Almost non interacting control by measurement feedback

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ALMOST NON INTERACTING CONTROL BY MEASUREMENT FEEDBACK

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ABSTRACT

Consider a linear system Σ that, apart from a control input and a measurement output, has two exogenous inputs and two exogenous outputs. Controlling such a system by means of a measurement feedback compensator Σ_c results in a closed loop system with two inputs and two outputs. Hence, the closed loop transfer matrix can be partitioned as a two by two block matrix.
The problem addressed in this paper consists of the following.
Given Σ and any positive number ε, is it possible to find Σ_c such that the off-diagonal blocks of the closed loop transfer matrix, in a suitable norm, are smaller than ε?
For the solvability of this problem necessary and sufficient conditions will be derived.

Keywords & Phrases
Almost non interacting control, measurement feedback, common solution to a pair of linear matrix equations.

1. Introduction

In this paper we shall be concerned with a control problem that arises in the field of almost (or approximate) non interacting control by measurement feedback.

We consider a plant (a system) which, in addition to a control input and a measurement output, has two exogenous inputs and two exogenous outputs. Controlling such a plant by means of a measurement feedback compensator results in a closed loop system which has two inputs and two outputs.

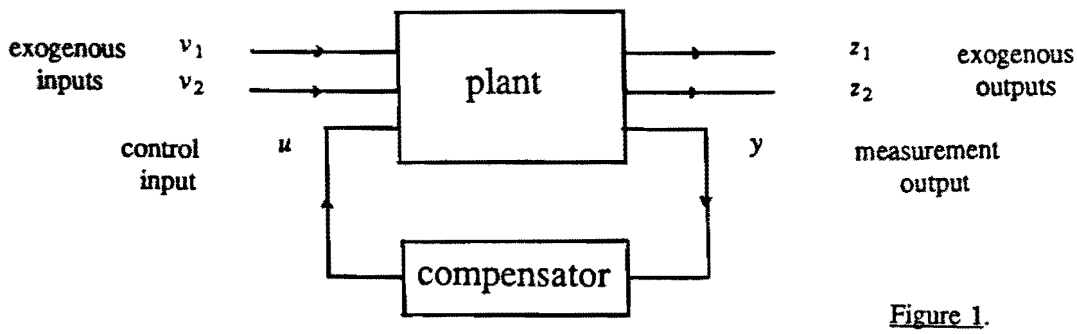

Figure 1.

Prior to the formulation of the main problem of this paper we shall mention a problem that appears in the context of *exact* non interacting control by measurement feedback. However, we note that this problem is *not* yet solved and will not be solved here. It is just meant to be an introduction to our problem.

For a given plant we shall say that the problem of exact non interacting control by measurement feedback is solvable if there exists a compensator such that in the closed loop system of plant and compensator the exogenous output z_1 (respectively z_2) is *not* influenced by the exogenous input v_2 (respectively v_1).

Stated in terms of transfer matrices the problem of exact non interacting control by measurement feedback reads as follows. Given a plant, find a measurement feedback compensator such that the transfer matrix of the resulting closed loop system, when partitioned according to the exogenous inputs and outputs, is block diagonal. If the problem is solvable the corresponding compensator is said to achieve *exact* non interaction.

As anounced, the statement of this problem merely serves as an introduction to and a motivation for the problem considered in this paper. The latter will be the almost version of the above mentioned problem and will be formulated as follows. Given a plant as above, we shall say that the problem of *almost* non interacting control by measurement feedback is solvable if for any positive number ε a measurement feedback compensator can be found such that the transfer matrix of the resulting closed loop system, partitioned according to the exogenous input and outputs, has off-diagonal blocks which, in a suitable norm, are smaller than ε. If the problem is solvable, it will be said that it is possible to achieve *almost* non interaction.

At this point we want to make clear that the main contribution of the present paper lies in the fact that instead of allowing (dynamic) *state* feedback in order to achieve almost non interaction (cf. Willems [8], Trentelman & van der Woude [7]), in this paper we require almost non interaction to be achieved by *measurement* feedback.

As will be clear from the problem formulation (or from Figure 1) the non interacting control problem (both the exact as well as the almost version) is a self dual problem. Roughly speaking this means that the formulation of the non interacting control problem by measurement feedback for our plant has the same structure as the formulation of the same problem for the "dual" plant. (Reversing the directions of the arrows in Figure 1 does not really change the structure of Figure 1.) This idea of self duality will also show up in the necessary and sufficient conditions for the solvability of our main problem.

Also, by Figure 1, it is clear that the point of view on non interacting control as exposed in this paper is completely different from the so-called "classical" approach to non interacting control (cf. Wonham [11], Hautus & Heyman [2]). The latter requires the plant, apart from a control input and a measurement output, only to have k exogenous outputs, where k is an integer larger than or equal to two. Now the problem of *exact* non interacting control by measurement feedback in the "classical" context is said to be solvable if there exists a compensator with $k+1$ inputs, (one of which is the measurement output of the plant) and with one output (serving as the control input of the plant) such that the transfer matrix of the closed loop system, when partitioned with respect to the remaining inputs and outputs, is block diagonal.

The problem of *almost* non interacting control by measurement feedback in the "classical" sense can be formulated in a similar fashion.

The set up of this paper is as follows.

In Section 2 we shall give the mathematical formulation of the almost non interacting control problem by measurement feedback. Also, in Section 2, we give notation and recall some well-known results. Section 3 contains some new results, fundamental to the solution of our problem. These results concern the existence of a common solution to a pair of linear matrix equations. In Section 4 the main result of this paper is stated. It provides necessary and sufficient conditions for the solvability of the almost non interacting control problem by measurement feedback for plants as considered in this paper. In Section 5, an algorithm is given to obtain (when possible) a compensator that achieves almost non interaction with a prescribed accuracy. Furthermore, in Section 5, conclusions and remarks are given.

2. Problem formulation

Consider the finite-dimensional linear time-invariant system Σ given by

$$\dot{x}(t) = A\,x(t) + B\,u(t) + G_1 v_1(t) + G_2 v_2(t)\,, \tag{1a}$$

$$y(t) = C\,x(t)\,, \tag{1b}$$

$$z_1(t) = H_1 x(t)\,, \quad z_2(t) = H_2 x(t)\,, \tag{1c}$$

with $x(t) \in \mathbb{R}^n$ the state of the system, $u(t) \in \mathbb{R}^m$ the control input, $v_1(t) \in \mathbb{R}^{q_1}$, $v_2(t) \in \mathbb{R}^{q_2}$ the exogenous inputs, $y(t) \in \mathbb{R}^p$ the measurement output and $z_1(t) \in \mathbb{R}^{r_1}$, $z_2(t) \in \mathbb{R}^{r_2}$ the exogenous outputs of Σ.

In the above A, B, C, G_1, G_2, H_1 and H_2 are real matrices of the appropriate dimensions.

In this paper we shall assume that the system (1) is controlled by means of a measurement feedback compensator Σ_c described by

$$\dot{w}(t) = K\,w(t) + L\,y(t) \tag{2a}$$

$$u(t) = M\,w(t) + N\,y(t) \tag{2b}$$

with $w(t) \in \mathbb{R}^w$ the state of the compensator and K, L, M and N real matrices of appropriate dimensions. Sometimes we shall denote Σ_c by the matrices K, L, M and N, i.e. $\Sigma_c = (K, L, M, N)$.

Interconnection of Σ and Σ_c yields a closed loop system with two exogenous inputs $v_1(t)$ and $v_2(t)$, and with two exogenous outputs $z_1(t)$ and $z_2(t)$. This closed loop system is described by

$$\dot{x}_e(t) = A_e\,x_e(t) + G_{1,e}\,v_1(t) + G_{2,e}\,v_2(t)\,, \tag{3a}$$

$$z_1(t) = H_{1,e}\,x_e(t)\,, \quad z_2(t) = H_{2,e}\,x_e(t)\,. \tag{3b}$$

Here we have denoted

$$x_e(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \quad A_e = \begin{bmatrix} A + BNC & BM \\ LC & K \end{bmatrix}, \quad G_{i,e} = \begin{bmatrix} G_i \\ 0 \end{bmatrix}, \quad i = 1,2,$$

$$H_{i,e} = [H_i \quad 0]\,, \quad i = 1,2\,.$$

In the frequency domain, the plant Σ is described by

$$y(s) = C(sI - A)^{-1}B\,u(s) + C(sI - A)^{-1}G_1 v_1(s) + C(sI - A)^{-1}G_2 v_2(s)\,,$$

$$z_1(s) = H_1(sI - A)^{-1}B\,u(s) + H_1(sI - A)^{-1}G_1 v_1(s) + H_1(sI - A)^{-1}G_2 v_2(s)\,,$$

$$z_2(s) = H_2(sI - A)^{-1}B\,u(s) + H_2(sI - A)^{-1}G_1 v_1(s) + H_2(sI - A)^{-1}G_2 v_2(s)\,,$$

while the compensator Σ_c is described by

$$u(s) = F(s)\,y(s)$$

where $F(s) = M(sI - K)^{-1}L + N$.

Consequently, the closed loop transfer matrix between the input $v_j(s)$ and the output $z_i(s)$ (i.e. $H_{i,e}(sI-A_e)^{-1}G_{j,e}$) satisfies:

$$H_{i,e}(sI-A_e)^{-1}G_{j,e} = H_i(sI-A)^{-1}B\,X(s)\,C(sI-A)^{-1}G_j + H_i(sI-A)^{-1}G_j \ ,$$

where

$$X(s) = (I-F(s)\,C(sI-A)^{-1}B)^{-1}F(s)\ .$$

Note that the inverse in the latter expression indeed exists since $(I-F(s)C(sI-A)^{-1}B)$ is a bicausal rational matrix (cf. Hautus & Heyman [1]), which also implies that $X(s)$ is a proper rational matrix. Furthermore, $F(s) = X(s)(I+C(sI-A)^{-1}B\,X(s))^{-1}$.

In order to give a precise mathematical formulation of the problem considered in this paper we have to introduce the following.

Let $S \cong \mathbb{R}^t$ be a real t-dimensional linear space with norm $\|\cdot\|$, let $s : \mathbb{R}^+ \to S$ be a measurable function with values in S, where $\mathbb{R}^+ = [0,\infty)$ and let $1 \le p \le \infty$. We shall say that $s \in L_p(\mathbb{R}^+,S)$ if and only if the L_p-norm of s is finite (i.e. $\|s\|_p < +\infty$). Here $\|s\|_p := \left[\int_0^\infty \|s(t)\|^p\,dt\right]^{1/p}$ when $1 \le p < \infty$ and $\|s\|_\infty := \operatorname{ess.sup}_{t\in\mathbb{R}^+} \|s(t)\|$.

We can now give the following definition.

Definition 2.1.

Let Σ be given and let $1 \le p \le \infty$. *The almost non interacting control problem by measurement feedback in the* L_p-*sense* (ANICPM)$_p$ is said to be solvable if for all $\varepsilon > 0$ there exists $\Sigma_c = (K,L,M,N)$ such that with $(x(0),w(0)) = 0$ there holds $\|z_1\|_p \le \varepsilon\|v_2\|_p$ and $\|z_2\|_p \le \varepsilon\|v_1\|_p$.

In the remainder of this section we shall introduce further notation and recall some known results.

Let $\mathbb{R}(s)$ denote the field of rational functions with real coefficients and let $\mathbb{R}_0(s)$ (respectively $\mathbb{R}_+(s)$) denote the class of proper (respectively strictly proper) rational functions with real coefficients. $\mathbb{R}^{k\times l}(s)$ (respectively $\mathbb{R}_0^{k\times l}(s)$, $\mathbb{R}_+^{k\times l}(s)$) will denote the set of $k \times l$ matrices with entries in $\mathbb{R}(s)$ (respectively $\mathbb{R}_0(s)$, $\mathbb{R}_+(s)$). Likewise, $\mathbb{R}^k(s)$ (respectively $\mathbb{R}_0^k(s)$, $\mathbb{R}_+^k(s)$) denotes the set of all k-vectors with entries in $\mathbb{R}(s)$ (respectively $\mathbb{R}_0(s)$, $\mathbb{R}_+(s)$).

Let $G(s) \in \mathbb{R}_+^{k\times l}(s)$ be asymptotically stable and let $1 \le p < \infty$. Then $\|G(s)\|_p$ will denote the L_p-norm of the inverse-Laplace transform $L^{-1}G(t)$ of $G(s)$, i.e. $\|G(s)\|_p = \|L^{-1}G\|_p$. $\|G(s)\|_\infty$ denotes the H^∞-norm of $G(s)$, i.e. $\|G(s)\|_\infty = \sup_{\mathrm{Re}\,s\ge 0}\|G(s)\|$. With these definitions the following turns out to hold (cf. Trentelman [6]).

Proposition 2.2.

(1) Let $p \in \{1,\infty\}$. Then $(\text{ANICPM})_p$ is solvable if and only if for all $\varepsilon > 0$ there exists $X(s) \in \mathbb{R}_0^{m \times p}(s)$ such that

$$\| H_1(sI-A)^{-1}B\,X(s)\,C(sI-A)^{-1}G_2 + H_1(sI-A)^{-1}G_2 \|_1 \le \varepsilon$$

and

$$\| H_2(sI-A)^{-1}B\,X(s)\,C(sI-A)^{-1}G_1 + H_2(sI-A)^{-1}G_1 \|_1 \le \varepsilon \, .$$

(2) $(\text{ANICPM})_2$ is solvable if and only if for all $\varepsilon > 0$ there exists $X(s) \in \mathbb{R}_0^{m \times p}(s)$ such that

$$\| H_1(sI-A)^{-1}B\,X(s)\,C(sI-A)^{-1}G_2 + H_1(sI-A)^{-1}G_2 \|_\infty \le \varepsilon$$

and

$$\| H_2(sI-A)^{-1}B\,X(s)\,C(sI-A)^{-1}G_1 + H_2(sI-A)^{-1}G_1 \|_\infty \le \varepsilon \, .$$

Consider the rational matrix equation

(RME) $A(s)x(s) = b(s)$

with $A(s) \in \mathbb{R}_+^{k \times l}(s)$, $b(s) \in \mathbb{R}_+^k(s)$ and the unknown rational l-vector $x(s)$. The following theorem is due to Willems [10] and plays a crucial role in the proof of our main result.

Theorem 2.3.

Let $1 \le p < \infty$. The following statements are equivalent:

(1) For every $\varepsilon > 0$ there exists $x(s) \in \mathbb{R}_0^l(s)$ such that $\|A(s)x(s)-b(s)\|_p \le \varepsilon$.

(2) For every $\varepsilon > 0$ there exists $x(s) \in \mathbb{R}_0^l(s)$ such that $\|A(s)x(s)-b(s)\|_\infty \le \varepsilon$.

(3) There exists $x(s) \in \mathbb{R}^l(s)$ such that $A(s)x(s) = b(s)$.

Remark 2.4.

Consider the rational matrix equation

(RME)' $A(s)X(s)B(s) = C(s)$

with $A(s) \in \mathbb{R}_+^{k \times l}(s)$, $B(s) \in \mathbb{R}_+^{p \times q}(s)$, $C(s) \in \mathbb{R}_+^{k \times q}(s)$ and the unknown rational $l \times p$ matrix $X(s)$. It is well known that (RME)' can be rewritten as a rational matrix equation of type (RME) by means of Kronecker products (cf. Macduffee [3]).

By Theorem 2.3 and Remark 2.4 the following result follows readily.

Theorem 2.5.

Let $A_1(s) \in \mathbb{R}_+^{k_1 \times n}(s)$, $A_2(s) \in \mathbb{R}_+^{k_2 \times n}(s)$, $B_1 \in \mathbb{R}_+^{m \times l_1}(s)$, $B_2 \in \mathbb{R}_+^{m \times l_2}(s)$, $C_1(s) \in \mathbb{R}_+^{k_1 \times l_1}(s)$ and $C_2(s) \in \mathbb{R}^{k_2 \times l_2}(s)$. Let $1 \leq p \leq \infty$.

Then the following holds.

For all $\varepsilon > 0$ there exists $X(s) \in \mathbb{R}_0^{n \times m}(s)$ such that

$$\| A_1(s)X(s)B_1(s) - C_1(s) \|_p \leq \varepsilon \quad \text{and} \quad \| A_2(s)X(s)B_2(s) - C_2(s) \|_p \geq \varepsilon$$

if and only if there exists $X(s) \in \mathbb{R}^{n \times m}(s)$ such that

$$A_1(s)X(s)B_1(s) = C_1(s) \quad \text{and} \quad A_2(s)X(s)B_2(s) = C_2(s) .$$

Proof. (Sketch) Replace the two equations of type (RME)' (i.e. $A_1(s)X(s)B_1(s) = C_1(s)$ and $A_2(s)X(s)B_2(s) = C_2(s)$) by two equations of type (RME) denoted by

$$\hat{A}_1(s)\hat{x}(s) = \hat{b}_1(s) \quad \text{and} \quad \hat{A}_2(s)\hat{x}(s) = \hat{b}_2(s) .$$

Observe that a common rational solution $\hat{x}(s)$ to these two equations exists if and only if there exists a rational solution $\hat{x}(s)$ to the equation

$$\begin{bmatrix} \hat{A}_1(s) \\ \hat{A}_2(s) \end{bmatrix} \hat{x}(s) = \begin{bmatrix} \hat{b}_1(s) \\ \hat{b}_2(s) \end{bmatrix} .$$

Now application of Theorem 2.3 and carrying out the above described procedure in reversed order, using some elementary properties of norms, proves the theorem. □

3. On a common solution to a pair of linear matrix equations

In this section we shall derive some results that are of great importance in the proof of our main result. The results of the present section will be stated in terms of an arbitrary field $\mathbb{F}$ and deal with the existence of a common solution to a pair of linear matrix equations over $\mathbb{F}$. The use of the results of this section in connection with the previous section is obvious when $\mathbb{F}$ is replaced by $\mathbb{R}(s)$. In the sequel $\mathbb{F}^{k \times l}$ denotes the set of all $k \times l$ matrices with entries in $\mathbb{F}$ and $\mathbb{F}^k$ denotes the set of all k-vectors with entries in $\mathbb{F}$.

For a given $A \in \mathbb{F}^{k \times l}$, we will say that the rank of A is q, i.e. $\operatorname{rank} A = q$, if there exists a q^{th} order minor of A *not* equal to 0 ($\in \mathbb{F}$), while every $(q+1)^{\text{th}}$ order minor of A is equal to 0 ($\in \mathbb{F}$).

In order to obtain the main result of this section, we shall need the following two lemmas.

Lemma 3.1.

Let $A \in \mathbb{F}^{p \times n}$, $B \in \mathbb{F}^{m \times q}$ and $C \in \mathbb{F}^{p \times q}$ be given. The following statements are equivalent:

(1) There exists $X \in \mathbb{F}^{n \times m}$ such that $AXB = C$.

(2) $\operatorname{Im} C \subseteq \operatorname{im} A$, $\ker B \subseteq \ker C$.

(3) $\operatorname{Rank} A = \operatorname{rank}\,[A, C]$, $\operatorname{rank} B = \operatorname{rank} \begin{bmatrix} B \\ C \end{bmatrix}$.

Proof.

(1) $\Leftrightarrow$ (2). See Willems [10].

(2) $\Leftrightarrow$ (3). See any textbook on matrix theory.
For instance Macduffee [3]. □

Lemma 3.2.

Let $A \in \mathbb{F}^{p \times n}$, $B \in \mathbb{F}^{m \times q}$ and $C \in \mathbb{F}^{p \times q}$ be given. The following statements are equivalent:

(1) There exist $X \in \mathbb{F}^{n \times q}$, $Y \in \mathbb{F}^{p \times m}$ such that $AX + YB = C$.

(2) $\operatorname{Rank} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$.

(3) $C \ker B \subseteq \operatorname{im} A$.

Proof.

(1) $\Leftrightarrow$ (2). See Roth [5].

(1) $\Rightarrow$ (3). Take any vector $u \in \ker B$. Then $Cu = (AX + YB)u = AXu$, from which it is clear that $C \ker B \subseteq \operatorname{im} A$.

(3) $\Rightarrow$ (1). Let $\{v_1, v_2, \ldots, v_q\}$ be a basis of $\mathbb{F}^q$ such that $\{v_1, v_2, \ldots, v_t\}$ with $t \leq q$ is a basis of $\ker B$. Note that $\{Bv_{t+1}, Bv_{t+2}, \ldots, Bv_q\}$ is a set of linearly independent vectors in $\mathbb{F}^m$. Extend this set with vectors $b_i \in \mathbb{F}^m$ where $i = 1, 2, \ldots, m-q+t$ such that the set $\{Bv_{t+1}, Bv_{t+2}, \ldots, Bv_q\,; b_1, b_2, \ldots, b_{m-q+t}\}$ is a basis of $\mathbb{F}^m$. By $C \ker B \subseteq \operatorname{im} A$ it follows that for every $i = 1, \ldots, t$ there exists a vector $w_i \in \mathbb{F}^n$ such that $Cv_i = Aw_i$.

For $i = t+1, \dots, q$ choose vectors $w_i \in \mathbb{F}^n$ arbitrary. Now define the matrix $Y \in \mathbb{F}^{p \times m}$ by

$$Y \{Bv_{t+1}, Bv_{t+2}, \dots, Bv_q ; b_1, b_2, \dots, b_{m-q+t}\} =$$

$$= \{Cv_{t+1} - Aw_{t+1}, Cv_{t+2} - Aw_{t+2}, \dots, Cv_q - Aw_q ; z_1, z_2, \dots, z_{m-q+t}\}$$

where for $i = 1,2, \dots, m-q+t$ the vectors $z_i \in \mathbb{F}^p$ are chosen arbitrary. Finally, define the matrix $X \in \mathbb{F}^{n \times q}$ by $X \{v_1, v_2, \dots, v_q\} = \{w_1, w_2, \dots, w_q\}$. Now it follows that $(AX + YB) \{v_1, v_2, \dots, v_q\} = C \{v_1, v_2, \dots, v_q\}$. from which it is immediate that $AX + YB = C$. □

We are now able to state the main result of this section. The result provides new necessary and sufficient conditions for the existence of a common solution to a pair of linear matrix equations over a field $\mathbb{F}$. In our opinion the presented conditions are relatively simpler than the conditions found by Mitra [4], since the latter involve complicated expressions with generalized inverses of matrices.

Theorem 3.3.

Let, for $i = 1,2$, $A_i \in \mathbb{F}^{p_i \times n}$, $B_i \in \mathbb{F}^{m \times q_i}$ and $C_i \in \mathbb{F}^{p_i \times q}$ be given. The following statements are equivalent:

(1) There exists $X \in \mathbb{F}^{n \times m}$ such that $A_1 X B_1 = C_1$ and $A_2 X B_2 = C_2$.

(2) For $i = 1,2$: $\operatorname{rank} A_i = \operatorname{rank} [A_i, C_i]$, $\operatorname{rank} B_i = \operatorname{rank} \begin{bmatrix} B_i \\ C_i \end{bmatrix}$,

and

$$\operatorname{rank} \begin{bmatrix} A_1 & 0 & 0 \\ A_2 & 0 & 0 \\ 0 & B_1 & B_2 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A_1 & C_1 & 0 \\ A_2 & 0 & -C_2 \\ 0 & B_1 & B_2 \end{bmatrix}.$$

(3) For $i = 1,2$: $\operatorname{im} C_i \subseteq \operatorname{im} A_i$, $\ker B_i \subseteq \ker C_i$,

and

$$\begin{bmatrix} C_1 & 0 \\ 0 & -C_2 \end{bmatrix} \ker [B_1, B_2] \subseteq \operatorname{im} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.$$

Proof.

(2) ⇔ (3). Follows from Lemma 3.1 and 3.2.

(1) ⇒ (3). From Lemma 3.1 it follows that for $i = 1,2$: $\operatorname{im} C_i \subseteq \operatorname{im} A_i$, $\ker B_i \subseteq \ker C_i$.

Take any vector $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \ker [B_1, B_2]$ where $u_1 \in \mathbb{F}^{q_1}$ and $u_2 \in \mathbb{F}^{q_2}$. Then $B_1 u_1 = -B_2 u_2 =: b$. Observe that

$$\begin{bmatrix} C_1 & 0 \\ 0 & -C_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} C_1 u_1 \\ -C_2 u_2 \end{bmatrix} = \begin{bmatrix} A_1 X B_1 u_1 \\ -A_2 X B_2 u_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} Xb ,$$

from which it is immediate that

$$\begin{bmatrix} C_1 & 0 \\ 0 & -C_2 \end{bmatrix} \ker [B_1, B_2] \subseteq \operatorname{im} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.$$

(3) $\Rightarrow$ (1). Let $\bar{B}_0 \in \mathbb{F}^{m \times r_0}$ with $\operatorname{rank} \bar{B}_0 = r_0$ be such that $\operatorname{im} \bar{B}_0 = \operatorname{im} B_1 \cap \operatorname{im} B_2$. Determine $\bar{B}_1 \in \mathbb{F}^{m \times r_1}$ and $\bar{B}_2 \in \mathbb{F}^{m \times r_2}$ such that $\operatorname{im} B_1 = \operatorname{im} [\bar{B}_0, \bar{B}_1]$, $\operatorname{im} B_2 = [\bar{B}_0, \bar{B}_2]$, $\operatorname{rank} [\bar{B}_0, \bar{B}_1] = r_0 + r_1$ and $\operatorname{rank} [\bar{B}_0, \bar{B}_2] = r_0 + r_2$. Note that $\operatorname{rank} [\bar{B}_0, \bar{B}_1, \bar{B}_2] = r_0 + r_1 + r_2$. Hence, $[\bar{B}_0, \bar{B}_1, \bar{B}_2]$ has a left inverse.

So there exist $W_1 \in \mathbb{F}^{q_1 \times q_1}$ and $W_2 \in \mathbb{F}^{q_2 \times q_2}$ with $\operatorname{rank} W_1 = q_1$ and $\operatorname{rank} W_2 = q_2$ such that $B_1 W_1 = [\bar{B}_0, \bar{B}_1, 0]$ with $0 \in \mathbb{F}^{m \times (q_1 - r_0 - r_1)}$ and $B_2 W_2 = [\bar{B}_0, \bar{B}_2, 0]$ with $0 \in \mathbb{F}^{m \times (q_2 - r_0 - r_2)}$. Partition $C_1 W_1$ and $C_2 W_2$ in the corresponding way.

Hence $C_1 W_1 = [C_1', \bar{C}_1, \hat{C}_1]$, $C_2 W_2 = [C_2', \bar{C}_2, \hat{C}_2]$ with $C_1' \in \mathbb{F}^{p_1 \times r_0}$, $C_2' \in \mathbb{F}^{p_2 \times r_0}$, $\bar{C}_1 \in \mathbb{F}^{p_1 \times r_1}$, $\bar{C}_2 \in \mathbb{F}^{p_2 \times r_2}$, $\hat{C}_1 \in \mathbb{F}^{p_1 \times (q_1 - r_0 - r_1)}$ and $\hat{C}_2 \in \mathbb{F}^{p_2 \times (q_2 - r_0 - r_2)}$.

Because $\ker B_1 \subseteq \ker C_1$ and $\ker B_2 \subseteq \ker C_2$ it is obvious that $\ker B_1 W_1 \subseteq \ker C_1 W_1$ and $\ker B_2 W_2 \subseteq \ker C_2 W_2$. Also it follows that

$$\begin{bmatrix} C_1 & 0 \\ 0 & -C_2 \end{bmatrix} \ker [B_1, B_2] = \begin{bmatrix} C_1 W_1 & 0 \\ 0 & -C_2 W_2 \end{bmatrix} \ker [B_1 W_1, B_2 W_2] =$$

$$= \begin{bmatrix} C_1' & \bar{C}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -C_2' & -\bar{C}_2 & 0 \end{bmatrix} \ker [\bar{B}_0, \bar{B}_1, 0, \bar{B}_0, \bar{B}_2, 0] =$$

$$= \begin{bmatrix} C_1' & \bar{C}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{C}_2 & -C_2' & 0 & 0 \end{bmatrix} \ker [\bar{B}_0, \bar{B}_1, \bar{B}_2, \bar{B}_0, 0, 0] =$$

$$= \begin{bmatrix} C_1' & \bar{C}_1 & 0 & -C_1' & 0 & 0 \\ 0 & 0 & -\bar{C}_2 & -C_2' & 0 & 0 \end{bmatrix} \ker [\bar{B}_0, \bar{B}_1, \bar{B}_2, 0, 0, 0] = \operatorname{im} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix}$$

where the last equality is due to the fact that $[\bar{B}_0, \bar{B}_1, \bar{B}_2]$ has full column rank. Since $\begin{bmatrix} C_1 & 0 \\ 0 & -C_2 \end{bmatrix} \ker [B_1, B_2] \subseteq \operatorname{im} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ it is clear that $\operatorname{im} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$. Because $\bar{B}_0$ has full column rank it is also obvious that $\ker \bar{B}_0 \subseteq \ker \begin{bmatrix} C_1' \\ C_2' \end{bmatrix}$. By Lemma 3.1 it is clear that there exists $X_0 \in \mathbb{F}^{n \times m}$ such that $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} X_0 \bar{B}_0 = \begin{bmatrix} C_1' \\ C_2' \end{bmatrix}$.

Since $\operatorname{im} C_1 \subseteq \operatorname{im} A_1$ and $\operatorname{im} C_2 \subseteq \operatorname{im} A_2$ also $\operatorname{im} C_1 W_1 \subseteq \operatorname{im} A_1$ and $\operatorname{im} C_2 W_2 \subseteq \operatorname{im} A_2$. By Lemma 3.1 it follows that there exist $X_1, X_2 \in \mathbb{F}^{n \times m}$ such that $A_1 X_1 B_1 W_1 = C_1 W_1$ and $A_2 X_2 B_2 W_2 = C_2 W_2$.

Define $X \in \mathbb{F}^{n \times m}$ by $X [\bar{B}_0, \bar{B}_1, \bar{B}_2] = [X_0 \bar{B}_0, X_1 \bar{B}_1, X_2 \bar{B}_2]$.

Now for $i = 1,2$ we have $A_i X B_i W_i = A_i X [\bar{B}_0, \bar{B}_i, 0] = A_i [X_0 \bar{B}_0, X_i \bar{B}_i, 0] = [C_i', \bar{C}_i, 0] = C_i W_i$. Hence, since W_1 and W_2 are invertible, X is such that $A_1 X B_1 = C_1$ and $A_2 X B_2 = C_2$. □

4. Main result

In this section we shall state the main result of this paper. The result provides necessary and sufficient conditions for the solvability of the main problem of this paper in terms of some rank tests. Note that we use the notion of rank as introduced in Section 3 with $\mathbb{F} = \mathbb{R}(s)$.

Theorem 4.1.

Let $p \in \{1,2,\infty\}$. The following statements are equivalent:

(1) $(\text{ANICPM})_p$ is solvable.

(2) For $(i,j) = (1,2),(2,1)$: $\operatorname{rank} H_i(sI-A)^{-1}B = \operatorname{rank} H_i(sI-A)^{-1}[B, G_j]$,

$$\operatorname{rank} C(sI-A)^{-1}G_j = \operatorname{rank} \begin{bmatrix} C \\ H_i \end{bmatrix} (sI-A)^{-1}G_j ,$$

and

$$\operatorname{rank} \begin{bmatrix} H_1(sI-A)^{-1}B & 0 & 0 \\ H_2(sI-A)^{-1}B & 0 & 0 \\ 0 & C(sI-A)^{-1}G_2 & C(sI-A)^{-1}G_1 \end{bmatrix} =$$

$$= \operatorname{rank} \begin{bmatrix} H_1(sI-A)^{-1}B & H_1(sI-A)^{-1}G_2 & 0 \\ H_2(sI-A)^{-1}B & 0 & -H_2(sI-A)^{-1}G_1 \\ 0 & C(sI-A)^{-1}G_2 & C(sI-A)^{-1}G_1 \end{bmatrix} .$$

Proof. Follows immediately from Proposition 2.2, Theorem 2.5 and Theorem 3.3. □

5. Remarks and conclusions

5.1. Let the system (1) be given and assume that for some $p \in \{1,2,\infty\}$ $(\text{ANICPM})_p$ is solvable. The following scheme describes how a compensator (2) can be obtained that achieves almost non interaction with a prescribed accuracy.

Let $\varepsilon > 0$ express the desired degree of almost non interaction.
Compute $X(s) \in \mathbb{R}_0^{m \times p}(s)$ such that

$$\| H_1(sI-A)^{-1} B\, X(s)\, C(sI-A)^{-1} G_2 + H_1(sI-A)^{-1} G_2 \|_p \le \varepsilon$$

and

$$\| H_2(sI-A)^{-1} B\, X(s)\, C(sI-A)^{-1} G_1 + H_2(sI-A)^{-1} G_1 \|_p \le \varepsilon .$$

The latter can be done using Kronecker products and the procedure as described in Willems [10] (Section 6, Comments 1).
Set $F(s) = X(s)(I + C(sI-A)^{-1} B\, X(s))^{-1}$.
Now any realization of $F(s)$ yields a compensator (2) that achieves the desired degree of almost non interaction.

5.2. Denote system (1) by its matrices, $\Sigma = (A, B, C, G_1, G_2, H_1, H_2)$ and define the dual system $\Sigma' := (A', B', C', G_1', G_2', H_1', H_2')$ with $A' := A^{\mathsf{T}}, B' := C^{\mathsf{T}}$, $C' := B^{\mathsf{T}}$, $G_i' := H_i^{\mathsf{T}}$, $H_i' := G_i^{\mathsf{T}}$, where $^{\mathsf{T}}$ stands for transpose. Now the following, expressing the selfduality of $(\text{ANICPM})_p$, is immediate from the problem formulation as well as from Theorem 4.1.
Let $p \in \{1,2,\infty\}$ be given. Then $(\text{ANICPM})_p$ formulated for Σ is solvable if and only if $(\text{ANICPM})_p$ formulated for Σ' is solvable.

5.3. Consider the linear system

$$\dot{x}(t) = A\, x(t) + B\, u(t) + G\, v(t); \quad y(t) = C\, x(t); \quad z(t) = H\, x(t)$$

with state space $\mathbb{R}^n$.
Define the following subspaces in $\mathbb{R}^n$:

$$V_b^*(\ker H) := \{x_0 \in \mathbb{R}^n \mid \text{for all } \varepsilon > 0 \text{ there exist } \xi(s) \in \mathbb{R}_+^n(s) \text{ and}$$

$$w(s) \in \mathbb{R}_+^m(s) \text{ such that } x_0 = (A - sI)\xi(s) + B\, w(s) \text{ with } \| H\, \xi(s) \|_p \le \varepsilon\} .$$

(To indicate that $V_b^*(\ker H)$ is computed relatively A and B we sometimes denote $V_b^*(A, B; \ker H) := V_b^*(\ker H)$.)

$$S_b^*(\operatorname{im} G) := (V_b^*(A^{\mathsf{T}}, C^{\mathsf{T}}; \ker G^{\mathsf{T}}))^{\perp} .$$

Here $^{\perp}$ denotes the orthogonal complement.
It can be shown that the subspaces $V_b^*(\ker H)$ and $S_b^*(\operatorname{im} G)$ can actually be computed (cf. Willems [9], Trentelman [6]). Also it is proved in Willems [10] that the following statements are equivalent:

(1) $\operatorname{im} G \subseteq V_b^*(\ker H)$.

(2) There exists $X(s) \in \mathbb{R}^{m \times q}(s)$ such that $H(sI - A)^{-1} B\, X(s) = H(sI - A)^{-1} G$.

(3) $\operatorname{Rank} H(sI - A)^{-1} B = \operatorname{rank} H(sI - A)^{-1} [B \quad G]$.

Analogous results can be derived with respect to $S_b^*(\operatorname{im} G)$ and $\ker H$. With these results the following statement is obviously equivalent to either one of the statements of Theorem 4.1.

For $(i,j) = (1,2),(2,1)$: $\operatorname{im} G_j \subseteq V_b^*(\ker H_i)$, $S_b^*(\operatorname{im} G_j) \subseteq \ker H_i$,
and

$$\operatorname{rank} \begin{bmatrix} H_1(sI-A)^{-1}B & 0 & 0 \\ H_2(sI-A)^{-1}B & 0 & 0 \\ 0 & C(sI-A)^{-1}G_2 & C(sI-A)^{-1}G_1 \end{bmatrix} =$$

$$= \operatorname{rank} \begin{bmatrix} H_1(sI-A)^{-1}B & H_1(sI-A)^{-1}G_2 & 0 \\ H_2(sI-A)^{-1}B & 0 & -H_2(sI-A)^{-1}G_1 \\ 0 & C(sI-A)^{-1}G_2 & C(sI-A)^{-1}G_1 \end{bmatrix}.$$

5.4. Consider the closed loop system (3), obtained by the interconnection of system (1) and the measurement feedback compensator (2). From the frequency domain description it is clear that $H_{1,e}(sI - A_e)^{-1} G_{2,e} = 0$ and $H_{2,e}(sI - A_e)^{-1} G_{1,e} = 0$ if and only if there exists a proper rational matrix $X(s)$ such that

$$H_1(sI-A)^{-1} B\, X(s) C(sI-A)^{-1} G_2 + H_1(sI-A)^{-1} G_2 = 0$$

and

$$H_2(sI-A)^{-1} B\, X(s) C(sI-A)^{-1} G_1 + H_2(sI-A)^{-1} G_1 = 0 .$$

Observe that all transfer matrices appearing in the two equations are proper rational matrices (in fact, they are strictly proper rational matrices). Furthermore, note that the set of proper rational functions $\mathbb{R}_0(s)$ with the usual addition and multiplication forms a principal ideal domain.

Consequently, if we have a result, generalizing Theorem 3.3, concerning the existence of a common solution to a pair of linear matrix equations over a principal ideal domain, then the problem of *exact* non interacting control by measurement feedback for systems as considered in this paper can be solved.

5.5. While in this paper we assume the system Σ to have two exogenous inputs and two exogenous outputs, it is interesting to consider systems that have k exogenous inputs and k exogenous outputs where k is some integer larger than or equal to two. For these systems it is possible to formulate the problem of non interacting control by measurement feedback, both the exact as well as the almost version. As may be expected, the solvability of these non interacting control problems appears to be intimately related to the existence of a common solution to k linear matrix equations of type (RME)' over the field $\mathbb{R}(s)$ for the almost version and over the principal ideal domain $\mathbb{R}_0(s)$ for the exact version.

The latter together with Remark 5.4 clearly motivates the search for conditions in the spirit of Theorem 3.3 for the existence of a common solution to k linear matrix equations of type (RME)' both over the field $\mathbb{R}(s)$ as well as over the principal ideal domain $\mathbb{R}_0(s)$.

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