On some multiple integrals involving determinants

ON SOME MULTIPLE INTEGRALS INVOLVING DETERMINANTS

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1. Introduction. The following formula is wellknown [1]

$$\int \cdots \int_{a \leqslant x_1 < \ldots < x_n \leqslant b} \det_{1 \leqslant i,j \leqslant n} \phi_i(x_j) . \det_{1 \leqslant i,j \leqslant n} \psi_i(x_j)\, dx_1 \ldots dx_n$$

$$= \det_{1 \leqslant i,j \leqslant n} \int_a^b \phi_i(x)\, \psi_j(x)\, dx. \tag{1.1}$$

It is analogous to the Lagrange formula for det MM^T, where M is a rectangular matrix and M^T its transpose. It may be remarked that (1.1) is easily proved in two steps : (i) if we replace the range $a \leqslant x_1 < \ldots < x_n \leqslant b$ by $a \leqslant x_1 \leqslant b, \ldots, a \leqslant x_n \leqslant b$, then the integral is multiplied by $n!$, and after that, (ii) the first determinant det $\phi_i(x_j)$ can be replaced by $n!$ times its main term $\phi_1(x_1) \ldots \phi_n(x_n)$. It seems that, as yet, no attempt has been made to deal with similar integrals (or sums) containing only one determinant factor:

$$\Omega = \int \cdots \int_{a \leqslant x_1 < \ldots < x_n \leqslant b} \det_{1 \leqslant i,j \leqslant n} \phi_i(x_j)\, dx_1 \ldots dx_n. \tag{1 2}$$

The author was led to consider these by an integral proposed by G. Szekeres (see §5 below). The result for (1.2) is similar to (1.1), but there are typical differences : (i) not Ω itself, but its square will be expressed as a determinant ; Ω itself can be expressed as a Pfaffian form (a Pfaffian form of even order is a square root of a skew determinant); (ii) there is a difference between the cases "n odd" and "n even".

For simplicity we write (a, b) for the range of each of the variables $x_1, \ldots, x_n$. However, our methods and results remain valid if the integration over (a, b) is replaced by integration over any ordered measure space. In particular we may replace the multiple integration over the range $a \leqslant x_1 < \ldots < x_n \leqslant b$ by $-\infty < x_1 < \ldots < x_n < \infty$, or by multiple summation over the set of n-tuples $x_1, \ldots, x_n$ defined by $x_1 < x_2 < \ldots < x_n$, $x_1 \in D, \ldots, x_n \in D$, where D is any ordered finite or countable set. It has to be understood that the functions $\phi_i(x)$ are absolutely integrable over the measure space under consideration.

The integral (1.2) is closely related to the *signature function* $E(x_1, \ldots, x_n)$, defined as follows. Let S be any ordered set. Then, if $x_1 \in S, \ldots, x_n \in S$, we define E by the properties (i) $E(x_1, \ldots, x_n) = 1$ if $x_1 < x_2 < \ldots < x_n$, and (ii) $E(x_1, \ldots, x_n)$ is alternating in $x_1, \ldots, x_n$. In other words

$$E(x_1, \ldots, x_n) = \Pi_{1 \leqslant i < j \leqslant n} \operatorname{sgn}(x_j - x_i).$$

In particular, $E(x_1, \ldots, x_n) = 0$ if two of the x's are equal. In the case $n = 1$ we have $E(x_1) = 1$ for all $x_1 \in S$.

Szekeres' integral is one of the coefficients in a multiple Fourier expansion of $E(x_1, \ldots, x_n)$ in the case that S is the interval $(0, 1)$. We shall also consider expansions of E or functions generated by E in other cases. If S is the interval $(-\infty, \infty)$ we consider the multiple Fourier integral (§8). If S is the interval $(0, \infty)$ we evaluate the Laplace transform (§9). If S is the countable set $(0, 1, 2, \ldots)$, we evaluate the multiple power series generated by E (§10). If S is the countable set $(\ldots, -1, 0, 1, \ldots)$, we express E as a coefficient of a multiple Fourier series (§8). If S is a finite set, we develop in terms of roots of unity (§8).

The Fourier expansions of E suggest another general formula for a multiple integral involving a determinant. That one will be given in §7.

2. The function $E(x_1, \ldots, x_n)$. In order to evaluate multiple sums and integrals involving $E(x_1 \ldots, x_n)$, we express this function of n variables in terms of functions of two variables, viz. the function $E(x_1, x_2) = \operatorname{sgn}(x_2 - x_1)$. One easily finds

$$E(x_1, x_2, x_3) = E(x_1, x_2) + E(x_2, x_3) + E(x_3, x_1). \qquad (2.1)$$

$$E(x_1, x_2, x_3, x_4) = E(x_1, x_2)\, E(x_3, x_4) + E(x_2, x_3)\, E(x_1, x_4) + \\ + E(x_3, x_1)\, E(x_2, x_4), \quad (2.2)$$

and the following general formula presents itself : if m is the integral part of $\frac{1}{2}n$, we shall prove

$$E(x_1, \dots, x_n) = \frac{1}{2^m m!} \sum_{j_1=1}^{n} \dots \sum_{j_n=1}^{n} E(j_1, \dots, j_n)\, E(x_{j_1}, x_{j_2}) \times \\ \times E(x_{j_3}, x_{j_4}) \dots E(x_{j_{2m-1}}, x_{j_{2m}}). \quad (2.3)$$

Here $E(j_1, \dots, j_n)$ is simply the signature of the permutation $(j_1, \dots, j_n)$ (it vanishes if $(j_1, \dots, j_n)$ is not a permutation of $(1, \dots, n)$). The formula can be proved in several ways, e.g. by induction. However, as it resembles the so-called Pfaffian form, it is only natural to study it from that point of view. In fact the right-hand side of (2.3) equals the Pfaffian of the matrix X, where $x_{ij} = \operatorname{sgn}(x_j - x_i)$ (see (3.7)).

We remark that the formula for $2m - 1$ variables immediately follows from the one for $2m$ variables, by making $x_{2m} \to \infty$, as we observe on comparing (2.1) with (2.2).

Moreover, we notice that both sides of (2.3) are alternating functions of $x_1, \dots, x_n$. Furthermore, the values of the E-functions do not depend on the actual values of $x_1, \dots, x_n$, but only on their order of succession. Hence we may restrict ourselves to the case $x_1 = 1, x_2 = 2, \dots, x_n = n$, n even. The proof will be given in §3.

3. Pfaffians. As to the classical definition (3.1) and the formulas (3.4), (3.5) we may refer to H. Weyl [7, p. 165]. Nevertheless the presentation given below is complete in itself. Usually Pfaffians are defined only for skew matrices of even order ; but it has some advantages also to consider odd orders.

If $A = (a_{ij})$ is any skew $n \times n$ matrix ($a_{ij} = -a_{ji}$, $a_{ii} = 0$), then its Pfaffian Pf (A) is defined by

$$\operatorname{Pf}(A) = \frac{1}{2^m m!} \sum_{j_1=1}^{n} \dots \sum_{j_n=1}^{n} E(j_1, \dots, j_n)\, a_{j_1 j_2} a_{j_3 j_4} \dots a_{j_{2m-1} j_{2m}}, \quad (3.1)$$

where $m = [\frac{1}{2} n]$.

Now assume n to be even, $n = 2m$. We observe that each non-zero term of (3.1) involves n distinct j's, divided into m pairs. There are $2^m m\,!$ sets $j_1, \dots, j_n$ which give the same dissection into pairs (both order in the pairs and order of the pairs are disregarded). Each of these $2^m m\,!$ terms corresponding to one and the same dissection into pairs has the same value. A consequence to be used in the sequel is as follows. If in (3.1) we omit the summation over j_n, and fix the value of j_n arbitrarily, $j_n = \nu$, then the sum does not depend on ν (and so equals $\mathrm{Pf}(A)/2m$). For, each dissection into pairs gives $2^m m\,!/2m$ terms with $j_n = \nu$.

Let C denote the standard non-singular $n \times n$ skew matrix defined by $c_{12} = c_{34} = \dots = c_{n-1,n} = 1$, $c_{21} = c_{43} = \dots = c_{n,n-1} = -1$, $c_{ij} = 0$ otherwise. It is easily seen that $\mathrm{Pf}(C) = 1$.

Let B denote the matrix

$$B = (b_{ij}),\ b_{ij} = \mathrm{sgn}\,(j - i), \quad (1 \leqslant i, j \leqslant n). \tag{3.2}$$

Then clearly the proof of (2.3), in view of the simplification explained at the end of §2, reduces to the proof of

$$\mathrm{Pf}(B) = 1. \tag{3.3}$$

It is well known that, if P is an arbitrary $n \times n$ matrix, and P^T its transpose, then

$$\mathrm{Pf}(PAP^T) = \mathrm{Pf}(A).\det P, \quad (n \text{ even}). \tag{3.4}$$

(A proof of this theorem is contained in §4 as a special case.)

We have $\mathrm{Pf}(C) = 1$, $\det C = 1$, and any non-singular skew matrix A can be written as $A = PCP^T$. Therefore (3.4) generalizes the famous theorem

$$(\mathrm{Pf}(A)\,)^2 = \det A, \quad (n \text{ even}). \tag{3.5}$$

(If A is singular, we have to use a standard matrix of rank $< n$.)

In order to prove (3.3), we apply the standard procedure for the reduction of the bilinear from $\Sigma\Sigma\, b_{ij} x_i y_j$. We see that $B = QCQ^T$, where Q is described by $q_{ij} = 1$ if $i > j$, $q_{12} = q_{34} = \dots = q_{n-1,n} = -1$,

$q_{ij} = 0$ otherwise. The determinant of Q is easily seen to be $+1$, and therefore $\mathrm{Pf}(B) = \mathrm{Pf}(C)$. $\det Q = 1$. This proves (3.3) and (2.3).

If n is odd, then (3.4) and (3.5) do not hold. Nevertheless (3.3) remains true ; (3.3) is still equivalent to (2.3), and the truth of (2.3) for n odd follows from the case n even (see §2). There is also a more general aspect. Let K be a skew $(n-1) \times (n-1)$ matrix, where n is odd. The $n \times n$ matrix K^+ arises from K by adding an nth column consisting of $n-1$ elements 1, an nth row consisting of $n-1$ elements -1, whereas the element $k^+_{nn} = 0$. Now we can show

$$\mathrm{Pf}(K^+) = \mathrm{Pf}(K). \tag{3.6}$$

In order to prove this, we write $K^+ = A$, and we express $\mathrm{Pf}(K^+)$ by (3.1). As n is even, we may fix the value of $j_n : j_n = n$, if we only replace the factor $(2^m m!)^{-1}$ by $(2^{m-1}(m-1)!)^{-1}$ (see the remark made after (3.1)). As $j_n = n$, the last factor $a_{j_{2m-1},n}$ in each term equals 1, by the definition of K^+. Omitting this factor, we arrive at the expression for $\mathrm{Pf}(K)$. This proves (3.6).

In the trivial case $n = 1$ there is only one skew matrix, viz. the zero matrix. In order to maintain (3.4) it has to be understood that its Pfaffian is 1. This corresponds to the fact that $E(x_1) = 1$ for all x_1.

It may be remarked that $\mathrm{Pf}(A)$ can be defined also by (3.1) if A is not skew. It is, however, hardly worth while to consider this possibility, as $\mathrm{Pf}(A) = \mathrm{Pf}(\frac{1}{2}(A - A^T))$, and $\frac{1}{2}(A - A^T)$ is skew for all A.

Let $x_1, \dots, x_n$ be elements of an ordered set. Then it follows from (2.3) and (3.1) that if $X = (x_{ij})$ is the matrix given by $x_{ij} = x_j - x_i$, then

$$\mathrm{Pf}(X) = E(x_1, \dots, x_n). \tag{3.7}$$

This holds both for n odd and n even. If n is odd, we can generalize (3.7). Let $p_1, \dots, p_n$ be real or complex numbers, and let $Y = (y_{ij})$ be defined by $y_{ij} = \{ \operatorname{sgn}(x_j - x_i) \} + p_j - p_i$. Then again

$$\mathrm{Pf}(Y) = E(x_1, \dots, x_n), \qquad (n \text{ odd}). \tag{3.8}$$

This can be proved as follows: let P be the $(n+1) \times (n+1)$ matrix arising from the unit matrix on replacing the $(n+1)$-th column $0, \dots, 0, 1$ by $p_1, \dots, p_n, 1$. Then we have $Y^+ = PX^+P^T$. Hence, by (3.4), (3,6) and (3.7),

$$\mathrm{Pf}(Y) = \mathrm{Pf}(X^+) = \mathrm{Pf}(X)\, E(x_1, \dots, x_n).$$

4. The integral Ω and a generalization. If in (1.2) we multiply the integrand by $E(x_1, \dots, x_n)$ then its value is obviously unaltered. After this operation the integrand becomes symmetric in $x_1, \dots, x_n$, so that it may be written as

$$\Omega = (n\,!)^{-1} \int_a^b \cdots \int_a^b E(x_1, \dots, x_n) \det \phi_i\,(x_j)\, dx_1 \dots dx_n. \quad (4.1)$$

(Points where some of the x's are equal can be disregarded since $E = 0$ in such points.) The determinant in (4.1) consists of $n!$ terms, and each term gives the same contribution to the integral. Therefore

$$\Omega = \int_a^b \cdots \int_a^b E(x_1, \dots, x_n)\, \phi_1\,(x_1) \dots \phi_n(x_n)\, dx_1 \dots dx_n. \quad (4.2)$$

Now if we expand $E(x_1, \dots, x_n)$ according to (2.3), then the integral reduces to a sum of products of m double integrals and, if n is odd, one further single integral. The result can be written as the Pfaffian of a matrix of order n if n is even, and of a matrix of order $n+1$ if n is odd. Define the matrix $A = (a_{ij})$ by

$$a_{ij} = \int_a^b \int_a^b \phi_i(x)\, \phi_j(y) \operatorname{sgn}(y - x)\, dx\, dy, \quad (i, j = 1, \dots, 2m), \quad (4.3)$$

and, if n is odd $(n = 2m+1)$; moreover

$$a_{i,2m+2} = -a_{2m+2,i} = \int_a^b \phi_i(x)\, dx, \quad (i = 1, \dots, 2m+1);\ a_{2m+2,2m+2} = 0. \quad (4.4)$$

Then we immediately find $\Omega = \mathrm{Pf}(A)$ if n is even, and $\Omega = \mathrm{Pf}(A^+)$ if n is odd. Therefore, by (3.6) we have in both cases

$$\Omega = \mathrm{Pf}(A). \quad (4.5)$$

There is an obvious generalization. Let $s(x, y)$ satisfy $s(x, y) = -s(y, x)$, so that the matrix $S = S(x_1, \dots, x_n)$ defined by $s_{ij} = s(x_i, x_j)$ $(i, j = 1, \dots, n)$ is skew. Now define Ω_s by

$$\Omega_s = \int_{a \leqslant x_1 < \dots < x_n \leqslant b} \cdots \int \mathrm{Pf}(S(x_1, \dots, x_n)) \det_{1 \leqslant i,j \leqslant n} \phi_i(x_j)\, dx_1 \dots dx_n. \quad (4.6)$$

If we follow the same procedure as above, we obtain

$$\Omega_s = (n\,!)^{-1} \int_a^b \cdots \int_a^b \mathrm{Pf}(S(x_1, \dots, x_n)) \det \phi_i(x_j)\, dx_1 \dots dx_n$$
$$= \int_a^b \cdots \int_a^b \mathrm{Pf}(S(x_1, \dots, x_n))\, \phi_1(x_1) \dots \phi_n(x_n)\, dx_1 \dots dx_n.$$

Now expand the Pfaffian according to its definition (3.1), and again reduce the integral to a sum of products of double integrals (and one further single integral, if n is odd). We then find

$$\Omega_s = \mathrm{Pf}(A_s), \quad (4.7)$$

where A_s is defined by

$$a_{ij} = \int_a^b \int_a^b \phi_i(x)\, \phi_j(y)\, s(x,y)\, dx\, dy, \quad (i, j = 1, \dots, 2m), \quad (4.8)$$

whereas, if n is odd $(n = 2m + 1)$, formula (4.4) completes the definition.

If $s(x, y) = \mathrm{sgn}\,(y - x)$, (4.7) specializes into (4.5). It is also interesting to note that, if n is even, (3.4) is a special case of (4.6). For, replace the integration over (a, b) by summation over a set of n elements $1, \dots, n$. Now in (4.6) we have a multiple sum consisting of just one term $\mathrm{Pf}(S) \det \Phi$, where S is the matrix $s(i, j)$, and Φ is the square matrix $\phi_i(j)$. On the other hand, A_s becomes, by (4.8), $A_s = \Phi\, S\, \Phi^T$; hence we have $\mathrm{Pf}(\Phi\, S\, \Phi^T) = \mathrm{Pf}(S) \det \Phi$, that is (3.4). If n is odd we can see, in a similar manner, that (3.6) is a special case of (4.7) : if we take for Φ the $n \times n$ unit matrix, then A_s becomes S^+.

5. Szekeres' integral. In a preceding paper in this Journal, G. Szekeres [6] expressed the volume of the space of unitary symmetric matrices in terms of the following integral

$$J_n = \pi^n \int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} dx_1 \prod_{1 \leqslant \mu < \nu \leqslant n} 2 \sin \pi (x_\nu - x_\mu). \quad (5.1)$$

(We applied a trivial transformation $\phi_1 = 2\pi x_n, \ldots, \phi_n = 2\pi x_1$). We shall find that

$$J_n = \pi^{\frac{1}{2}(n+1)} / \Gamma(\tfrac{1}{2}(n+1)), \quad (n = 2, 3, 4, \ldots). \quad (5.2)$$

The product in (5.1) equals a Vandermonde determinant

$$(-i)^{\frac{1}{2}n(n-1)} \det_{1 \leqslant \lambda, \mu \leqslant n} \exp\{2\pi i (\lambda - 1 - \tfrac{1}{2}(n-1)) x_\mu\},$$

and therefore (cf. (4.2)) we have

$$J_n = \pi^n (-i)^{\frac{1}{2}n(n-1)} \times$$
$$\times H\{-\tfrac{1}{2}(n-1), -\tfrac{1}{2}(n-3), \ldots, \tfrac{1}{2}(n-3), \tfrac{1}{2}(n-1)\}, \quad (5.3)$$

where

$$H(\lambda_1, \ldots, \lambda_n) = \int_0^1 \cdots \int_0^1 E(x_1, \ldots, x_n) \times$$
$$\times \prod_{j=1}^n \exp(2\pi i \lambda_j x_j)\, dx_1 \ldots dx_n. \quad (5.4)$$

We shall evaluate (5.4) assuming that $\lambda_1, \ldots, \lambda_n$ are integers if n is odd, and half-odd integers if n is even. Hence, in both cases, $\lambda_1 + \frac{1}{2}(n-1), \ldots, \lambda_n + \frac{1}{2}(n-1)$ are integers. We shall not explicitly use (4.5), but we use the procedure by which (4.5) was obtained.

We shall need the following result (cf. (4.3)):

$$f(\lambda, \mu) = \int_0^1 \int_0^1 e^{2\pi i \lambda x + 2\pi i \mu y} \operatorname{sgn}(y - x)\, dx\, dy = (\pi i \lambda)^{-1} \delta_{\lambda - \mu},$$

(δ is the Kronecker symbol) in the following cases: (i) if both λ and μ are integers, and $\lambda \neq 0$, $\mu \neq 0$, and (ii) if both 2λ and 2μ are odd integers.

As E is an alternating function of $x_1, \ldots, x_n$, it follows from (5.4) that $H(\lambda_1, \ldots, \lambda_n)$ is an alternating function of the λ's. So we may and do assume that $\lambda_1, \ldots, \lambda_n$ are distinct (otherwise $H = 0$).

Substituting (2.3) into (5.4), and integrating, we obtain

$$H(\lambda_1, \dots, \lambda_n) = \frac{1}{2^m m!} \sum_{j_1=1}^{n} \dots \sum_{j_n=1}^{n} E(j_1, \dots, j_n) f(\lambda_{j_1}, \lambda_{j_2}) \dots \\ \dots f(\lambda_{j_{2m-1}}, \lambda_{j_{2m}})\, g(\lambda_{j_n}), \qquad (5.5)$$

where $g(\lambda) = 1$ if n is even (then the factor does not even arise), and $g(\lambda) = \int_0^1 e^{2\pi i\lambda x}\, dx = \delta_{\lambda,0}$, if n is odd and λ is an integer.

First consider the case in which n is even, and $2\lambda_1, \dots, 2\lambda_n$ are odd. Then all terms in (5.5) vanish unless the λ's occur in opposite pairs. So it is sufficient to evaluate $H(\mu_1, -\mu_1, \mu_2, -\mu_2, \dots, \mu_m, -\mu_m)$, where the μ's are half-odd integers, and $|\mu_1|, \dots, |\mu_m|$ are all different. In this case, the matrix A of which H is the Pfaffian according to (4.5), is, apart from the factors $(\pi i \mu)^{-1}$, equal to the standard matrix C (see §3), so that

$$H(\mu_1, -\mu_1, \dots, \mu_m, -\mu_m) = (\pi i)^{-m} (\mu_1 \dots \mu_m)^{-1}.$$

Taking $\mu_1 = \frac{1}{2}$, $\mu_2 = 3/2$, $\mu_3 = 5/2, \dots, \mu_m = \frac{1}{2}(n-1)$, and applying a simple permutation, we now reduce (5.3) to (5.2).

If n is odd, things are only slightly more complicated. We assume the λ's to be distinct integers. $H = 0$ unless one λ is zero, for otherwise the factor g in (5.5) always vanishes. Assume $\lambda_n = 0$; then we only need to consider the terms with $j_n = n$, and $j_1, \dots, j_{2m}$ all different from n. In the other factors of such a term we have $f(\lambda, \mu)$'s with $\lambda \neq 0$, $\mu \neq 0$. The only possibility to have a term $\neq 0$ is that $\lambda + \mu = 0$ in all these factors. So it turns out that $H = 0$ unless one λ is zero and the others occur in opposite pairs. We easily evaluate

$$H(\mu_1, -\mu_1, \dots, \mu_m, -\mu_m, 0) = (\pi i)^{-m} (\mu_1 \dots \mu_m)^{-1},$$

if $\mu_1, \dots, \mu_m$ are integers, and $|\mu_1|, \dots, |\mu_m|$, 0 all distinct. The special case $\mu_1 = 1$, $\mu_2 = 2, \dots, \mu_m = m$ leads again to (5.2).

6. Orthogonal expansions. Let the function $f(x_1, \dots, x_n)$ be alternating in all its variables (i.e. by interchanging two x's f is transformed into its opposite). Let $\phi_1(x), \phi_2(x), \dots$ be an orthogonal system for the measure space $[a, b]$. Now the products $\phi_{\nu_1}(x) \dots \phi_{\nu_n}(x_n)$

form an orthogonal system for the product space $a \leqslant x_1 \leqslant b, \ldots, a \leqslant x_n \leqslant b$. Denote the coefficients in the multiple expansion of f by c:

$$c_{\nu_1,\ldots,\nu_n} = \int_a^b \cdots \int_a^b f(x_1, \ldots, x_n)\, \phi_{\nu_1}(x_1) \ldots \phi_{\nu_n}(x_n)\, dx_1 \ldots dx_n.$$

It is easily seen that c also is alternating in all its variables. Therefore, in the (formal) orthogonal expansion of f those terms which correspond to the same set of indices can be assembled into a determinant. The result can be described as follows. Let S denote any arbitrary set of n distinct indices (taken from the set of indices of the orthogonal system, which may be finite). If S consists of the indices $\nu_1, \ldots, \nu_n (\nu_1 < \ldots < \nu_n)$, then to S we make correspond the function Φ_S, defined by

$$\Phi_S = \det_{1 \leqslant i,j \leqslant n} \phi_{\nu_i}(x_j).$$

Then the functions $(n!)^{-\frac{1}{2}}\, \Phi_S$ form an orthogonal system in the product space :

$$\int_a^b \cdots \int_a^b \Phi_S . \Phi_T\, dx_1 \ldots dx_n = \delta_{ST}\, n!.$$

This orthogonal system belongs to the class of alternating functions; the formal development of f is

$$f(x_1, \ldots, x_n) = \Sigma_S\, c_S\, \Phi_S, \text{ where } n!\, c_S = \int_a^b \cdots \int_a^b f.\ \Phi_S\, dx_1 \ldots dx_n.$$

We shall specialize by taking $f = E$, and shall consider orthogonal systems of the Fourier type. It turns out that many coefficients vanish : only those c_S remain where S consists of pairs corresponding to cosines and sines of the same argument (and, if n is odd, one further index corresponding to the constant function).

We start from the results obtained in §5. After we know in what form the expansions appear, it will not be difficult to start anew from the other end (§7 and §8), and to produce several similar expansions.

If $\lambda_1, \dots, \lambda_n$ are integers, then (5.4) is a Fourier coefficient of $E(x_1, \dots, x_n)$. If $\lambda_1, \dots, \lambda_n$ are half-odd integers, then (5.4) is a Fourier coefficient of $\exp(\pi i(x_1 + \dots + x_n)) . E(x_1, \dots, x_n)$; hence it is related to the orthogonal development of E in terms of sines and cosines of half-odd multiples of $2\pi x_1, \dots, 2\pi x_n$. The difference between the two cases may be emphasized as follows. The additive group of real numbers mod 1 is, topologically, a circle, on which there is an obvious orientation. If n is odd, and $x_1, \dots, x_n$ represent points on this circle, then $E(x_1, \dots, x_n)$ can be defined. For if the order of $x_1, \dots, x_n$ corresponds to the order of the circle, then any odd permutation of the x's will disturb this order. If n is even, this is no longer true.

In order to be able to write down the Fourier developments concisely, we define $\Delta(x_1, \dots, x_n;\ \mu_1, \dots, \mu_m)$ as the determinant of the $n \times n$ matrix, whose kth row is

$$\cos 2\pi\, \mu_1 x_k,\ \sin 2\pi\, \mu_1 x_k, \dots,\ \cos 2\pi\, \mu_m x_k,\ \sin 2\pi\, \mu_m x_k,$$

if n is even ($n = 2m$). If n is odd, $n = 2m + 1$, this sequence defines the first $n - 1$ elements of the kth row, and we take the last element of each row to be 1.

The evaluation of H in §5 results in the following development :

$$E(x_1, \dots, x_n) = \frac{2^m}{\pi^m m!} \sum_{\mu_1} \cdots \sum_{\mu_m} \frac{\Delta(x_1, \dots, x_n;\ \mu_1, \dots, \mu_m)}{\mu_1 \cdots \mu_m}, \qquad (6.1)$$

where $m = [\frac{1}{2} n]$, $n > 1$, and where each μ runs through

(i) all positive half-odd integers, if n is even,

(ii) all positive integers, if n is odd.

These results will be obtained by another method in §8, and it will turn out that case (i) also holds if n is odd.

We remark that the terms in (6.1) are symmetric in $\mu_1, \dots, \mu_m$, and vanish if two μ's are equal. Hence we may restrict the summation to $\mu_1 < \mu_2 < \dots < \mu_m$, provided that we multiply by $m!$

Our arguments showed that (6.1) is the formal Fourier development of E. From the theory of Fourier series, we know, however, that the sum represents E in any point inside the cube $0 < x_1 < 1, \dots,$ $0 < x_n < 1$ where E is locally constant. This is always the case, unless two of the x's are equal but then both sides of (6.1) trivially vanish, hence (6.1) remains true. If, however, one or two of the x's are 0 or 1, then (6.1) may be false. It is easily seen to remain valid if n is odd, $0 \leqslant x_1 < 1, \dots, 0 \leqslant x_n < 1$ and if the summation is taken according to case (ii).

7. Another multiple integral involving a determinant. In (6.1) we have a multiple sum over a determinant. Trying to evaluate this sum directly, we are led to consider a more general case. We replace the summation by integration over any ordered measure space (cf. § 1), for which we write simply $\int_c^d$.

Let $n > 1$, $m = [\frac{1}{2} n]$, and let $\phi_1, \dots, \phi_n, \psi_1, \dots, \psi_n$ be functions of t defined for $c \leqslant t \leqslant d$. We define $D = D(\phi_1, \dots, \phi_n, \psi_1, \dots, \psi_n;$ $t_1, \dots, t_m)$ as the determinant of the $n \times n$ matrix whose k-th row is

$$\phi_k(t_1),\ \psi_k(t_1),\ \phi_k(t_2),\ \psi_k(t_2),\ \dots,\ \phi_k(t_m),\ \psi_k(t_m),$$

if n is even. If n is odd, we add an element 1 at the end of each of these rows, so that again a square matrix is obtained. The determinant Δ, defined in § 6, is the special case $\phi_k(\mu) = \cos\,(2\pi\,\mu\,x_k)$, $\psi_k(\mu) = \sin\,(2\pi\,\mu\,x_k)$.

We shall evaluate

$$\Lambda = \int_c^d \dots \int_c^d D\, dt_1 \dots dt_m. \tag{7.1}$$

We notice that, contrary to (1.2), the integrand D is symmetric in the variables $t_1, \dots, t_m$, as interchanging t_i and t_j in D amounts to the interchanging of two *pairs* of columns.

Applying the definition of a determinant, we can write

$$D = \sum \dots \sum E(j_1, \dots, j_n)\, \phi_{j_1}(t_1)\, \psi_{j_2}(t_1) \dots \phi_{j_{2m-1}}(t_m)\, \psi_{j_{2m}}(t_m),$$

where the sum is extended over $j_1, \dots, j_n$, each running from 1 to n.

Now it is easy to integrate over $t_1, \dots, t_m$. Writing

$$k_{ij} = k(i,j) = \int_c^d \phi_i(t)\, \psi_j(t)\, dt, \quad (i, j = 1, \dots, n), \tag{7.2}$$

we obtain

$$\Lambda = \sum \cdots \sum E(j_1, \dots, j_n)\, k(j_1, j_2) \dots k(j_{2m-1}, j_{2m}),$$

so that $\Lambda = 2^m m!\ \mathrm{Pf}(K)$ $(K = (k_{ij}))$. As K need not be skew, we prefer to write (see the end of §3)

$$\Lambda = 2^m m!\ \mathrm{Pf}(\tfrac{1}{2}K - \tfrac{1}{2}K^T). \tag{7.3}$$

Our arguments hold both for n odd and n even. For reduction of a Pfaffian of odd order to a Pfaffian of even order see §3.

8. Some special orthogonal expansions of $E(x_1, \dots, x_n)$. Let the functions $\phi(x, t)$, $\psi(x, t)$ be defined on $a \leqslant x \leqslant b$, $c \leqslant t \leqslant d$, and assume that

$$\int_c^d \phi(x, t)\, \psi(y, t)\, dt = \mathrm{sgn}\,(y - x) + h(x, y), \ (a \leqslant x, y \leqslant b), \tag{8.1}$$

where h is some symmetric function: $h(x, y) = h(y, x)$. Let $x_1, \dots, x_n$ all belong to the interval $[a, b]$, and write $\phi(x_k, t) = \phi_k(t)$, $\psi(x_k, t) = \psi_k(t)$. With these functions we form the determinant D as in §7, and we evaluate Λ. We have (see (7.2) and (8.1)) $\frac{1}{2}(k_{ij} - k_{ji}) = \mathrm{sgn}\,(x_j - x_i)$, and the Pfaffian of this matrix is $E(x_1, \dots, x_n)$ (see (3.7)).

Therefore, by (7.3),

$$E(x_1, \dots, x_n)$$
$$= (2^m m!)^{-1} \int_c^d \cdots \int_c^d D\{\phi(x_1, t), \dots, \psi(x_n, t); t_1, \dots, t_m\}\, dt_1 \dots dt_m. \tag{8.2}$$

The formula holds for all integers $n > 1$, if $a \leqslant x_1 \leqslant b, \dots,$ $a \leqslant x_n \leqslant b$. We quote some special cases.

a. Let $[a, b]$ represent the unit interval $0 \leqslant x \leqslant 1$. We have the formula

$$\frac{4}{\pi}\left\{\sin \pi(y - x) + \frac{\sin 3\pi(y - x)}{3} + \frac{\sin 5\pi(y - x)}{5} + \dots\right\} = \mathrm{sgn}(y - x)$$

if $-1 < y - x < 1$; so certainly if $0 \leqslant x < 1$, $0 \leqslant y < 1$. Now we take, instead of integration over t, summation over the values

$\frac{1}{2}$, 3/2, 5/2, ... Further, we take $\phi(x, t) = 4(\pi\ t)^{-1} \cos(2\pi\ tx)$, $\psi(x, t) = \sin(2\pi tx)$. Then (8.2) becomes the first case of (6.1) (summation over all half-odd integers), and it now turns out that this formula holds also if n is odd. In the latter case, however, we have not a proper multiple orthogonal expansion (as the function $\phi \equiv 1$ does not belong to the orthogonal system).

b. Let $[a, b]$ represent the interval $-\infty < x < \infty$. We now use Dirichlet's formula $\int_0^\infty t^{-1} \sin \lambda t\, dt = \frac{1}{2}\pi \operatorname{sgn} \lambda$. Therefore, if $\phi(x, t) = 4(\pi\ t)^{-1} \cos(2\pi\ tx)$, $\psi(x, t) = \sin(2\pi\ tx)$, then (8.1) is again satisfied, provided that $\int_c^d$ is understood to be $\int_0^\infty$. Now (8.2) gives the expansion of E in terms of a Fourier integral :

$$E(x_1, \dots, x_n) = \frac{2^m}{\pi^m m!} \int_0^\infty \dots \int_0^\infty \frac{\Delta(x_1, \dots, x_n; t_1, \dots, t_m)}{t_1 \dots t_m} dt_1 \dots dt_m \quad (8.3)$$

($n = 2, 3, \dots$; $m = [\frac{1}{2}n]$; $-\infty < x_1 < \infty, \dots, -\infty < x_n < \infty$; Δ is defined as in §6). The integral can be interpreted as a repeated integral; the order in which the integrations are carried out is irrelevant. There is obviously no absolute convergence.

c. Instead of $[a, b]$ we take the discrete set of all integers. As we have

$$\int_0^1 \cot \pi\, t \sin(2\pi\, kt)\, dt = \operatorname{sgn} k, \quad (k = 0, \pm 1, \pm 2, \dots),$$

we see that (8.1) is true for all integers x, y, provided that $\int_c^d$ is interpreted as $\int_0^1$, and $\phi(x, t) = 2 \cot \pi t \cos(2\pi\, xt)$, $\psi(x, t) = \sin(2\pi\, xt)$.

Then (8.2) gives

$$E(x_1, \dots, x_n) = (m!)^{-1} \int_0^1 \dots \int_0^1 \cot \pi t_1 \dots \cot \pi t_m \times$$
$$\times \Delta(x_1, \dots, x_n; t_1, \dots, t_m) dt_1, \dots dt_m, \quad (8.4)$$

for all integers $x_1, \dots, x_n$ ($n = 2, 3, \dots$; $m = [\frac{1}{2}n]$).

Properly speaking, we have hardly the right to call (8.3) a Fourier integral and (8.4) the representation of E as Fourier coefficient. The transforms are highly singular, so much so that the n-fold integrals which occur in the regular case, are reduced to m-fold integrals.

d. Instead of $[a, b]$ we take a set S consisting of K consecutive integers (K is any positive integer). We now use the following formula

$$\sum \sin (\pi \sigma (y-x)/K) \cot (\pi \sigma / 2K) = K \operatorname{sgn} (y-x) \quad (x \in S, y \in S),$$

where the summation is taken over the values $\sigma = 1, 3, 5, \ldots, 2K-1$. Again applying the same procedure, we obtain

$$E(x_1, \ldots, x_n) = \frac{1}{K^m m!} \sum_{\sigma_1} \cdots \sum_{\sigma_m} \cot \frac{\pi \sigma_1}{2K} \cdots \cot \frac{\pi \sigma_m}{2K} \times$$
$$\times \Delta\left(x_1, \ldots, x_n; \frac{\sigma_1}{2K}, \ldots, \frac{\sigma_m}{2K} \right), \quad (8.5)$$

where $x_1 \in S, \ldots, x_n \in S$, $n = 2, 3, \ldots$, $m = [\frac{1}{2} n]$, and in the multiple sum each σ runs through the values $1, 3, 5, \ldots, 2K-1$. The cases a, b, c are limit cases of (8.5).

Case (ii) of formula (6.1) is not related to (8.1) itself, but to a slightly more complicated case, essentially restricted to odd values of n. Instead of (8.1), we assume that ϕ and ψ satisfy

$$\int_c^d \phi(x, t)\, \psi(y, t)\, dt = \operatorname{sgn} (y - x) + p(y) - p(x) + h(x, y), \quad (8.6)$$

where $p(x)$ is an arbitrary function, and h is symmetric. We now again arrive at (8.2), this time by virtue of (3.8) (take $p_i = p(x_i)$). As an example we take the case that, as in case d, the x's are taken from a set S of K consecutive integers. We shall obtain an expansion of E in terms of K-th roots of unity, analogous to the Fourier expansion (6.1) (ii). We use the formula

$$\sum_{\nu=1}^{k-1} \cot (\pi \nu / K). \sin (2\pi i k \nu / K) = K \operatorname{sgn} k - 2k$$
$$(k = -K + 1, \ldots, K + 1).$$

Therefore, (8.6) holds if

$$\phi(x, \nu) = 2 \cot (\pi \nu / K) \cos (2\pi \nu x / K),$$
$$\psi(x, \nu) = K^{-1} \sin (2\pi \nu x / K), \quad p(x) = -2x/K,$$

and if $\int_c^d dt$ is interpreted as summation over $\nu (\nu = 1, \ldots, K-1)$.

Hence we obtain

$$E(x_1, \dots, x_n) = \frac{1}{K^m m!} \sum_{\nu_1=1}^{K-1} \cdots \sum_{\nu_m=1}^{K-1} \cot(\pi \nu_1/K) \dots \times$$

$$\times \cot(\pi \nu_m/K)\, \Delta\left(\frac{x_1}{K}, \dots, \frac{x_n}{K}; \nu_1, \dots, \nu_m\right), \quad (8\ 7)$$

($x_1 \in S, \dots, x_n \in S$, n odd, $m = [\frac{1}{2} n]$). It is easily seen that (6.1) ii, (8.3) and (8.4) are limit cases of (8.7).

9. Laplace transform of $E(x_1, \dots, x_n)$. We now consider the range $0 \leqslant x_i < \infty$. The Laplace transform of E is

$$\Omega = \int_0^\infty \cdots \int_0^\infty E(x_1, \dots, x_n) \exp(-p_1 x_1 - \dots - p_n x_n)\, dx_1 \dots dx_n,$$

$$(\operatorname{Re} p_1 > 0, \dots, \operatorname{Re} p_n > 0). \quad (9.1)$$

In other words (cf. § 4)

$$\Omega = \int \cdots \int \det_{1 \leqslant i,j \leqslant n} (\exp(-p_i x_j))\, dx_1 \dots dx_n, \qquad (9.2)$$

where the range of integration is given by $0 < x_1 < x_2 < \dots < x_n < \infty$. By (4.4) we have $\Omega = \mathrm{Pf}(A)$, where A is defined as in § 4, with $\phi_j(x) = \exp(-p_j x)$.

First consider the simpler case, viz. n even, so that a_{ij} is defined by (4.3) only. By elementary calculus we obtain

$$\int_0^\infty \int_0^\infty e^{-px-qy} \operatorname{sgn}(y - x)\, dx\, dy = (p - q)/(pq(p + q)),$$

and therefore

$$p_i\, p_j\, a_{ij} = h_{ij}, \text{ where } h_{ij} = (p_i - p_j)/(p_i + p_j), \ (i, j = 1, \dots, n).$$

The determinant of h_{ij} has been determined by K. Rohn [5] who dealt with the more general $\det \{(x_i - y_j)/(x_i + y_j)\}$. (The further generalization $\det \{(x_i + z_j)/(x_i + y_j)\}$ was dealt with by H. J. A. Duparc [2]; the present special case was issued as a problem in [3]). The result is

$$\det \{(p_i - p_j)/(p_i + p_i)\} = \prod_{i<j} \{(p_i - p_j)^2/(p_i + p_j)^2\}, \ (n \text{ even}). \quad (9.3)$$

As $\det A = \mathrm{Pf}(A)^2$, we now know the value of $\mathrm{Pf}(A)$, and hence the value of Ω, apart from its sign. The sign can be determined in several ways. The Pfaffian of (h_{ij}) is a rational function of $p_1, \dots, p_n$. Hence its value is, by (9.3), either always $P = \Pi_{i<j}(p_i - p_j)/(p_i + p_j)$ or always $-P$. Making $p_1, \dots, p_n$ tend to infinity such that $p_1/p_2 \to \infty, p_2/p_3 \to \infty, \dots, p_{n-1}/p_n \to \infty$, the matrix (h_{ij}) tends to B (see §3), whose Pfaffian is $+1$, and P also tends to $+1$. Therefore $\mathrm{Pf}(H) = +P$. Another method to determine the sign is by application of a theorem of Polya's (see [4], Vol. 2, Abschnitt 5, problem 76, pages 49 and 235) which states that the integrand of (9.2) is positive if $p_1 > p_2 > \dots > p_n > 0$; therefore Ω is positive in that case.

Thus we have proved, for n even, that

$$\Omega = \prod_{1 \leqslant i < j \leqslant n} \frac{p_i - p_j}{p_i + p_j} \prod_{1 \leqslant j \leqslant n} p_j^{-1}. \tag{9.4}$$

It turns out that (9.4) remains true if n is odd. For then we have, by §4, $\Omega = \mathrm{Pf}(A^*)$, where A^* is obtained by bordering the matrix $a_{ij} = (p_i - p_j)/(p_i + p_j)\, p_i\, p_j$ by an $(n+1)$th column $p_1^{-1}, \dots, p_n^{-1}$, 0 and an $(n+1)$th row $-p_1^{-1}, \dots, -p_n^{-1}$, 0. In order to evaluate $\mathrm{Pf}(A^*)$, we multiply, for each $i \leqslant n$, both the ith column and the ith row of A^* by p_i. The new matrix can be described as $(p_i - p_j)/(p_i + p_j)$ $(i, j = 1, \dots, n+1)$, where $p_{n+1} = 0$. As $n+1$ is even, its Pfaffian has already been proved to be $\Pi_{1 \leqslant i < j \leqslant n+1}$ $(p_i - p_j)/(p_i + p_j)$. In the latter expression the factors with $j = n+1$ have no influence, as $p_{n+1} = 0$. Omitting these factors, we obtain $\Pi_{1 \leqslant i < j \leqslant n}(p_i - p_j)/(p_i + p_j)$, and we easily verify (9.4). We quote one special case of (9.4). Take $p_k = n + 1 - k$ $(k = 1, \dots, n)$, and carry out the substitution $\exp(-x_j) = y_j$. The integrand becomes a Vandermonde determinant, and (9.4) becomes.

$$\int_0^1 \cdots \int_0^1 \prod\nolimits_{1 \leqslant i < j \leqslant n} |\, y_i - y_j \,|\, dy_1 \dots dy_n = \frac{\{1!\ 2!\ 3! \dots (n-1)!\}^2\, n!}{1!\ 3!\ 5! \dots (2n-1)!},$$

valid for $n = 2, 3, \dots$.

10. Power series whose coefficients are given by E. For the range of the variables of $E(x_1, \dots, x_n)$ we now take the set of all integers $\geqslant 0$. We shall determine the sum of the multiple power series

$$f(z_1, \dots, z_n) = \sum_0^\infty \cdots \sum_0^\infty E(\nu_1, \dots, \nu_n)\, z_1^{\nu_1} \dots z_n^{\nu_n}, \qquad (10.1)$$

which is absolutely convergent if $|z_1| < 1, \dots, |z_n| < 1$. We shall show that, for $n = 1, 2, 3, \dots$,

$$f(z_1, \dots, z_n) = \prod_{1 \leqslant i, \leqslant j \leqslant n} \{(z_i - z_j)/(1 - z_i z_j)\} \prod_{1 \leqslant j \leqslant n} (1 - z_j)^{-1}. \qquad (10.2)$$

The proof follows the same lines as the proof of (9.4). (Actually (9.4) can be considered as a limit case of (10.2)). We have

$$\sum_0^\infty \sum_0^\infty z_i^\mu z_j^\nu \operatorname{sgn}(\nu - \mu) = (z_j - z_i)/\{(1 - z_i)(1 - z_j)(1 - z_i z_j)\},$$

and we infer that, if n is even, $f(z_1, \dots, z_n)$ equals $\Pi_{j=1}^n (1 - z_j)^{-1}$ times the Pfaffian of the matrix $(z_j - z_i)/(1 - z_i z_j)$. This matrix arises from the matrix $(p_i - p_j)/(p_i + p_j)$ by the substitution $p_i = (z_i - 1)/(z_i + 1)$, $(i = 1, \dots, n)$, and so its Pfaffian is, by § 9,

$$\prod_{1 \leqslant i < j \leqslant n} \frac{p_i - p_j}{p_i + p_j} = \prod_{1 \leqslant i < j \leqslant n} \frac{x_j - x_i}{1 - x_i x_j}.$$

Now (10.2) easily follows. Furthermore, the case that n is odd can be discussed in the same way as it was done in §9.

The following consequence of (10.2) is amusing: the function $f(z_1, \dots, z_n)\, \Pi_1^n(1 - z_i)$ is invariant under the group of simultaneous substitutions

$$z_1 = (a\,\zeta_1 + b)/(b\,\zeta_1 + a), \dots, z_n = (a\,\zeta_n + b)/(b\,\zeta_n + a),$$

where a and b are complex numbers, with the restriction $a^2 \neq b^2$.

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