

Now, the right-hand side is independent of x^k for $k \neq i, j$ so must the left-hand side be. Therefore, ψ_k is constant, $\dot{A}_k = 0$ or $\frac{\dot{A}_i}{A_i} - \frac{\dot{A}_j}{A_j} = 0$. The function ψ_k cannot be constant as f must depend on x^k by assumption. Hence, $\dot{A}_k = 0$ or $\frac{\dot{A}_i}{A_i} = \frac{\dot{A}_j}{A_j}$. As the reasoning holds for all combinations of distinct $i, j, k \in \{1, 2, 3\}$, we must have $\dot{A}_k = 0$ or $\frac{\dot{A}_i}{A_i} = \frac{\dot{A}_j}{A_j}$ for all k, where i and j are the other values in $\{1, 2, 3\}$ that are not k.

The first possible situation is that $\dot{A}_i = 0$ for all $i = 1, 2, 3$, but then the A_i's are all constant and we effectively have the Minkowski metric as we showed in section 4.1. The second case we consider is $\dot{A}_i = 0$ for some $i \in \{1, 2, 3\}$ and $\frac{\dot{A}_i}{A_i} = \frac{\dot{A}_j}{A_j}$ for all $j \neq i$. Then, $\dot{A}_i = 0$ for all $i = 1, 2, 3$ and we are in the same situation as before. Thirdly, it is also possible that $\frac{\dot{A}_i}{A_i} = \frac{\dot{A}_j}{A_j}$ for all combinations of i and j. However, $\frac{A_i}{A_j}$ must then be constant for all i and j and multiplying the coordinates by suitable constants, we can simply take $A_1 = A_2 = A_3 = A$ for some function $A = A(t)$. Hence, we are in the case of section 4.2, where we found a suitable 1-form. Since the VGR line element we found actually comes from the Minkowski metric, we could probably use this to find a more general 1-form which also gives a Berwald spacetime using our VSR line element. However, we are not necessarily interested in finding all suitable 1-forms. Therefore, we will be fine with our purely time-dependent 1-form. The last possibility is $\dot{A}_i = \dot{A}_j = 0$ for some i and j and $\frac{\dot{A}_i}{A_i} = \frac{\dot{A}_j}{A_j}$. Obviously, the latter equality is satisfied when $\dot{A}_i = \dot{A}_j = 0$. We already considered this situation in section 4.1, and there we found a 1-form $a\,dx^i + b\,dx^j$ for some functions a and b. Again, there might be more 1-forms that for example do depend on time, but we are not necessarily interested in them. Now, we considered all possible cases such that equation (4.49) is satisfied as we want.

Hence, there are no new possibilities for the functions $A_i, i = 1, 2, 3$, such that the VGR line element describes a Berwald spacetime where there are no vanishing components in the 1-form. It is possible that in the situations we investigated in sections 4.1 and 4.2, there are more suitable 1-forms, but we are not necessarily interested in finding all 1-forms if we have a suitable one.

Chapter 5

Conclusion

In chapter 1, we saw that the Bianchi type-I metrics and the Kasner metric in particular play an important role in cosmology. We were interested to know what happens when we generalize the Kasner metric and the general Bianchi type-I metrics to Finsler geometry. To do so, the VGR line element from equation (2.21) has been used. Equation (2.22) states the necessary and sufficient condition such that the VGR line element induces a Berwald spacetime, which we wanted to simplify the generalized field equation. At the end of chapter 2, in section 2.4, we found that *the 1-form in equation (2.21) must always be closed.*

In chapter 3, we found that it is *impossible for the VGR line element with the Kasner metric to induce a Berwald spacetime for nontrivial Kasner exponents.* Only for $p_1 = p_2 = 0$ and $p_3 = 1$, the VGR line element can describe a Berwald spacetime, but this Kasner metric is a transformation of the Minkowski metric, which is something we were not really interested in. Then we effectively have a VSR spacetime.

Since it was impossible to find a VGR generalization of the Kasner metric to Berwald spacetimes, we wanted to know whether the same holds for general Bianchi type-I metrics. Unfortunately, section 4.3 has no final result. Therefore, the case of a 1-form with one or two vanishing spatial component(s) should be investigated further. For simplicity, we did not try to find all 1-forms such that the VGR line element constitutes a Berwald spacetime when we had one 1-form. It would also be good to complete the results by finding all suitable 1-forms. Nevertheless, the results from chapter 4 are summarized in table 5.1. For the Kasner metric, we see it does not fit in any of the cases from table 5.1 and in section 3.2 we also found that a vanishing spatial component is not possible.

Table 5.1: The possibilities for the Bianchi type-I metrics such that the VGR line element constitutes a Berwald spacetime.

Case	**Possible 1-form**
Minkowski ($A_1 = A_2 = A_3 = 1$)	$n_\mu dx^\mu$ for constants n_μ
$A_1 = 1$	cdx^1 for constant c
$A_1 = A_2 = 1$	$c_1 dx^1 + c_2 dx^2$ for constants c_1, c_2
Isotropic ($A_1 = A_2 = A_3 = A$)	$cA^{-\frac{1}{b}}dt$ for constant c
Section 4.3 has no final result	Independent of one or two spatial component(s)

What could also be interesting are Landsberg spacetimes. Landsberg spacetimes are a little bit more general than Berwald spacetimes, while the generalized field equation will stay the same. This can be shown easily in the same way as is done for Berwald spacetimes in [22]. Therefore, it would be interesting to know whether it is possible for the VGR line element with the Kasner metric and the entire family of Bianchi type-I metrics to induce a Landsberg spacetime. If so, the second step would be to find out whether it satisfies the generalized field equation.

Lastly, it would be possible to consider general 1-forms in general Finsler geometries, but then the generalized field equation is much more complicated. Maybe, there are solutions to these generalized field equations that are not of Berwald/Landsberg-type. However, it will probably be difficult to find the solutions to these field equations.

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Appendix A

Mathematica notebook

On the next pages, a Mathematica notebook is shown that has been used to calculate certain quantities in Finsler manifolds. It has both been used to get a feeling with the project and to calculate certain quantities in the Finsler manifolds involved in this research. For brevity, only the definitions of the functions are shown.

Define functions

In[1]:=
```
MakeSym[g_] := Module[{rtmp}, rtmp = Table[g[[i, j]], {i, Length[g]}, {j, i}];
  MapThread[Join, {rtmp, Rest /@ Flatten[rtmp, {{2}, {1}}]}]]
```

In[2]:=
```
FundamentalTensor[F_, xx_, yy_] := Block[{n, dF, res}, n = Length[yy];
  dF = Table[D[1/2 * F[xx, yy]^2, yy[[i]]], {i, n}];
  res = Table[D[dF[[i]], yy[[j]]], {i, n}, {j, i}];
  res = MakeSym[res];
  FullSimplify[res]]
FundamentalTensor::usage =
  "FundamentalTensor[F, xx, yy] computes the fundamental tensor
    (g_ij) from the Finsler structure F at (xx,yy)∈TM.";
```

In[4]:=
```
InverseFundamentalTensor[g_] := FullSimplify[Inverse[g]]
InverseFundamentalTensor::usage =
  "InverseFundamentalTensor[g] computes the inverse fundamental
    tensor (g^ij) from the fundamental tensor g_ij.";
```

In[6]:=
```
CartanTensor[F_, g_, xx_, yy_] := Block[{n, res}, n = Length[yy];
  res = Table[F[xx, yy]/2 * D[g[[i, j]], yy[[k]]], {i, n}, {j, n}, {k, n}];
  FullSimplify[res]]
CartanTensor::usage =
  "CartanTensor[F, g, xx, yy] computes the Cartan tensor (A_ijk) from the
    Finsler structure F and the fundamental tensor g_ij at (xx,yy)∈TM.";
```

In[8]:=
```
DistinguishedSection[F_, xx_, yy_] := FullSimplify[Table[yy[[i]]/F[xx, yy], {i, Length[yy]}]]
DistinguishedSection::usage =
  "DistinguishedSection[F, xx, yy] computes the components of the distinguished
    section (l^i) from the Finsler structure F and at (xx,yy)∈TM.";
```

In[10]:=
```
HilbertForm[F_, xx_, yy_] := FullSimplify[Table[D[F[xx, yy], yy[[i]]], {i, Length[yy]}]]
HilbertForm::usage =
  "HilbertForm[F, xx, yy] computes the components of the Hilbert form
    (l_i) from the Finsler structure F and at (xx,yy)∈TM,
    which is the dual of the distinguished section l^i.";
```

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In[12]:=
```
ChristoffelSymbols[g_, xx_] := Block[{n, ig, dg, res}, n = Length[xx];
  ig = InverseFundamentalTensor[g];
  dg = Table[D[g[[i, j]], xx[[k]]], {i, n}, {j, n}, {k, n}];
  res =
   Table[(1/2) * Sum[ig[[i, s]] * (dg[[s, j, k]] - dg[[j, k, s]] + dg[[k, s, j]]), {s, n}],
    {i, n}, {j, n}, {k, j}];
  res = Table[MakeSym[res[[i]]], {i, n}];
  FullSimplify[res]]
ChristoffelSymbols::usage =
  "ChristoffelSymbols[g, xx] computes the formal Christoffel symbols
    of the second kind (γ^i_jk) from the fundamental tensor g_ij and xx.";
```

In[14]:=
```
NonlinearConnection[F_, g_, xx_, yy_] :=
 Block[{n, ig, dF2dx, dF2dxdy, res}, n = Length[yy];
  ig = InverseFundamentalTensor[g];
  dF2dx = Table[D[F[xx, yy]^2, xx[[k]]], {k, n}];
  dF2dxdy = Table[D[F[xx, yy]^2, yy[[k]], xx[[l]]], {l, n}, {k, n}];
  res =
   Table[1/4 * D[Sum[ig[[i, k]] * (Sum[yy[[l]] * dF2dxdy[[l, k]], {l, n}] - dF2dx[[k]]),
        {k, n}], yy[[j]]], {i, n}, {j, n}];
  FullSimplify[res]]
NonlinearConnection[F_, g_, A_, γ_, xx_, yy_] := Block[{n, ig, res}, n = Length[yy];
  ig = InverseFundamentalTensor[g];
  res = Table[Sum[γ[[i, j, k]] * yy[[k]], {k, n}] -
      Sum[(ig[[i, l]] * A[[i, j, k]] * γ[[k, r, s]] * yy[[r]] * yy[[s]]) / F[xx, yy],
       {l, n}, {k, n}, {r, n}, {s, n}], {i, n}, {j, n}];
  FullSimplify[res]]
NonlinearConnection[F_, g_, G_, xx_, yy_] := Block[{n, res}, n = Length[yy];
  res = Table[(1/2) * D[G[[i]], yy[[j]]], {i, n}, {j, n}];
  FullSimplify[res]]
NonlinearConnection::usage =
  "NonlinearConnection[F, g, xx, yy] computes the nonlinear connection (N^i_j)
    from the Finsler structure F and the fundamental tensor g_ij at (xx,yy).
NonlinearConnection[F, g, A, γ, xx, yy] does the same job but also wants the
    Cartan tensor A_ijk and the formal Christoffel symbols γ^i_jk as input.
NonlinearConnection[F, g, G, xx, yy] does the same job but also
    needs the geodesic spray coefficients G^i as input.";
```

In[18]:=
```
HorizontalDerivative[f_, F_, g_, xx_, yy_] := Block[{n, NC, res}, n = Length[xx];
  NC = NonlinearConnection[F, g, xx, yy];
  res = Table[D[f, xx[[j]]] - Sum[NC[[i, j]] * D[f, yy[[i]]], {i, n}], {j, n}];
  FullSimplify[res]]
HorizontalDerivative[f_, F_, g_, NC_, xx_, yy_] := Block[{n, res}, n = Length[xx];
  res = Table[D[f, xx[[j]]] - Sum[NC[[i, j]] * D[f, yy[[i]]], {i, n}], {j, n}];
  FullSimplify[res]]
HorizontalDerivative::usage =
  "HorizontalDerivative[f, F, g, xx, yy] computes the horizontal derivatives of f
    from the Finsler structure F and the fundamental tensor g_ij at (xx,yy).
HorizontalDerivative[f, F, g, N, xx, yy] does the same job but
    also wants the nonlinear connection N^i_j as input.";
```

In[21]:=
```
ChernConnection[F_, g_, xx_, yy_] := Block[{n, ig, ghd, res}, n = Length[xx];
  ig = InverseFundamentalTensor[g];
  ghd = FullSimplify[
    Table[HorizontalDerivative[g[[i, j]], F, g, xx, yy][[k]], {i, n}, {j, n}, {k, n}]];
  res = Table[(1/2) * Sum[ig[[i, s]] * (ghd[[s, j, k]] - ghd[[j, k, s]] + ghd[[k, s, j]]),
      {s, n}], {i, n}, {j, n}, {k, n}];
  FullSimplify[res]]
ChernConnection[F_, g_, NC_, xx_, yy_] := Block[{n, ig, ghd, res}, n = Length[xx];
  ig = InverseFundamentalTensor[g];
  ghd = FullSimplify[Table[
     HorizontalDerivative[g[[i, j]], F, g, NC, xx, yy][[k]], {i, n}, {j, n}, {k, n}]];
  res = Table[(1/2) * Sum[ig[[i, s]] * (ghd[[s, j, k]] - ghd[[j, k, s]] + ghd[[k, s, j]]),
      {s, n}], {i, n}, {j, n}, {k, n}];
  FullSimplify[res]]
ChernConnection[F_, g_, γ_, A_, NC_, xx_, yy_] := Block[{n, ig, res}, n = Length[xx];
  ig = InverseFundamentalTensor[g];
  res =
   Table[γ[[i, j, k]] - Sum[ig[[i, l]] * (Sum[A[[l, j, s]] * (NC[[s, k]] / F), {s, n}] - Sum[
          A[[j, k, s]] * (NC[[s, l]] / F), {s, n}] +
         Sum[A[[k, l, s]] * (NC[[s, j]] / F), {s, n}]), {l, n}], {i, n}, {j, n}, {k, n}];
  FullSimplify[res]]
ChernConnection::usage =
  "ChernConnection[F, g, xx, yy] computes the Chern connection coefficients Γ^i_jk
    from the Finsler structure F and the fundamental tensor g_ij at (xx,yy).
ChernConnection[F, g, N, xx, yy] does the same job but also
    wants the nonlinear coefficients N^i_j as input.
ChernConnection[F, g, γ, A, N, xx, yy] does the same job but also
    wants the formal Christoffel symbols γ^i_jk, the Cartan
    tensor A_ijk and the nonlinear connection N^i_j as input.";
```

In[25]:=
```
hhcurvature[F_, g_, xx_, yy_] := Block[{n, Γ, Γhd, res}, n = Length[xx];
  Γ = ChernConnection[F, g, xx, yy];
  Γhd = FullSimplify[Table[HorizontalDerivative[Γ[[i, j, l]], F, g, xx, yy][[k]],
     {i, n}, {j, n}, {l, n}, {k, n}]];
  res = Table[Γhd[[i, j, l, k]] - Γhd[[i, j, k, l]] + Sum[Γ[[i, h, k]] * Γ[[h, j, l]],
      {h, n}] - Sum[Γ[[i, h, l]] * Γ[[h, j, k]], {h, n}], {j, n}, {i, n}, {k, n}, {l, n}];
  FullSimplify[res]]
hhcurvature[F_, g_, NC_, Γ_, xx_, yy_] := Block[{n, Γhd, res}, n = Length[xx];
  Γhd = FullSimplify[Table[HorizontalDerivative[Γ[[i, j, l]], F, g, NC, xx, yy][[k]],
     {i, n}, {j, n}, {l, n}, {k, n}]];
  res = Table[Γhd[[i, j, l, k]] - Γhd[[i, j, k, l]] + Sum[Γ[[i, h, k]] * Γ[[h, j, l]],
      {h, n}] - Sum[Γ[[i, h, l]] * Γ[[h, j, k]], {h, n}], {j, n}, {i, n}, {k, n}, {l, n}];
  FullSimplify[res]]
hhcurvature::usage =
  "hhcurvature[F, g, xx, yy] computes the hh-curvature tensor R_j^i_kl from
    the Finsler structure F and the fundamental tensor g_ij at (xx,yy).
hhcurvature[F, g, N, Γ, xx, yy] does the same job but also needs the
    nonlinear connection N^i_j and the Chern connection Γ^i_jk as input.";
```

In[28]:=
```
hvcurvature[F_, g_, xx_, yy_] := Block[{n, Γ, res}, n = Length[yy];
  Γ = ChernConnection[F, g, xx, yy];
  res = Table[-F[xx, yy] * D[Γ[[i, j, k]], yy[[l]]], {j, n}, {i, n}, {k, n}, {l, n}];
  FullSimplify[res]]
hvcurvature[F_, g_, Γ_, xx_, yy_] := Block[{n, res}, n = Length[yy];
  res = Table[-F[xx, yy] * D[Γ[[i, j, k]], yy[[l]]], {j, n}, {i, n}, {k, n}, {l, n}];
  FullSimplify[res]]
hvcurvature::usage =
  "hvcurvature[F, g, xx, yy] computes the hv-curvature tensor P_j^i_kl from
    the Finsler structure F and the fundamental tensor g_ij at (xx,yy).
hvcurvature[F, g, Γ, xx, yy] does the same job but also needs
    the Chern connection Γ^i_jk as input.";
```

In[31]:=
```
GeodesicSpray[F_, g_, xx_, yy_] := Block[{n, ig, res}, n = Length[yy];
  ig = InverseFundamentalTensor[g];
  res =
   Table[1/2 * Sum[ig[[i, l]] * (Sum[yy[[j]] * D[F[xx, yy]^2, xx[[j]], yy[[l]]], {j, n}] -
          D[F[xx, yy]^2, xx[[l]]]), {l, n}], {i, n}];
  FullSimplify[res]]
GeodesicSpray[F_, g_, γ_, xx_, yy_] := Block[{n, res}, n = Length[yy];
  If[Length[Dimensions[γ]] == 3,
   res = Table[Sum[γ[[i, j, k]] * yy[[j]] * yy[[k]], {j, n}, {k, n}], {i, n}]
   , res = Table[Sum[γ[[i, j]] * yy[[j]], {j, n}], {i, n}]];
  FullSimplify[res]]
GeodesicSpray::usage = "GeodesicSpray[F, g, xx, yy]
    computes the Geodesic spray coefficients G^i from F, g, xx and yy.
GeodesicSpray[F, g, γ, xx, yy] does the same job but also needs the formal
    Christoffel symbols γ^i_jk or the Chern connection Γ^i_jk as input.";
```

In[34]:=
```
PredecessorFlagCurvature[F_, g_, xx_, yy_] := Block[{n, l, NC, auxd, res}, n = Length[xx];
  l = DistinguishedSection[F, xx, yy];
  NC = NonlinearConnection[F, g, xx, yy];
  auxd = Table[HorizontalDerivative[NC[[i, j]] / F[xx, yy], F, g, xx, yy][[k]],
    {k, n}, {i, n}, {j, n}];
  res = Table[Sum[l[[j]] * (auxd[[k, i, j]] - auxd[[j, i, k]]), {j, n}], {i, n}, {k, n}];
  FullSimplify[res]]
PredecessorFlagCurvature[F_, g_, l_, NC_, xx_, yy_] :=
 Block[{n, auxd, res}, n = Length[xx];
  auxd = Table[HorizontalDerivative[NC[[i, j]] / F[xx, yy], F, g, xx, yy][[k]],
    {k, n}, {i, n}, {j, n}];
  res = Table[Sum[l[[j]] * (auxd[[k, i, j]] - auxd[[j, i, k]]), {j, n}], {i, n}, {k, n}];
  FullSimplify[res]]
PredecessorFlagCurvature[F_, g_, G_, xx_, yy_] := Block[{n, res}, n = Length[xx];
  res = Table[(1 / F[xx, yy]^2) * (D[G[[i]], xx[[k]]] -
        (1 / 4) * Sum[D[G[[i]], yy[[j]]] * D[G[[j]], yy[[k]]], {j, n}] -
        (1 / 2) * Sum[yy[[j]] * D[G[[i]], yy[[k]], xx[[j]]], {j, n}] +
        (1 / 2) * Sum[G[[j]] * D[G[[i]], yy[[k]], yy[[j]]], {j, n}]), {i, n}, {k, n}];
  FullSimplify[res]]
PredecessorFlagCurvature::usage =
  "PredecessorFlagCurvature[F, g, xx, yy] computes the
    predecessor of the flag curvature R^i_k.
PredecessorFlagCurvature[F, g, l, N, xx, yy] does the same job but also needs
    the distinguished section l^i and the nonlinear connection N^i_j as input.
PredecessorFlagCurvature[F, g, G, xx, yy] does the same job but
    also needs the Geodesic spray coefficients G^i as input.";
```

In[38]:=
```
RicciScalar[F_, g_, xx_, yy_] := Block[{n, R, res}, n = Length[yy];
  R = PredecessorFlagCurvature[F, g, xx, yy];
  res = Sum[R[[s, s]], {s, n}];
  FullSimplify[res]]
RicciScalar[F_, g_, R_, xx_, yy_] := Block[{n, res}, n = Length[yy];
  res = Sum[R[[s, s]], {s, n}];
  FullSimplify[res]]
RicciScalar::usage =
  "RicciScalar[F, g, xx, yy] returns the Ricci scalar Ric in Finsler geometry.
RicciScalar[F, g, R, xx, yy] does the same job but also
    needs the predecessor of the flag curvature R^i_k as input.";
```

In[41]:=

```
RicciTensor[F_, g_, xx_, yy_] := Block[{n, Ric, res}, n = Length[yy];
  Ric = RicciScalar[F, g, xx, yy];
  res = Table[D[(1/2) * F^2 * Ric, yy[[k]], yy[[i]]], {i, n}, {k, n}];
  FullSimplify[res]]
RicciTensor[F_, g_, Ric_, xx_, yy_] := Block[{n, res}, n = Length[yy];
  res = Table[D[(1/2) * F[xx, yy]^2 * Ric, yy[[k]], yy[[i]]], {i, n}, {k, n}];
  FullSimplify[res]]
RicciTensor::usage =
  "RicciTensor[F, g, xx, yy] calculates the Ricci tensor in Finsler geometry.
RicciTensor[F, g, Ric, xx, yy] does the same job
    but also needs the Ricci scalar as input.";
```

Overview of defined functions

Kasner Metric

Very special relativity

Finsler pp-waves

FLRW metric

Special case $p_1 = p_2 = 0,\ p_3 = 1$

B(t)dt

From f

From 1-form

Bianchi Type-I