

# Analysing multiprogramming queues by generating functions

**Citation for published version (APA):**

Adan, I. J. B. F., Wessels, J., & Zijm, W. H. M. (1993). Analysing multiprogramming queues by generating functions. *SIAM Journal on Applied Mathematics*, 53(4), 1123-1131. <https://doi.org/10.1137/0153056>

**DOI:**

[10.1137/0153056](https://doi.org/10.1137/0153056)

**Document status and date:**

Published: 01/01/1993

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

**General rights**

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

[www.tue.nl/taverne](http://www.tue.nl/taverne)

**Take down policy**

If you believe that this document breaches copyright please contact us at:

[openaccess@tue.nl](mailto:openaccess@tue.nl)

providing details and we will investigate your claim.

## ANALYSING MULTIPROGRAMMING QUEUES BY GENERATING FUNCTIONS\*

I. J. B. F. ADAN†, J. WESSELS‡, AND W. H. M. ZIJM§

**Abstract.** The generating function approach for analysing queueing systems has a long-standing tradition. One of the highlights is the seminal paper by Kingman [*Ann. Math. Statist.*, 32(1961), pp. 1314–1323] on the shortest-queue problem, where the author shows that the equilibrium probabilities  $p_{m,n}$  of the queue lengths can be written as an infinite sum of products of powers. The same approach is used by Hofri [*Internat. J. Computer and Information Sci.*, 7 (1978), pp. 121–155] to prove that, for a multiprogramming model with two queues, the boundary probability  $p_{0,j}$  can be expressed as an infinite sum of powers. This paper shows that the latter representation does not always hold, which implies that the multiprogramming problem is essentially more complicated than the shortest-queue problem. However, it appears that the generating function approach is very well suited to show when such a representation is available and when it is not.

**Key words.** equilibrium distribution, generating functions, multiprogramming, queues, two-dimensional random walk

**AMS(MOS) subject classifications.** 60K25, 90B22

**1. Introduction.** There is a long tradition of using generating functions for analysing exponential queueing models. A seminal paper in this area is Kingman's paper [6] on the shortest-queue model, in which the author shows that the generating function for the equilibrium probabilities  $p_{i,j}$  for the queue lengths is meromorphic. This implies that a partial fraction decomposition is possible, which shows that the equilibrium probabilities can be expressed as a countable sum of products of powers. Kingman gives the first term of this expansion explicitly, and Flatto and McKean [4] give the second. Hofri [5] uses the same approach for a multiprogramming problem with essentially two queues. The equilibrium equations for the multiprogramming problem are similar to the equilibrium equations for the shortest-queue problem, and therefore it seems likely that the same approach works. By concentrating on the boundary probabilities  $p_{0,j}$  rather than on the general probabilities, Hofri is able to get the expansion in a more explicit form.

The aim of this paper is to show that the multiprogramming problem is less similar to the shortest-queue problem than it first appears. It seems that the representation of  $p_{0,j}$  as an infinite sum of powers does not necessarily hold for every  $j$ , but only from some  $j$  onward. It also appears that the generating-function approach is a good tool to handle this complication.

The complication mentioned above stems from a feature that does not occur in the shortest-queue problem, at least not in the symmetric version as treated by Kingman. Recently, another approach has been developed for the shortest-queue problem, leading to more explicit representations of the equilibrium probabilities (see [1]). The extension of this new approach is the asymmetric shortest-queue problem (see [2]) encounters a complication similar to the one studied by Hofri. In fact, for this new approach, the analogy of the shortest-queue model with Hofri's multiprogramming model can be exploited for the analysis of the latter (see [3]).

---

\* Received by the editors July 31, 1991; accepted for publication (in revised form) July 8, 1992.

† Eindhoven University of Technology, Department of Mathematics and Computing Science, P.O. Box 513, 5600 MB Eindhoven, the Netherlands.

‡ International Institute for Applied Systems Analysis, A-2361 Laxenburg, Austria.

§ University of Twente, Department of Mechanical Engineering, P.O. Box 217, 7500 AE Enschede, the Netherlands.

The paper is organised as follows. In § 2 the model is introduced, together with the equilibrium equations for the probabilities  $p_{i,j}$  and with the corresponding functional equation for the generating-function  $G(z, u)$ . For the solution of the functional equation, it is crucial to determine  $G(0, u)$ . Hofri's approach for the determination of  $G(0, u)$ , together with some corrections and simplifications, is outlined in § 3. The final part of that determination is the partial fraction decomposition of  $G(0, u)$ . In this part, Hofri oversimplified the problem. Section 4 is devoted to the partial fraction decomposition of this generating function and hence to the conditions for the representation of the boundary probabilities  $p_{0,j}$  as a countable linear combination of powers. Section 5 contains concluding remarks.

**2. The multiprogramming model.** The multiprogramming system as introduced by Hofri in [5] has the following queueing properties (compare Fig. 1). In the queueing model, it is supposed that queue III of incoming jobs provides an infinite source of ever-available jobs. The multiprogramming system consists of an input-output (IO) unit and a central processor (CP). Incoming jobs start at the IO unit with an exponentially distributed service time with parameter  $\mu'$ . Subsequently, the job leaves the system with probability  $p$  or proceeds to queue II at the CP with probability  $1 - p$ . At the CP, a job has an exponentially distributed service time with parameter  $\mu$ . Next, the job is recycled to the IO unit where it joins queue I. The IO unit treats the jobs in queue I with non-preemptive priority with respect to the new jobs in queue III.

The system may be represented by a continuous-time Markov process with states  $(i, j)$ ,  $i = 0, 1, \dots$  and  $j = 1, 2, \dots$ , where  $i$  and  $j$  are the lengths of the queues II and I, respectively (including the jobs being served). Let  $\{p_{i,j}\}$  be the equilibrium distribution of the Markov chain, which exists if  $(1 - p)\mu' < \mu$  (see [5, App. A]). This distribution satisfies the following recursion relations obtained by equating in each state the rate into and the rate out of the state:

- (1)  $\lambda p_{0,1} = \eta p_{0,2},$
- (2)  $(\mu + \lambda)p_{i,1} = \lambda p_{i-1,1} + \lambda p_{i-1,2} + \eta p_{i,2}, \quad i \geq 1,$
- (3)  $\mu' p_{0,j} = \eta p_{0,j+1} + \lambda p_{1,j-1}, \quad j \geq 2,$
- (4)  $\kappa p_{i,j} = \lambda p_{i-1,j+1} + \mu p_{i+1,j-1} + \eta p_{i,j+1}, \quad i \geq 1, \quad j \geq 2,$

where  $\lambda = (1 - p)\mu'$ ,  $\eta = p\mu'$ , and  $\kappa = \mu + \lambda + \eta$ . As explained below, these relations are quite similar to the ones for the shortest-queue problem.

In the shortest-queue model, jobs arrive according to a Poisson stream with rate  $\omega$  at a system consisting of two parallel queues. On arrival, a job joins the shortest queue. All service times are exponentially distributed with parameter  $\nu$ . This system may be represented as a Markov process with states  $(i, j)$ ,  $i, j = 0, 1, \dots$ , where  $i$  is a the length

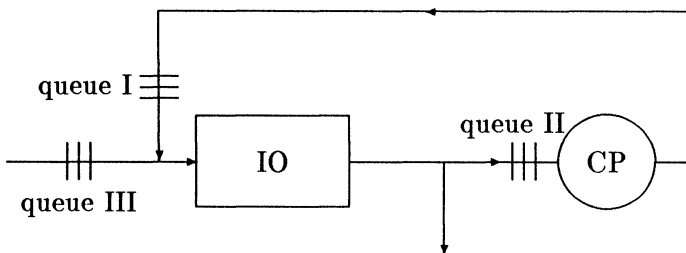


FIG. 1. Queueing model for a multiprogramming system.

of the shortest queue and  $j$  is the difference between the lengths of the longest and the shortest queue. Therefore the state space also includes the points  $(i, 0)$ ,  $i = 0, 1, \dots$ , which give rise to additional recursion relations for the equilibrium distribution  $\{p_{i,j}\}$ , and the form of the relations in the points  $(i, 1)$ ,  $i = 0, 1, \dots$ , is different from (1), (2). However, the relations in all other points are given by (3), (4) with  $\lambda, \mu, \mu'$ , and  $\eta$  replaced by  $\omega, \nu, \omega + \nu$ , and  $\nu$ , respectively. Therefore, it seems likely that the generating-function technique used by Kingman [6] to analyse the shortest-queue problem also works for the present problem.

The generating-function  $G(z, u)$  of the equilibrium distribution  $\{p_{i,j}\}$  is defined by

$$G(z, u) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} p_{i,j} z^i u^j.$$

The recursion relations for  $\{p_{i,j}\}$  can now be transformed to the following functional equation for  $G(z, u)$ :

$$(5) \quad G(z, u) = \frac{\mu u(z - u)G(0, u) + zu(u - 1)(\eta + \lambda z)G(z, 1)}{\kappa uz - \mu u^2 - \lambda z^2 - \eta z}.$$

The analysis of this equation is the main topic of [5]. In § 3 we sketch the part of the generating-function analysis that is crucial to our discussion. For other parts and more details, refer to the extensive treatment provided by Hofri in [5].

**3. Generating-function analysis.** The functional equation (5) for  $G(z, u)$  relates  $G(z, u)$  to the boundary values  $G(z, 1)$  and  $G(0, u)$ . Clearly,  $G(0, u)$  is the (one-dimensional) generating function of the boundary probabilities  $p_{0,j}$ ,  $j = 1, 2, \dots$ . The determination of  $G(0, u)$  is crucial. It will be proved that  $G(0, u)$  may be continued to a meromorphic function and that the poles and residues can be found. This provides, by partial fraction decomposition of  $G(0, u)$ , the possibility of expressing  $p_{0,j}$  as an infinite sum of powers. In § 4 we show that this partial fraction decomposition is more complicated than suggested by Hofri and that this decomposition does not necessarily lead to explicit expressions for all  $p_{0,j}$ . Below, the determination of  $G(0, u)$  is first outlined in more detail. This outline closely follows Hofri's exposition in [5], and it also includes some corrections and simplifications of the results in [5].

The following result will be needed. A proof is included, since the one in [5, App. B] seems to move in cycles.

LEMMA 1.  $G(z, u)$  exists in  $|z| \leq 1$ ,  $|u| < \mu'/\mu$  and in  $|z| < \mu/\lambda$ ,  $|u| \leq 1$ .

*Proof* (cf. the proof of Lemma 2 in [6]). Addition of the equilibrium equations in the points  $(i, k)$ ,  $i = 0, 1, \dots$  and  $k = 1, \dots, j$  yields that, for all  $j \geq 1$ ,

$$\mu' \sum_{i=0}^{\infty} p_{i,j+1} - \mu \sum_{i=0}^{\infty} p_{i,j} = -p_{0,j} \leq 0,$$

so that

$$\sum_{i=0}^{\infty} p_{i,j+1} \leq \left(\frac{\mu}{\mu'}\right)^j \sum_{i=0}^{\infty} p_{i,1}.$$

Hence, for  $|z| \leq 1$  and  $|u| < \mu'/\mu$ ,

$$|G(z, u)| \leq \sum_{j=1}^{\infty} \left( \sum_{i=0}^{\infty} p_{i,j} |u|^j \right) < \infty,$$

which proves the first part of the lemma. Addition of the equilibrium equations in the points  $(k, j)$ ,  $k = 0, \dots, i$  and  $j = 1, 2, \dots$  yields that, for all  $i \geq 0$ ,

$$(6) \quad \sum_{j=1}^{\infty} p_{i+1,j} = \frac{\lambda}{\mu} \sum_{j=1}^{\infty} p_{i,j} = \dots = \left(\frac{\lambda}{\mu}\right)^i \sum_{j=1}^{\infty} p_{0,j}.$$

from which the second part readily follows.  $\square$

It follows that, whenever  $z$  and  $u$  satisfy  $|z| \leq 1$ ,  $|u| < \mu'/\mu$ , or  $|z| < \mu/\lambda$ ,  $|u| \leq 1$ , and

$$(7) \quad \kappa uz - \mu u^2 - \lambda z^2 - \eta z = 0,$$

then the numerator in (5) also vanishes, so that

$$(8) \quad z(\eta + \lambda z)G(z, 1) = \mu \frac{z - u}{1 - u} G(0, u).$$

For each  $z$ , the quadratic equation (7) in  $u$  is solved by

$$u_{\pm} (\kappa z \pm \sqrt{\kappa^2 z^2 - 4\mu(\eta + \lambda z)z})/2\mu.$$

It can be shown that  $u_{\pm}$  may be written as

$$(9) \quad u_+ = h(\zeta), \quad u_- = h(\alpha\zeta)$$

for some  $\zeta$ , where the mapping  $h(\zeta)$  is defined by

$$(10) \quad h(\zeta) = a + \phi(\zeta + \zeta^{-1})$$

with

$$(11) \quad a = \frac{\kappa\eta}{R^2}, \quad \phi = \frac{\eta\sqrt{\mu\lambda}}{R^2}, \quad a = \frac{(\kappa + R)^2}{4\mu\lambda} > 1, \quad R = \sqrt{\kappa^2 - 4\mu\lambda},$$

and vice versa, for each  $\zeta$  there exists a  $z$  for which  $u_{\pm}$  satisfy (9). Now, (8) and representation (9) are exploited to determine  $G(0, u)$ .

If  $|u_{\pm}| < (\mu + \mu')/2\mu$ , then the corresponding  $z$  given by  $z = \mu(u_- + u_+)/\kappa$ , satisfies

$$|z| \leq \mu(|u_-| + |u_+|)/\kappa \leq 1,$$

and, accordingly, if  $|u_{\pm}| \leq 1$ , then  $z$  satisfies  $|z| \leq 2\mu/(\mu + \mu') < \mu/\lambda$ . Hence, for  $u_{\pm}$  in  $|u_{\pm}| < \max \{1, (\mu + \mu')/2\mu\}$ , we obtain from (8) that

$$(12) \quad \frac{z - u_+}{1 - u_+} G(0, u_+) = \frac{z - u_-}{1 - u_-} G(0, u_-).$$

Using  $z = \mu(u_- + u_+)/\kappa$  to eliminate  $z$  in (12) and then substituting (9) gives, after some calculations, that

$$(13) \quad \frac{\bar{G}(\alpha\zeta)}{\bar{G}(\zeta)} = \frac{\alpha \zeta - \zeta_1/\alpha}{\beta \zeta - 1/\zeta_1},$$

where  $\bar{G}(\zeta)$  is defined by  $\bar{G}(\zeta) = G(0, h(\zeta))$  and

$$(14) \quad \frac{1}{\zeta_1} = \frac{2\sqrt{\mu\lambda}}{\kappa} \frac{\mu\alpha - \mu'}{\mu' - \mu + R} > 1, \quad \beta = \frac{\mu - \mu'\alpha}{\mu\alpha - \mu'}.$$

Relation (13) is valid in  $|h(\zeta)|, |h(\alpha\zeta)| < \max \{1, (\mu + \mu')/2\mu\}$ . It is easily verified that  $h(\zeta)$  is a conformal mapping from  $|\zeta| > 1$  on the whole  $u$ -plane, excluding the interval  $[a - 2\phi, a + 2\phi]$ . The unit circle  $|\zeta| = 1$  is mapped two-to-one on the interval

$[a - 2\phi, a + 2\phi]$ . Let  $r > 1$  be the largest number such that  $h(\zeta)$  maps the annular  $1/r < |\zeta| < r$  into the disk  $|u| < \max \{1, (\mu + \mu')/2\mu\}$ . So relation (13) is valid in  $1/r < |\zeta| < |\alpha\zeta| < r$ , and hence in

$$(15) \quad 1/r < |\zeta| < r/\alpha,$$

which can be shown to be nonempty.  $\bar{G}(\zeta)$  is regular in the annulus (15), and may therefore be expanded in a Laurent series  $\sum_{-\infty}^{\infty} a_n \zeta^n$ . Substitution of this series into (13) yields a recursion relation for  $a_n$ , which implies that  $a_{\pm n} \sim C_{\pm}(\zeta_1/\alpha)^{|n|}$  as  $n \rightarrow \infty$ . So  $\bar{G}(\zeta)$  is regular in  $\zeta_1/\alpha < |\zeta| < \alpha/\zeta_1$ . Relation (13) is now used to define  $\bar{G}(\zeta)$  over  $|\zeta| > 1$  recursively as a regular function, except for simple poles at

$$(16) \quad \zeta = \bar{\zeta}_j = \frac{\alpha^{j-1}}{\zeta_1}, \quad j = 2, 3, \dots,$$

with corresponding residues  $\bar{g}_j$ , which satisfy the recursion relation

$$(17) \quad \bar{g}_{j+1} = \bar{g}_j \frac{\alpha^2}{\beta} \frac{\alpha^{j-1}/\zeta_1 - \zeta_1/\alpha}{\alpha^{j-1}/\zeta_1 - 1/\zeta_1}, \quad j = 2, 3, \dots$$

The residue  $\bar{g}_2$  follows by letting  $\zeta \rightarrow 1/\zeta_1$  in (13) and using that  $\bar{G}(1/\zeta_1) = G(0, 1) = 1 - \lambda/\mu$  (see (6)). This results in (cf. the complicated expression (102) in [5])

$$\bar{g}_2 = \left(1 - \frac{\lambda}{\mu}\right) \frac{\alpha^2}{\beta} \left(\frac{1}{\zeta_2} - \frac{\zeta_1}{\alpha}\right).$$

Equivalently,  $\bar{G}(\zeta)$  may be continued over  $0 < |\zeta| < 1$ . Each of these two continuations may be exploited for the calculation of  $G(0, u)$ ; namely, any  $u \notin [a - 2\phi, a + 2\phi]$  is associated with two values of  $\zeta$  for which  $h(\zeta) = u$ . One value of  $\zeta$  satisfies  $|\zeta| > 1$ , and the other satisfies  $|\zeta| < 1$ . Since neither of the two values of  $\zeta$  has any advantages over the other, we decide arbitrarily to always take that value satisfying  $|\zeta| > 1$ . Denoting by  $h^{\leftarrow}(u)$  the inverse of  $h(\zeta)$  from the whole  $u$ -plane, excluding  $[a - 2\phi, a + 2\phi]$ , to  $|\zeta| > 1$ , it follows that  $\bar{G}(h^{\leftarrow}(u))$  is a regular function, except for simple poles at

$$(18) \quad u = u_j = h(\bar{\zeta}_j) = a + \phi \left( \frac{\alpha^{j-1}}{\zeta_1} + \frac{\zeta_1}{\alpha^{j-1}} \right), \quad j = 2, 3, \dots,$$

with corresponding residues

$$(19) \quad g_j = \phi \bar{g}_j \frac{\bar{\zeta}_j^2 - 1}{\bar{\zeta}_j^2}, \quad j = 2, 3, \dots$$

Since  $\bar{G}(h^{\leftarrow}(u))$  and  $G(0, u)$  coincide on the interior of the ellipse  $|h^{\leftarrow}(u)| = r$ , excluding  $[a - 2\phi, a + 2\phi]$ , it follows that  $\bar{G}(h^{\leftarrow}(u))$  is the analytic continuation of  $G(0, u)$  over the region  $|u| \geq \max \{1, (\mu + \mu')/2\mu\}$ .

So far, it has been proved that  $G(0, u)$  can be continued to a meromorphic function over the whole  $u$ -plane with simple poles at the points  $u = u_j$  and corresponding residues  $g_j, j = 2, 3, \dots$ . To obtain expressions for the boundary probabilities  $p_{0,i}$ , the meromorphic function  $G(0, u)$  is decomposed into partial fractions. This partial fraction decomposition is the main topic of § 4.

**4. Partial fraction decomposition of the generating function.** To decompose  $G(0, u)$  into partial fractions, we use the approach in [9, § 7.4]. Let  $E_j$  be the ellipse in

the  $u$ -plane corresponding to  $|h^{\leftarrow}(u)| = |\zeta| = (1 + \alpha)\bar{\zeta}_l/2$  for  $l = 2, 3, \dots$ . Since

$$\bar{\zeta}_l < \frac{1 + \alpha}{2} \bar{\zeta}_l < \bar{\zeta}_{l+1},$$

no ellipse  $E_l$  passes through any poles of  $G(0, u)$ . It is not essential for the curves  $E_l$  to be ellipses, but it appears that this choice facilitates the proof of Lemma 2 in this section.

If  $u$  is not a pole of  $G(0, z)$  and if  $l$  is sufficiently large such that  $E_l$  encloses  $u$ , then, since the only poles of the integrand are the poles of  $G(0, z)$  and the point  $z = u$ , we have by the Theorem of Residues that

$$(20) \quad \frac{1}{2\pi i} \int_{E_l} \frac{G(0, z)}{z - u} dz = G(0, u) + \sum_{k=2}^l \frac{g_k}{u_k - u}.$$

However, inserting the expansion

$$\frac{1}{z - u} = \frac{1}{z} + \frac{u}{z^2} + \dots + \frac{u^{n-1}}{z^n} + \frac{u^n}{z^n(z - u)},$$

where  $n$  is some nonnegative integer, yields that

$$(21) \quad \begin{aligned} \frac{1}{2\pi i} \int_{E_l} \frac{G(0, z)}{z - u} dz &= \sum_{k=0}^{n-1} \frac{1}{2\pi i} \int_{E_l} \frac{G(0, z)u^k}{z^{k+1}} dz + \frac{u^n}{2\pi i} \int_{E_l} \frac{G(0, z)}{z^n(z - u)} dz, \\ &= \sum_{k=1}^{n-1} p_{0,k}u^k + \sum_{k=0}^{n-1} \sum_{j=2}^l \frac{g_j u^k}{u_j^{k+1}} + \frac{u^n}{2\pi i} \int_{E_l} \frac{G(0, z)}{z^n(z - u)} dz. \end{aligned}$$

Hence, by (20) and (21),

$$(22) \quad G(0, u) = \sum_{k=1}^{n-1} p_{0,k}u^k + \sum_{j=2}^l \frac{g_j u^n}{u_j^n(u - u_j)} + \frac{u^n}{2\pi i} \int_{E_l} \frac{G(0, z)}{z^n(z - u)} dz.$$

If we can now prove that, as  $l \rightarrow \infty$ ,

$$\int_{E_l} \frac{G(0, z)}{z^n(z - u)} dz \rightarrow 0,$$

then, by letting  $l \rightarrow \infty$  in (22), we obtain the following decomposition of  $G(0, u)$ :

$$(23) \quad G(0, u) = \sum_{k=1}^{n-1} p_{0,k}u^k + \sum_{j=2}^{\infty} \frac{g_j u^n}{u_j^n(u - u_j)}.$$

In expression (23),  $n - 1$  unknown probabilities occur. It is, of course, desirable to keep this number as small as possible.

DEFINITION. Let  $m$  be the smallest nonnegative integer such that  $1/|\beta|\alpha^{m-1} < 1$ .

LEMMA 2.  $G_l = \sup_{u \in E_l} (|G(0, u)|/|u^m|) \rightarrow 0$  as  $l \rightarrow \infty$ .

Proof. Since by (10),  $u = h(\zeta) \sim \phi\zeta$  as  $|\zeta| \rightarrow \infty$ , it is sufficient to prove that

$$\bar{G}_l = \sup_{|\zeta| = (1 + \alpha)\bar{\zeta}_l/2} \frac{|\bar{G}(\zeta)|}{|\zeta^m|} = \sup_{0 \leq \theta < 2\pi} \frac{|\bar{G}((1 + \alpha)\bar{\zeta}_l e^{i\theta}/2)|}{|((1 + \alpha)\bar{\zeta}_l e^{i\theta}/2)^m|} \rightarrow 0 \quad \text{as } l \rightarrow \infty.$$

It holds that

$$\frac{\bar{G}_l}{\bar{G}_{l-1}} \leq \frac{\bar{\zeta}_{l-1}^m}{\bar{\zeta}_l^m} \sup_{0 \leq \theta < 2\pi} \frac{|\bar{G}((1 + \alpha)\bar{\zeta}_l e^{i\theta}/2)|}{|\bar{G}((1 + \alpha)\bar{\zeta}_{l-1} e^{i\theta}/2)|}.$$

Inserting that  $\bar{\xi}_l = \alpha \bar{\xi}_{l-1}$ , by (16) and then applying relation (13) yields

$$\frac{\bar{G}_l}{\bar{G}_{l-1}} \leq \frac{1}{|\beta| \alpha^{m-1}} \frac{(1 + \alpha) \bar{\xi}_{l-1}/2 + \xi_1/\alpha}{(1 + \alpha) \bar{\xi}_{l-1}/2 - 1/\xi_1}.$$

Hence, since  $1/|\beta| \alpha^{m-1} < 1$  and  $\bar{\xi}_{l-1} \rightarrow \infty$  as  $l \rightarrow \infty$ , there exists a positive number  $K$ , strictly less than unity, such that, for all  $l$  sufficiently large,  $\bar{G}_l/\bar{G}_{l-1} \leq K$ , which proves that  $\bar{G}_l$  tends to zero as  $l$  tends to infinity.  $\square$

*Remark.* It can be shown that  $m$  is always strictly positive and that  $m$  is possibly larger than unity. For instance, for  $\mu' = 1$ ,  $\mu = 2$ , and  $p = 3/25$ , we obtain from (11) and (14) that  $\alpha = 11/4$  and  $\beta = -1/6$ ; so, in this case,  $m = 3$ .

Now, as  $l \rightarrow \infty$ ,

$$\int_{E_l} \frac{G(0, z)}{z^m(z - u)} dz = \mathcal{C}(G_l),$$

and so, by Lemma 2, this integral tends to zero as  $l \rightarrow \infty$ . Therefore, inserting  $n = m$  in (23) and next letting  $l \rightarrow \infty$  yields the following theorem.

**THEOREM.** *It holds that*

$$G(0, u) = \sum_{k=1}^{m-1} p_{0,k} u^k + \sum_{j=2}^{\infty} \frac{g_j u^m}{u_j^m (u - u_j)}.$$

By investigating the asymptotic behaviour of the poles  $u_j$  and the residues  $g_j$  as  $j \rightarrow \infty$ , it can be seen that  $n = m$  is indeed the smallest nonnegative integer, for which expression (23) is valid. From (18), we obtain that, as  $j \rightarrow \infty$ ,  $u_j \sim (\phi/\xi_1 \alpha) \alpha^j$ , and from (17) and (19) it is easy to show that, as  $j \rightarrow \infty$ ,

$$g_j \sim C \phi \bar{g}_2 \left( \frac{\alpha^2}{\beta} \right)^j,$$

where  $C$  is given by

$$C = \prod_{k=2}^{\infty} \frac{\alpha^{k-1}/\xi_1 - \xi_1/\alpha}{\alpha^{k-1}/\xi_1 - 1/\xi_1} > 0.$$

Hence, for any nonnegative integer  $n$ , as  $j \rightarrow \infty$ ,

$$(24) \quad \frac{g_j}{(u_j)^{n+1}} \sim C \phi \bar{g}_2 \left( \frac{\xi_1 \alpha}{\phi} \right)^{n+1} \left( \frac{1}{\beta \alpha^{n-1}} \right)^j.$$

So, if  $n < m$ , then the series in (23) is divergent.

We now show how the partial fraction decomposition of  $G(0, u)$  leads to the desired expressions for  $p_{0,j}$ . For  $|u| < 1$ , we obtain that (note that  $|u_j| \geq 1$  for all  $j$ )

$$(25) \quad \begin{aligned} \sum_{j=2}^{\infty} \frac{g_j u^m}{u_j^m (u - u_j)} &= - \sum_{j=2}^{\infty} \frac{g_j u^m}{u_j^{m+1}} \sum_{i=0}^{\infty} \frac{u^i}{u_j^i} \\ &= - \sum_{i=0}^{\infty} u^{m+i} \sum_{j=2}^{\infty} \frac{g_j}{u_j^{m+1+i}}, \end{aligned}$$

where interchanging of the summations is allowed, since

$$\sum_{j=2}^{\infty} \sum_{i=0}^{\infty} \frac{|g_j u^{m+i}|}{|u_j^{m+1+i}|} \leq \sum_{j=2}^n \frac{|g_j|}{|u_j|^{m+1}} \frac{1}{1 - |u|}$$



and the right-hand side converges by (24) with  $n = m$ . From (25) and the theorem, the corollary follows.

COROLLARY 1. *It holds that*

$$p_{0,i} = - \sum_{j=2}^{\infty} \frac{g_j}{u_j^{i+1}} \quad \text{for all } i \geq m.$$

From (24) it follows that this expression for  $p_{0,i}$  is not valid for  $i < m$ .

COROLLARY 2. *The series*

$$- \sum_{j=2}^{\infty} \frac{g_j}{u_j^{i+1}}$$

*is divergent for*  $i < m$ .

The feature that the series for  $p_{0,i}$  diverges for small values of  $i$  is not encountered in the shortest-queue problem. In fact, from a result of Kingman [6, Lemma 4], it follows for the shortest-queue problem that the analogue of the lemma formulated in this section is always valid for  $m = 1$ , which implies that the series for the probability  $p_{0,i}$  converges for all  $i \geq 1$ .

**5. Concluding remarks.** It has been shown in this paper that for a multiprogramming model originally studied by Hofri, the generating-function approach is very well suited to derive an expression for the boundary probabilities  $p_{0,j}$  in the form of an infinite sum of powers. The remarkable feature of the multiprogramming model appeared to be that such a representation does not necessarily converge for all  $i$  and  $j$ , whereas in the closely related shortest-queue problem this complication did not occur. By investigating the asymptotic behaviour of the generating-function  $G(0, u)$  of the boundary probabilities  $p_{0,j}$  and that of its poles and residues, it has been shown when such a representation is available and when it is not.

Similarly as for  $p_{0,j}$  series expressions can be obtained for the boundary probabilities  $p_{i,1}$ . In this paper, we did not seek to deduce the partial fraction decomposition for the two-dimensional generating-function  $G(z, u)$ , providing series expressions for  $p_{i,j}$ . Similarly, as for the shortest-queue problem, this analysis is much more complicated than the one for the one-dimensional generating-functions  $G(0, u)$  and  $G(z, 1)$ , and it leads to cumbersome expressions for  $p_{i,j}$ . However, in [3] it is shown that explicit expressions for  $p_{i,j}$  can be obtained by using a compensation approach, which is not based on generating functions, but directly applies to the equilibrium equations. In particular, it appears that the expressions for  $p_{i,j}$  are valid for those  $i$  and  $j$  with  $i + j \geq m$ , and not for smaller  $i$  and  $j$ .

The series expressions for the probabilities  $p_{i,j}$  are easily exploited for numerical calculations and provide insight in the asymptotic behaviour of  $p_{i,j}$ , although such insight may also be obtained more directly from the equilibrium equations; see, e.g., Knessl et al. [7] and [8].

#### REFERENCES

- [1] I. J. B. F. ADAN, J. WESSELS, AND W. H. M. ZIJM, *Analysis of the symmetric shortest queue problem*, Stochastic Models, 6 (1990), pp. 691–713.
- [2] ———, *Analysis of the asymmetric shortest queue problem*, Queueing Systems, 8 (1991), pp. 1–58.
- [3] I. J. B. F. ADAN, G. J. VAN HOUTUM, J. WESSELS, AND W. H. M. ZIJM, *A compensation procedure for multiprogramming queues*, OR Spectrum, 1993, to appear.

- [4] L. FLATTO AND H. P. MCKEAN, *Two queues in parallel*, *Comm. Pure Appl. Math.*, 30 (1977), pp. 255–263.
- [5] M. HOFRI, *A generating-function analysis of multiprogramming queues*, *Internat. J. Comput. Inform. Sci.*, 7 (1978), pp. 121–155.
- [6] J. F. C. KINGMAN, *Two similar queues in parallel*, *Ann. Math. Statist.*, 32 (1961), pp. 1314–1323.
- [7] C. KNESSL, B. J. MATKOWSKY, Z. SCHUSS, AND C. TIER, *Two parallel queues with dynamic routing*, *IEEE Trans. Comm.*, 34 (1986), pp. 1170–1175.
- [8] ———, *Two parallel  $M|G|1$  queues where arrivals join the system with the smaller buffer content*, *IEEE Trans. Comm.*, 35 (1987), pp. 1153–1158.
- [9] E. T. WHITTAKER AND G. N. WATSON, *A Course of Modern Analysis*, Cambridge University Press, Cambridge, UK, 1962.