Dissipativity-based framework for stability analysis of aperiodically sampled nonlinear systems with time-varying delay

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A B S T R A C T

In this paper, we provide novel conditions for stability analysis of aperiodically sampled nonlinear control systems subjected to time-varying delay. The proposed approach can also deal with cases in which delay is larger than the sampling interval. It is applicable to a general class of nonlinear systems and provides sufficient criteria for stability that aid in making trade-offs between control performance and the bounds on sampling interval and delay. As a stepping stone, a preliminary and generic result based on dissipativity, is introduced to analyse the exponential stability of a class of feedback-interconnected systems. The nonlinear sampled-data system is remodelled to consider the effects of sampling and delay in the dissipativity framework, as perturbations to the nominal closed-loop system. This leads to constructive stability conditions for a continuous time closed-loop system given by the feedback interconnection of the nominal closed-loop system and an operator(s) that captures the effects of sampling and delay. For Linear Time-Invariant (LTI) systems, we recover simple Linear Matrix Inequality (LMI) and frequency domain conditions previously proposed in the robust control framework.

1. Introduction

Currently, almost all sampled-data control systems are implemented numerically, and embedded in a networked environment where data is exchanged between sensors, controllers and actuators through digital communication channels (Hespanha, Naghshtabrizi, & Xu, 2007; Zhang, Branicky, & Phillips, 2001). Examples include mobile sensor networks, smart grids, highway systems, etc., see Hespanha et al. (2007).

However, in such control configurations, perturbation effects such as sampling jitter, data-packet dropouts, delays, etc., are often introduced in the network and this impacts the overall stability of the system (Åström & Wittenmark, 1996; Fujioka, 2009; Hespanha et al., 2007; Hetel et al., 2017; Zhang et al., 2001). From the point of view of control theory, such phenomena are considered as sampled-data systems with aperiodic sampling and/or time-varying delay, or more generally, as Networked Control Systems (NCS) (Zhang et al., 2001). In this paper, we focus on the stability analysis problem for aperiodically sampled nonlinear systems subjected to time-varying delay.

Existing literature provides various methods that deal with the stability analysis of sampled-data systems, with or without delay. An overview of different approaches in the case of aperiodic sampled-data systems can be found in Hetel et al. (2017). These approaches are broadly classified into four categories, i.e., the Time-delay approach, the Discrete-time approach, the Hybrid systems approach, and the Input–output approach. The Time-delay approach, has been largely used in the context of Linear Time Invariant (LTI) systems (Seuret, 2012). One of the advantages of this approach is that it can easily handle situations in which delay is greater than sampling period (van de Wouw, Naghshtabrizi, Cloosterman, & Hespanha, 2010). However, it is usually difficult to make a differentiation between sampling induced delay and actuation induced delay. The approach has also been extended to nonlinear systems (Karafyllis & Krstic, 2012; Mazenc, Malisoff, & Dinh, 2013). The Discrete-time approach, has been used for stability analysis of LTI systems (Cloosterman et al., 2010; Fujioka, 2009; van de Wouw et al., 2010) and in some cases, nonlinear
systems (Pepe, Pola, & Benedetto, 2018; van de Wouw, Nesić, & Heemels, 2012). Since it is based on the exact system discretization, it leads to very accurate numerical tools for stability analysis. However, inter-sampling behaviour has been taken into account only in the case of LTI systems, see for example, Cloosterman, van de Wouw, Heemels, and Nijmeijer (2006). Additionally, the application of such discretization-based approach is challenging for general nonlinear systems and for the large-delay case, see Mattioni, Monaco, and Normand-Cyrot (2018) and Polushin and Marquez (2004). The Hybrid system approach, was developed based on the fact that systems with sampling-and-hold in control and sensor signals can be modelled using impulsive systems (Heemels, Teel, van de Wouw, & Nesić, 2010). In the LTI systems case, by using Impulsive Delay Differential Equations, situations when delay is greater than the sampling interval were also studied (Liu, Fridman, & Hetel, 2012). However, for nonlinear systems, the analysis has only been done for cases in which delay is less than the sampling interval (Borgers, Geiselhart, & Heemels, 2017; Postoyan, Tabuada, Nesić, & Anta, 2015).

The Input–output approach, treats the error induced by sampling and/or delay as a perturbation to the continuous-time control system and captures its effects using an operator (Kao & Lincoln, 2004; Thomas, Hetel, Fiter, van de Wouw, & Richard, 2018). This approach is intuitively simple to develop and the stability analysis problem is related to the classical robust control framework (Fujioka, 2009; Mirkin, 2007). A primary advantage of this approach is that it can easily include perturbations as well as nonlinearities. However, in contrast to LTI systems, this approach has been used for stability analysis in the presence of sampling, and delay, only separately. The existing results only provide $L_2$–stability criteria for LTI systems. Generally, it can be shown that this implies asymptotic stability of the LTI sampled-data system. However, in such cases, it is difficult to describe the system performance, even in terms of the transient decay-rate. In the case of nonlinear systems, this approach has been employed to analyse stability only in the case of aperiodic sampling in the absence of delay (Omran, Hetel, Petreczky, Richard, & Lamnabhi-Lagarrigue, 2016). Providing constructive conditions for stability of nonlinear systems with aperiodic sampling and time-varying delay is largely an open problem.

In this paper, we provide a novel framework to analyse the stability of aperiodically sampled nonlinear systems subjected to time-varying delay, using an approach inspired from the notion of dissipativity (Willems, 2007). The main contributions of this paper are as follows. We introduce a constructive approach that is applicable to a general class of aperiodically sampled nonlinear systems with time-varying delays, even in the scenario when delay is greater than the sampling interval. We provide two tractable exponential stability conditions by taking into account the specific discontinuities in delay, as well as inter-sampling and inter-actuation behaviour. The dissipativity-based approach proposed in this paper leads to conditions in terms of dissipativity type properties of the associated continuous-time system, for which many results for classes of nonlinear systems exist in literature. Additionally, the approach provides bounds on operator(s) characterizing sampling, hold and delay effects. The proposed results also aid in deciding the trade-off between system decay-rate, and the bounds on sampling interval and delay. As a stepping stone, we introduce a primary result that provides exponential stability conditions for a class of feedback interconnected systems, which bear relevance to a range of problems in the robust control framework. The first criterion caters to the so-called ‘large delay case’, which delineates the situation arising often in information transmission over shared networks, where the delay introduced to the data packet exceeds the sampling interval of the sensors. The second criterion, a less conservative one, deals with the ‘small delay case’ where delay is less than the sampling period. This scenario has been studied in numerous theoretical as well as practical settings (see Cloosterman et al., 2006; Xiao, Shi, & Ren, 2018; Zhang et al., 2001). For example, in Cloosterman et al. (2006), it was shown that in the case of a single sensor sampling periodically, when the sampled-data experienced delays less than sampling-interval, the system was rendered unstable. The problem becomes much more complex when the sensors and actuators involved have aperiodic sampling and actuation frequencies. In our analysis for the small-delay case, two separate operators are used to capture the effects of sampling and delay. In the case of LTI systems, we recover simple LMI and frequency domain conditions previously proposed in the robust control framework (Kao & Lincoln, 2004; Mirkin, 2007).

The outline of this paper is as follows. In Section 2, we introduce the problem setting which comprises of a generic aperiodically sampled nonlinear system subjected to time-delay. In Section 3, a preliminary stability result in the exponential dissipativity framework is provided, for a class of feedback interconnected systems. Section 4 deals with the stability analysis of the nonlinear sampled-data system under the large-delay case. It begins with a model reformulation of the problem setting in terms of the feedback interconnection introduced in Section 3. Next, the remodelled system properties are exploited to formulate a required supply function that will be used to provide a stability criterion by employing the result introduced in Section 3. Section 5 introduces the stability analysis of the nonlinear sampled-data system in the small-delay case, and follows a similar outline as Section 4. In Section 6, examples are provided to corroborate the effectiveness of the proposed results in the nonlinear as well as linear case. Finally, conclusions and an insight into possible future work are given in Section 7. The proofs of the results introduced in this paper, if not given in the main body of the paper, are given in the Appendices.

Notations: Throughout the paper, we denote $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $\|x\|$. For a time-varying vector $z(t) \in \mathbb{R}^n$, $\dot{z}(t)$ is the Dini derivative given by $\dot{z}(t) \equiv \lim_{\Delta t \to 0^+} \sup \frac{z(t+\Delta t) - z(t)}{\Delta t}$. We denote $\mathbb{R}^m$ as the set of all piecewise continuous $n$-dimensional functions over $\mathbb{R}^+$. The notation $\mathbb{N}^*$ is used to denote the set $\{1, 2, \ldots\}$. The set of all continuously differentiable functions is denoted by $C^1$, and the set of all continuous functions is denoted by $C^0$. The maximum and minimum eigen values of a matrix $M \in \mathbb{R}^{n \times n}$ are denoted by $\delta_{\text{max}}$ and $\delta_{\text{min}}$, respectively. The Euclidean norm of a matrix $M$ is given by $\|M\|_2 = \sqrt{\lambda_{\text{max}}(M^T M)}$.

2. Problem statement

Consider the nonlinear system

$$\dot{x}_p(t) = f(x_p(t)) + g(x_p(t))u(t), \forall t \geq 0, \tag{1}$$

with the nonlinear sampled-data control

$$u(t) = \begin{cases} 0, & \forall t \in [0, a_0), \\ \kappa(x_p(s_k)), & \forall t \in [a_k, a_{k+1}), \ k \in \mathbb{N}, \end{cases} \tag{2}$$

where $x_p(t) \in \mathbb{R}^n$ is the system state, $x_p(0) = x_0^p$ and $u(t) \in \mathbb{R}^{m_p}$ is the control input based on the continuous time signal

$$u_c(t) = \kappa(x_p(t)), \forall t \geq 0, \tag{3}$$

subjected to sampling and delay. It is assumed that in the absence of sampling and delay, the origin of system (1) with $u(t) = u_c(t)$, is exponentially stable. The functions $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(0) = 0$, $g : \mathbb{R}^n \to \mathbb{R}^{n \times m_p}$ are globally Lipschitz, and the function $\kappa : \mathbb{R}^n \to \mathbb{R}^{m_p}$ belongs to $C^1$. The time instants $s_k$ and $a_k$ specify the sampling instants (when sensors send the measured state...
value to the controller) and actuation instants (when the control input is updated at the actuator level) respectively. We consider a sampling sequence \( s_k \in \mathbb{N} \) satisfying
\[
s_{k+1} = s_k + h_k, \quad \forall k \in \mathbb{N},
\] (4)
where the time-varying sampling interval \( h_k \) satisfies \( 0 < h_k \leq h \leq \bar{h}, \forall k \in \mathbb{N} \). Similarly, we consider the actuation sequence \( \{a_k\}_{k \in \mathbb{N}} \) such that
\[
a_k = s_k + \tau_k, \quad \forall k \in \mathbb{N},
\] (5)
where \( \tau_k \) is the time-varying delay between sampling and actuation instants and satisfies \( 0 \leq \tau \leq \tau_k \leq \bar{\tau}, \forall k \in \mathbb{N} \).

**Hypothesis 1.** The actuation instants satisfy
\[
a_k < a_{k+1}, \quad \forall k \in \mathbb{N}.
\] (6)
This assumption allows the bound on delay, \( \bar{\tau} \), to be greater than the bound on sampling interval, \( \bar{h} \), but under the constraint that the actuation instants occur in an order corresponding to the sampling instants. Without loss of generality, we consider that the first actuation occurs at time \( a_0 = \bar{\tau} + h \), while the first sampling instant is \( s_0 = a_0 - \tau_0 \). This assumption can also be ensured with a time-scale shift. Throughout the paper, \( \mathcal{P} \) denotes the nonlinear closed-loop sampled-data system defined by (1), (2), (4)–(6). The objective of this paper is to analyse the exponential stability of the system \( \mathcal{P} \).

### 3. Preliminary generic stability result

In this paper, we will use the fact that system \( \mathcal{P} \) can be remodelled as the feedback-interconnection given by
\[
\Sigma : \begin{cases}
\dot{x}(t) = f_0(x(t)) \\
y(t) = h_0(x(t))
\end{cases} \quad \forall t \in [0, a_0),
\] (7)
where \( x(t) \in \mathbb{R}^n \), \( y(t) \in \mathbb{R}^p \), and the operator \( \Delta : \mathbb{W}^p \mapsto \mathbb{V}^m \) such that
\[
\omega = \Delta \psi.
\] (8)
The function \( f_0 \) in (7) is considered to be globally Lipschitz, with a Lipschitz constant \( k_0 \) and \( f_0(0) = 0 \). Additionally, we consider that the functions \( f, g, h \) and \( \bar{l} \) are sufficiently smooth. We assume that solutions exist for the feedback interconnection \( \Sigma - \Delta \). We shall denote the feedback interconnection (7)–(8) by \( \Sigma - \Delta \). Such interconnection models will be introduced in Sections 4 and 5, wherein the functions introduced in (7) and (8) will also be detected. This will also establish the relation between the dimensions \( n \) introduced in (7) and \( n_0 \) introduced in (1). Prior to presenting such models, we will formulate, a technical result concerning exponential stability of \( \Sigma - \Delta \). This result will serve as a stepping stone for the stability analysis of systems of the form (1), (2), (4)–(6).

**Theorem 1.** Consider the feedback interconnection \( \Sigma - \Delta \) and the following assumptions:

#### Assumption 1: There exists a supply function \( S : \mathbb{R}^+ \times \mathbb{R}^p \times \mathbb{R}^m \mapsto \mathbb{R} \)
continuous in all parameters satisfying the integral constraint
\[
\int_0^1 S(\theta, \phi(\theta), (\Delta \phi)(\theta))d\theta \leq 0, \quad \forall t \geq 0, \phi \in \mathbb{W}^p.
\] (9)

#### Assumption 2: There exists a continuously differentiable storage function \( V : \mathbb{R}^m \mapsto \mathbb{R}^+ \) and scalars \( 0 < c_1 < c_2 \), and \( q > 0 \) such that
\[
c_1 \|x\|^q \leq V(x) \leq c_2 \|x\|^q.
\] (10)

**Assumption 3:** There exist scalars \( \lambda \in \mathbb{R} \) and \( \rho > 0 \) such that the inequalities
\[
-\delta(t, \gamma(t), \alpha(t)) \leq \rho (V(x(t))) \quad \forall t \in [0, a_0),
\] (11)
\[
\dot{V}(x(t)) \geq \lambda V(x(t)), \quad t \in [0, a_0),
\] (12)
\[
\dot{V}(x(t)) + \alpha V(x(t)) \leq e^{-\alpha(t-a_0)}\delta(t, \gamma(t), \alpha(t)) \quad \forall t \geq a_0,
\] (13)
are satisfied for some \( \alpha > 0 \), along the solutions of the system \( \Sigma - \Delta \).

Then \( \Sigma - \Delta \) is exponentially stable with a decay-rate of at least \( \alpha/\bar{q} \), i.e., \( \exists \delta > 0 : \forall t \geq 0, \|x(t)\| \leq e^{\alpha/\bar{q}}\|x(0)\| \).

### 4. Stability analysis for the large delay case

In this section, we provide a constructive approach for applying Theorem 1 to analyse the stability of system \( \mathcal{P} \) introduced in Section 2. The term ‘large delay’ signifies Hypothesis 1, which implies that the delay \( \tau_k \) can indeed be greater than the sampling interval \( h_k \), under the constraint that the actuation instants occur in order. Theorem 1 can be used in this scenario by reformulating the system \( \mathcal{P} \) as an interconnection of the form \( \Sigma - \Delta \) given by (7)–(8), so that the effects of sampling and delay are included as a perturbation. In order to do so, we define the perturbation induced by sampling and delay as
\[
e(t) = \begin{cases}
0, \quad \forall t \in [0, a_0),
\kappa(x_p(s_k)) - \kappa(x_p(t)) \quad \forall t \in [a_k, a_{k+1}), k \in \mathbb{N}.
\end{cases}
\] (14)
For all \( t \geq a_0 \), \( e(t) \) can be interpreted as the ‘error’ on the control action when compared to a continuous time controller as given in (3). We will introduce an operator \( \Delta \) that helps in expressing the error \( e(t) \) in an alternate manner. Additionally, we provide the functions introduced in (7), so that the dynamics of the interconnection \( \Sigma - \Delta \) and the sampled-data system \( \mathcal{P} \) are equivalent.

#### 4.1. System model reformulation

In this section, we introduce a particular case of operator \( \Delta \) in (8), with \( m = p = m_p \), that captures the perturbation (14). Subsequently, the system \( \mathcal{P} \) given by (1), (2), (4)–(6) is reformulated in terms of a feedback interconnection of the form \( \Sigma - \Delta \) in (7), (8).
Lemma 2. Consider the operator \( \Delta : \mathcal{W}^{mp} \mapsto \mathcal{W}^{mp} \) defined for any signal \( z \in \mathcal{W}^{mp} \) as
\[
(\Delta z)(t) = \begin{cases} 0, & \forall t \in [0, a_0), \\ \int_{a_k}^{a_{k+1}} z(s)ds, & \forall t \in [a_k, a_{k+1}), \ k \in \mathbb{N}, \end{cases}
\]
and the derivative of the continuous control in (3),
\[
\dot{u}_1(t) = \frac{d}{dt} \kappa(x_1(t)).
\]
Then, the sampling and delay induced error \( e \) defined in (14) can be expressed as \( e = \Delta \dot{u}_1 \).

We show next how the sampled-data system \( \mathcal{P} \) can be remodelled in the format \( \Sigma - \Delta \) given by (7), (8). This formulation in conjunction with Lemma 2 is used to prove the equivalence between the sampled-data system \( \mathcal{P} \) and the interconnection \( \Sigma - \Delta \).

Lemma 3. Consider the system \( \Sigma \) in (7), with
\[
\begin{align*}
\tilde{f}_0(x) &= f(x), \quad \tilde{h}_0(x) = \frac{d}{dx} f_0(x), \\
\tilde{f}(x) &= f(x) + g(x) v(x), \\
\tilde{g}(x) &= g(x), \quad \tilde{h}(x) = \frac{d}{dx} f(x), \quad \tilde{h}(x) = \frac{d}{dx} g(x),
\end{align*}
\]
and the derivative of the continuous control in (3),
\[
\dot{u}_1(t) = \frac{d}{dt} \kappa(x_1(t)).
\]
Then, the sampling and delay induced error \( e \) defined in (14) can be expressed as \( e = \Delta \dot{u}_1 \).

Remark. The aforementioned theorem provides (only) sufficient stability conditions based on the existence of a storage function. In the following sections, we will present how this can be used in a constructive manner based on LMI and Sum of Squares (SOS) criteria. In Section 6, we will illustrate with examples, how Theorem 5 can be used to provide stability conditions for nonlinear sampled-data systems of the form given by \( \mathcal{P} \). In Section 6.1, for an exemplary nonlinear system, we will show how the matrix \( \Delta \) characterizing the supply function, can be tuned by use of standard MATLAB functions.

4.3. Stability criteria for linear systems

Consider the linear sampled-data system \( \mathcal{P}_L \) given by
\[
\begin{align*}
x(t) &= A x(t) + B u(t), \quad \forall t \geq 0, \\
u(t) &= \begin{cases} 0, & \forall t \in [0, a_0), \\
K_0, & \forall t \in [a_0, a_{k+1}), \ k \in \mathbb{N},
\end{cases}
\end{align*}
\]
with \( x(0) = x_0 \), where \( x(t) \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times n}, \) and \( K_0 \in \mathbb{R}^{m \times n} \). Now, we provide a stability criterion for the linear sampled-data system \( \mathcal{P}_L \) in the form of tractable LMI.

Theorem 6. Consider \( \alpha \in \mathbb{R}^+ \). The linear sampled-data system \( \mathcal{P}_L \) is exponentially stable with a decay-rate \( \alpha/2 \) if there exists \( P > 0 \) and \( R = R^T > 0 \) such that
\[
\begin{bmatrix}
\tilde{A}^T P + \tilde{P}^T A + \alpha \tilde{P}^T B B^T \tilde{P} & PB^T \tilde{P} \\
K \tilde{A} & K \tilde{B}
\end{bmatrix} > 0,
\]
with \( \tilde{A} = A + BK, \) and \( \gamma^2 = (\hat{h} + \bar{\gamma})^2 e^{(\hat{h} + \bar{\gamma})} \).

Remark. Applying the Kalman–Yakubovich–Popov Lemma, we can infer that the LMI given by (22) is equivalent to the frequency-domain criterion \( ||G||_\infty < 1/\gamma \), where \( G \) is the operator defined by the transfer function \( \hat{G}(s) = K \tilde{A} (sI - \tilde{A} - \frac{\alpha}{2}I)^{-1}B + KB \). This result is fact a generalization of the results provided in Kao and Lincoln (2004) and Mirkin (2007). We have extended the results in Kao and Lincoln (2004) and Mirkin (2007) by providing stability conditions for non-linear sampled-data systems while guaranteeing a constant gain and a constant value of the exponential decay-rate. If \( \alpha = 0, \) and \( \hat{h} = 0, \) we recover the result in Kao and Lincoln (2004). Similarly, if \( \alpha = 0, \) and \( \bar{\gamma} = 0, \) we recover the result provided in Mirkin (2007).

5. Stability analysis for the small delay case

The large-delay case studied in Section 4 is more generic to processes communicating via a shared network, where traffic flow can increase considerably. However, in some cases, it has been shown that it is desirable to have delay less than sampling interval since sampled data arriving in a non-chronological order at the actuator can be hazardous from a control point of view (Åström & Wittenmark, 1996). Consequently, this would
make the implementations of algorithms and analysis much more complex. In this section, we will demonstrate how considering sampling and delay separately in the small-delay case, gives a less conservative stability criterion. The following assumption is considered throughout the section.

**Hypothesis 2.** The actuation based on the sampled state $x(s_k)$ is implemented before the next sampling instant $s_{k+1}$, i.e.,

$$\tau_k < h_k, \forall k \in \mathbb{N}. \quad (23)$$

Next, we re-formulate the sampled-data model for system $\mathcal{P}$ in order to include the effects of sampling and delay using two separate errors, denoted by $e_1(t)$ and $e_2(t)$, respectively. Consider the continuous-time control $u_c(t) = \kappa(x_c(t))$. The sampled version of this signal is $\bar{u}_c(t) = \kappa(x_c(s_k))$, $\forall t \in [s_k, s_{k+1})$, $k \in \mathbb{N}$. The sampling-induced error $e_1(t)$ is $e_1(t) = u_c(t) - u_c(t)$. Without loss of generality, we consider that $e_1(t) = 0$, $\forall t < s_0$. Then,

$$e_1(t) = \begin{cases} 0, & \forall t \in [0, s_0), \\ \kappa(x_c(s_k)) - \kappa(x_c(s_k)), & \forall t \in [s_k, s_{k+1}), k \in \mathbb{N}. \end{cases} \quad (24)$$

The delayed version of $u_c(t)$ is the control signal $u(t)$ applied to the level of the actuator. We introduce another error $e_2(t)$, which can be given by $u(t) - u_c(t)$. Note that we can define the error $e_2(t) = 0$, $\forall t < s_0$, since it bears no relevance. Formally, $e_2(t)$ is given by

$$e_2(t) = \begin{cases} 0, & \forall t \in [0, a_0), \\ 0, & \forall t \in [a_{k-1}, a_k), k \in \mathbb{N}^*, \\ \kappa(x_c(s_k)) - \kappa(x_c(s_k)), & \forall t \in [a_k, a_{k+1}), k \in \mathbb{N}^*. \end{cases} \quad (25)$$

Using this formulation for $e_1(t)$ and $e_2(t)$, given by (24) and (25), respectively, we proceed to reformulate the sampled-data system $\mathcal{P}$ in the form of $\Sigma - \Delta$.

### 5.1. System model reformulation

In this section, we introduce two different operators $\Delta_s$ and $\Delta_d$, which capture the errors induced by sampling and delay given in (24) and (25), respectively. In an approach similar to the one used in Section 4.1, system $\mathcal{P}$ under Hypothesis 2, i.e., (23), can be represented as a feedback interconnection of the form $\Sigma - \Delta$.

**Lemma 7.** Consider the operator $\Delta : \mathcal{V}_{2m+1} \mapsto \mathcal{V}_{2m+1}$

$$\Delta : \phi = \begin{pmatrix} v \\ w \end{pmatrix} \mapsto (\Delta \phi) = \begin{pmatrix} \Delta_s \phi \\ \Delta_d \phi \end{pmatrix}, \forall \phi \in \mathcal{V}_{2m+1}, w \in \mathcal{V}_{2m+1}, \quad (26)$$

under Hypothesis 2, i.e., (23), where

$$(\Delta_s \phi)(t) = \begin{cases} 0, & \forall t \in [0, s_0), \\ \int_{s_k}^{t} \phi(t) d\theta, & \forall t \in [s_k, s_{k+1}), k \in \mathbb{N}, \end{cases} \quad (27)$$

and

$$(\Delta_d \phi)(t) = \begin{cases} 0, & \forall t \in [0, a_0), \\ 0, & \forall t \in [a_{k-1}, a_k), k \in \mathbb{N}^*, \\ \int_{s_k}^{t} \phi(t) d\theta, & \forall t \in [a_k, a_{k+1}), k \in \mathbb{N}^*. \end{cases} \quad (28)$$

Then, the sampling and delay induced errors defined in (24) and (25), respectively, can be expressed as

$$\begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} = \begin{pmatrix} \Delta_s e_1 \\ \Delta_d e_2 \end{pmatrix}, \quad (29)$$

with $e_1(t)$ given by (16).

Analogous to the approach used in Section 4, we now proceed to reformulate the sampled-data system $\mathcal{P}$ under Hypothesis 2, i.e., (23), in the format $\Sigma - \Delta$ given by (7), (8). In the following lemma, by using such a model reformulation along with Lemma 7, we provide the equivalence between the sampled-data system $\mathcal{P}$ under Hypothesis 2, and the feedback interconnection $\Sigma - \Delta$.

**Lemma 8.** Consider the system $\Sigma$ in (7), with

$$\tilde{f}_0(x) = f(x), \quad \tilde{h}_0(x) = \begin{pmatrix} \frac{\partial x}{\partial x} f(x) \\ \frac{\partial x}{\partial x} g(x) \end{pmatrix},$$

$$\tilde{f}(x) = f(x) + g(x)\kappa(x), \quad \tilde{g}(x) = \begin{pmatrix} g(x) \\ g(x) \end{pmatrix}, \quad (30)$$

$$\tilde{h}(x) = \begin{pmatrix} \frac{\partial x}{\partial x} \tilde{f}(x) \\ \frac{\partial x}{\partial x} \tilde{g}(x) \end{pmatrix},$$

$n = n_p, m = p = 2m_p, x_0 = x^{\theta}_0$ and the operator $\Delta$ in (8), defined by (26), (27) and (28) under Hypothesis 2, i.e., (23). Then, the sampled-data system $\mathcal{P}$ can be expressed as the feedback interconnection $\Sigma - \Delta$, with $x = x^\theta$.

Lemmas 7 and 8 are used to provide constructive stability criterion for sampled-data system $\mathcal{P}$ under Hypothesis 2. To this end, the supply function $S$ given in Theorem 1 needs to be formulated. We proceed in this direction by studying the properties of operators $\Delta_s$ and $\Delta_d$.

#### 5.2. Stability analysis

In this section, we characterize the properties of $\Delta_s$ and $\Delta_d$ by functions $S_s$ and $S_d$, respectively. Consequently, we formulate the supply function $S = S_s + S_d$.

**Lemma 9.** Consider the operator $\Delta_s$ defined in (27), $\beta \in \mathbb{R}^+$ and $R_e \in \mathbb{R}^{m_p \times m_p}$ with $R_e = R_e^T > 0$. Then,

$$\int_0^t S_s(\theta, \nu(\theta), (\Delta_s v(\theta))) d\theta \leq 0, \quad \forall \theta \geq 0, \nu \in \mathcal{V}_{2m^p}, \quad (31)$$

where the function $S_s : \mathbb{R}^+ \times \mathbb{R}^{m_p} \times \mathbb{R}^{m_p} \mapsto \mathbb{R}$ is defined as

$$S_s : (\theta, v, \mu) \mapsto e^{\theta(\beta - \alpha_0)} \left[ \frac{v^T}{\mu} \right] \left[ \begin{array}{c} -\gamma_s^2 R_e \\ \gamma_s^2 \beta R_e \\ \gamma_s^2 \beta R_e \left( 1 - \gamma_s^2 \beta^2 R_e \right) R_e \end{array} \right] \left[ \begin{array}{c} v \\ \mu \end{array} \right], \quad (32)$$

with $\gamma_s = \frac{2h}{\pi}$.

**Lemma 10.** Consider $\Delta_d$ defined in (28) under Assumption 2, $\beta \in \mathbb{R}^+$ and $R_d \in \mathbb{R}^{m_p \times m_p}$ with $R_d = R_d^T > 0$. Then, for all $w \in \mathcal{V}_{2m^p}$,

$$\int_0^t S_d(\theta, w(\theta), (\Delta_d w(\theta))) d\theta \leq 0, \quad \forall \theta \geq 0, \quad (33)$$

where the function $S_d : \mathbb{R}^+ \times \mathbb{R}^{m_p} \times \mathbb{R}^{m_p} \mapsto \mathbb{R}$ is defined as

$$S_d : (\theta, w, \epsilon) \mapsto e^{\theta(\beta - \alpha_0)} \left[ \begin{array}{c} w^T \\ \epsilon \end{array} \right] \left[ \begin{array}{c} -\gamma_d R_d \\ 0 \\ R_d \end{array} \right] \left[ \begin{array}{c} w \\ \epsilon \end{array} \right], \quad (34)$$

with $\gamma_d = \frac{h^2 e^{h+\frac{1}{h}}}{\pi}$. The functions $S_s$ and $S_d$ given in Lemmas 9 and 10, provide the sampling and delay component, respectively, of the supply function $S = S_s + S_d$. As follows, we use the supply function $S = S_s + S_d$ to provide a general, more accurate stability criterion for the sampled-data system $\mathcal{P}$ under Hypothesis 2, i.e., when delay is less than sampling interval.
Theorem 11. Consider system $\mathcal{P}$, the interconnection $\Sigma - \Delta$ given by (7), (8), (26), (27), (28) and (30). If there exist functions $S = S_1 + S_2$ defined using (32) and (34), and $V : \mathbb{R}^n \mapsto \mathbb{R}^+$ that satisfy assumptions (10), (11), (12) and (13), then system $\mathcal{P}$ is exponentially stable with a decay-rate $\alpha/q$.

Proof. We exploit Lemma 8 to establish the equivalence between system $\mathcal{P}$ under Hypothesis 2 and $\Sigma - \Delta$ in (7), (8). Then, by Lemmas 9 and 10, Assumption 1 in Theorem 1 is satisfied for the operator $\Delta$ defined by (26), (27) and (28). Under the conditions of the theorem, Assumptions 2 and 3 of Theorem 1 are satisfied. Applying Theorem 1, $\Sigma - \Delta$ is proved to be exponentially stable and by equivalence, so is system $\mathcal{P}$. ■

The result presented in Theorem 6 holds for any positive symmetric definite matrices $R_1$ and $R_2$ characterizing the supply function. In Section 6.1, we will illustrate how Theorem 11 provides less conservative results for the sampled-data system $\mathcal{P}$ under Hypothesis 2, i.e., for the small delay case. The usage of numerical tools to tune matrices $R_1$ and $R_2$ will also be shown.

5.3. Stability criterion for linear systems

In this section, we recall the linear sampled-data system $\mathcal{P}_l$ described in Section 4.3 by (20). Based on Lemmas 9 and 10, we provide the following stability criterion for system $\mathcal{P}_l$ under Hypothesis 2.

Theorem 12. Consider a scalar $\alpha \in \mathbb{R}^+$ and Hypothesis 2. The linear sampled-data system $\mathcal{P}_l$ is exponentially stable with a decay-rate $\alpha/2$ if there exist $P = P^T > 0$, $R_1 = R_1^T > 0$, and $R_2 = R_2^T > 0$ such that

$$\begin{bmatrix} A^T P + PA + \alpha P & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} K A & K B \\ 0 & I \end{bmatrix}^T \begin{bmatrix} K A & K B \end{bmatrix} < 0,$$

with $A = A + BK$, $B = [B \ B]$, and

$$\Phi = \begin{bmatrix} \gamma^2 R_1 + \gamma \alpha R_2 & -\gamma^2 \frac{1}{2} R_2 \\ -\gamma^2 \frac{1}{2} R_2 & \gamma^2 \frac{1}{2} R_2 - \alpha \frac{1}{2} R_2 \end{bmatrix},$$

where $\gamma = \frac{2h}{\alpha}$ and $\gamma = \hat{h} \tau e^{\alpha(h+\hat{t})}$.

Remark. When $\alpha = 0$, $\hat{t} = 0$ (implying no delay component in $\mathcal{S}$), the LMI (35) translates to a form similar to LMI (22). Consequently, by virtue of the Kalman–Yakubovich–Popov lemma, we can recover the frequency domain condition introduced in Mirkin (2007), i.e., $\|G\|_{\infty} < \frac{2}{\alpha}$, where $G$ is the operator defined by the transfer function $G(s) = KA(s - A - \frac{u}{I})^{-1}B + KB$.

In Section 6.2, we will illustrate with examples, how the LMI (35) provides less conservative results for LTI systems under Hypothesis 2, i.e., for the small delay case.

6. Illustrative examples

In this section, we illustrate the effectiveness of our proposed results via examples. The provided examples highlight the difference between the single-error approach and the separate-error approach in terms of conservativeness and trade-offs between control performance and the bounds on sampling interval and delay. The result presented in this paper provides a foundation for deciding the trade-off between maximum delay $\hat{h}$, maximum sampling period $\hat{h}$, and decay-rate $\alpha$. By fixing one of the parameters, the trade-off between the remaining parameters can be obtained. For example, by fixing $\hat{t}$, and gridding over $\hat{h}$ and $\alpha$, a trade-off between the decay-rate and the maximum allowable sampling interval can be obtained. In a similar manner, fixing $\hat{h}$ will give the trade-off between $\alpha$ and $\hat{t}$, and so on.

6.1. Nonlinear system example

We consider the following example (Karafyllis & Kravaris, 2007; Nesic, Teel, & Carnevale, 2009; Omran et al., 2016),

$$\dot{x}(t) = dx(t)^2 - x(t) + u(t),$$

with a bounded time-varying parameter $|d(t)| \leq 1$, and a stabilizing control $u(t) = k(x(t)) = -2x(t)$ subjected to both sampling and delay. Since the function $f(x) = x^2 - x^3$ is locally Lipschitz, our results will only hold locally on any compact set containing the origin.

6.1.1. Large-delay case

Using the definition in (17), we reformulate the system model in the form $\Sigma - \Delta$, where $\Sigma$ is given by

$$\dot{x}(t) = dx(t)^2 - x(t) + u(t),$$

and $y(t) = -2dx(t)^2(t) - x(t) + u(t)$. We use a storage function of the form $V(x) = ax^2 + bx^4$ as given in Omran et al. (2016). Using (19), we obtain the supply function $S(\theta, y, w) = e^{\theta - \gamma} \left[Ru^2(\theta) - \gamma^2 Ry^2(\theta)\right]$, with $\gamma = (\hat{h} + \hat{t})^2 e^{\alpha(h+\hat{t})}$. For this case, from condition (13), we can infer that the values of $\hat{h} + \hat{t}$ satisfying the inequality

$$2ax + 4bx^3|x^2 - x^3 - 2x + w| + a(ax^2 + bx^4) - R\omega^2 + 4(\hat{h} + \hat{t})^2 e^{\alpha(h+\hat{t})} R(dx^2 - x^3 - 2x + w)^2 \leq 0,$$

will guarantee exponential stability. If (39) can be expressed as a Sum of Squares (SOS) for all the values of $(d, d^2)$ in $\{1, 0, 1, 1, -1, 0, -1, 1\}$, then it will be SOS for any time-varying $|d(t)| \leq 1$. Using SOSTOOLS (Papachristodoulou et al., 2013), Fig. 1 provides the feasible values of $\hat{h}$ and $\hat{t}$ (in red) for $\alpha = 0.1$, and all values of $(d, d^2)$. It can be seen from Fig. 1 that, for $\alpha = 0.1$, $\hat{h}$ and $\hat{t}$ satisfy a maximum bound $\hat{h} + \hat{t} \leq 0.45$, with $a = 0.7079$, $b = 0.1890$ and $R = 0.4268$. The parameters $a$, $b$ and $R$ are optimized using SOSTOOLS. Additionally, the trade-off between the desired decay-rate $\alpha/2$ and $\hat{h} + \hat{t}$ is shown in Fig. 2.
about 60% while using the result provided in the small-delay case. Additionally, when $h = 0.27$, the feasible values of $\bar{\tau}$ in the large and small-delay cases, are approximately up to 0.17 and 0.27, respectively, showing an improvement of about 59%. Using these numerical arguments, it can be concluded that for the small-delay case, capturing the effects of sampling and delay using two separate errors gives less conservative results. However, the amount of improvement in the small-delay case over the large-delay case depends on the parameter $\alpha$. We illustrate this in the following section for a linear system example.

**Remark.** The less-conservative nature of the results proposed in the small-delay case can also be justified from a theoretical perspective. In the large-delay case, the supply function was formulated using Jensen’s inequality, which introduces conservativeness (Briat, 2011). On the other hand, in the small-delay case, Wirtinger’s inequality has been used. For this case, the improvement over Jensen’s inequality is well known in the literature (Seuret & Gouaisbaut, 2013).

For the same example in the absence of delay, in Karafyllis and Kravaris (2007) and Nesic, Teel, and Carnevale (2009), upper-bounds of 0.368 and 0.143, respectively, were found. However, in Omran et al. (2016), an upper-bound of 0.72 was proposed for the system (37) without delay, with $\alpha = 0.1$. From the results proposed for the small-delay case, by setting $\bar{\tau} = 0$, indicating sampling without any delay, we can see in Fig. 1 that we obtain the same upper-bound of 0.72 on the sampling intervals, as proposed in Omran et al. (2016), with $\alpha = 2.9153 \times 10^{-6}$, $b = 7.29 \times 10^{-7}$, $R_a = 1.6964 \times 10^{-6}$ and $R_2 = 1.2465$. However, our results have added advantage that we provide tractable stability conditions for the nonlinear sampled-data system in the presence of time-varying delay.

6.2. Linear system example

Consider the system (20) characterized by the parameters (Zhang, 2001)

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}, \quad K = - \begin{bmatrix} 1 & 6 \end{bmatrix}.$$ (44)

By virtue of Theorem 6, we can compute the maximum allowable values of $\bar{h} + \bar{\tau}$ with respect to $\alpha$ from the LMI (22). The LMI (22) is solved using YALMIP, by optimizing parameters $P$ and $K$, for different values of $\alpha$ and $\bar{h} + \bar{\tau}$. The feasibility region thus obtained will aid in deciding the trade-off between a desired decay rate while taking into account the maximum bounds on sampling interval and delay. Considering $\alpha \in \{0.01, 1, 2\}$, we obtain the bounds on $\bar{h}$ and $\bar{\tau}$ as shown in Fig. 3 (in red solid, dashed and dotted lines). For the LTI system (44), if $\alpha = 0$ and $\bar{h} = 0$, we recover the bounds on $\bar{\tau}$ as given in Kao and Lincon (2004). For the chosen values of $\alpha \in \{0.01, 1, 2\}$, we also compute the bounds on $\bar{h}$ and $\bar{\tau}$ in the small-delay case (as shown in Fig. 3 in blue solid, dashed and dotted lines). Following a similar explanation as given in Section 6.1.2, we can conclude that for the small-delay case, differentiating the effects of sampling and delay using two separate errors, the LMI in (35) introduced in Theorem 12 provides less conservative results in comparison to the criterion provided in (22) (applied to the small-delay case). Fig. 3 also gives the dependence of the amount of improvement in the small-delay case over the large-delay case, on the parameter $\alpha$. If $\alpha = 0$ and $\bar{\tau} = 0$, we recover the bound on $\bar{h}$ as proposed in Mirkin (2007). Therefore, we can conclude that by applying our generic nonlinear tools to the linear case, we provide bounds on $\bar{h}$ and $\bar{\tau}$ that are not more conservative in comparison to the bounds.
provided in Kao and Lincoln (2004) and Mirkin (2007). Also, it has to be noted that despite the fact that the condition in (22) is more conservative when applied to the small-delay case, the result is still important since it is applicable to the more generic large-delay case.

7. Conclusion

In this paper, novel approaches for stability analysis of aperiodically sampled nonlinear systems with time-varying delay are provided. The framework introduced in this paper holds for a general class of nonlinear systems and provides tools that help in deciding required trade-offs between the system decay-rate and the bounds on sampling interval and delay. As a preliminary result, an approach inspired from the notion of exponential dissipativity is used to provide stability conditions for a class of feedback interconnected systems, while guaranteeing a desired decay-rate. The non-linear sampled-data system is modelled as a feedback interconnection of the nominal closed-loop system and an operator that captures the effects of sampling and delay, thereby leading to constructive stability conditions. The proposed approach leads to conditions on dissipativity properties of the system, for which many results exist in literature. When applying the results to LTI case, we see that they generalize existing frequency domain and LMI conditions in the robust stability framework. For the case when delay is less than sampling interval, a less conservative stability criterion is obtained by considering two separate operators to capture the effects of sampling and delay. The effectiveness of the proposed theoretical results have been corroborated via simulation results for an exemplary nonlinear system. We foresee numerous extensions. For example, a more realistic scenario would involve multiple sensors and actuators, each with unique bounds on sampling interval and delay (Fiter, Korabi, Etienne, & Hetele, 2018; Thomas et al., 2018).

Appendix A. Proof of Theorem 1

Let us first upper-bound the response $x(t)$ for all $t \geq a_0$. Consider the function

$$W(t) = e^{\alpha(t-a_0)} V(x(t)) - \int_{0}^{t} S(\theta, y(\theta), \omega(\theta)) d\theta, \forall t \geq a_0. \tag{A.1}$$

From condition (13), we have $W(t) \leq 0$, for all $t \geq a_0$ and therefore $W(t) \leq W(a_0)$, for all $t \geq a_0$, which can be stated as

$$e^{\alpha(t-a_0)} V(x(t)) - \int_{0}^{t} S(\theta, y(\theta), \omega(\theta)) d\theta \leq V(x(a_0)).$$

Then, for all $t \geq a_0$, we have

$$V(x(t)) \leq e^{-\alpha(t-a_0)} \left[ -\int_{0}^{a_0} S(\theta, y(\theta), \omega(\theta)) d\theta + \int_{0}^{t} S(\theta, y(\theta), \omega(\theta)) d\theta + V(x(a_0)) \right], \tag{A.2}$$

and by using (9), for all $\theta \geq 0$, we have

$$V(x(t)) \leq e^{-\alpha(t-a_0)} \left[ -\int_{0}^{a_0} S(\theta, y(\theta), \omega(\theta)) d\theta + V(x(a_0)) \right]. \tag{A.3}$$

By integrating condition (12) for all $t \in [0, a_0]$, we have

$$V(x(a_0)) \geq e^{\alpha(a_0-t)} V(x(t)), \forall t \in [0, a_0]. \tag{A.4}$$

Then, by integrating condition (11) and using (A.4) for all $t \in [0, a_0]$, we have

$$-\int_{0}^{a_0} S(\theta, y(\theta), \omega(\theta)) d\theta \leq \rho \int_{0}^{a_0} V(x(t)) d\theta \leq \rho \int_{0}^{a_0} e^{\alpha(\theta-a_0)} V(x(a_0)) d\theta = \eta V(x(a_0)),$$

where

$$\eta := \frac{e^{-\alpha a_0}}{\rho a_0},$$

with $\rho > 0$ (1 > 1. Then, from (10), we obtain for all $t \geq a_0$,

$$c_1 \|x(t)\|^q \leq V(x(t)) \leq e^{-\alpha(t-a_0)} c_2 V(x(a_0)) \leq e^{-\alpha(t-a_0)} c_2 \|x(a_0)\|^q,$$

and thus

$$\|x(t)\| \leq \sqrt{\frac{c_2}{c_1}} \|x(a_0)\|, \forall t \geq a_0. \tag{A.6}$$

Now, let us analyse the response in the interval $t \in [0, a_0]$. Using the definition of system $\Sigma$ in (7) for all $t \in [0, a_0]$, we have

$$x(t) = \tilde{f}_0(x(t)), \text{ where } \tilde{f}_0$$

is globally Lipschitz continuous with some constant $k_0$ and $\tilde{f}_0(0) = 0$. Hence, we have that $x(t) - x(0) = \int_{0}^{t} \tilde{f}_0(x(s)) ds$, implying, using the Triangular Inequality, that

$$\|x(t)\| \leq \|x(0)\| + \int_{0}^{t} \|\tilde{f}_0(x(s))\| ds. \tag{A.7}$$

Since $\tilde{f}_0$ is Lipschitz continuous and $\tilde{f}_0(0) = 0$, we have $\|\tilde{f}_0(x(s))\| = \|\tilde{f}_0(x(s)) - \tilde{f}_0(0)\| \leq k_0 \|x(s)\| - 0 = k_0 \|x(s)\|$. Consequently, (A.7) leads to

$$\|x(t)\| \leq \|x(0)\| + k_0 \int_{0}^{t} \|x(s)\| ds.$$
Appendix C. Proof of Lemma 3

Consider the system $\mathcal{P}$ in (1), (2), (4)–(6). Moreover, consider the following notational relations:

$$y(t) = \dot{u}_t(t), \quad (C.1)$$

with $\dot{u}_t(t)$ given by (16), and $\omega(t) = e(t)$, with $e(t)$ defined by (14). By virtue of Lemma 2, we have, $\omega(t) = e(t) = (\Delta \dot{u}_t)(t), \forall t \geq 0$.

(1) For all $t \in [0, a_0]$: As per the definition of system $\mathcal{P}$, we have

$$\dot{y}_t(t) = f(x(t)), \quad (C.2)$$

and using (16),

$$y(t) = \dot{u}_t(t) = \frac{d}{dt}k(x(t)) = \frac{d}{dx}\left(\frac{f(x(t))}{f_t(x(t))}\right). \quad (C.3)$$

Using (17), (C.2) and (C.3) this is equivalent to $\dot{y}_t(t) = \frac{d}{dx}\left(\frac{f(x(t))}{f_t(x(t))}\right)$.

(2) For all $t \in [a_0, a_{0+1}]$: The dynamics of system $\mathcal{P}$ is given by $\dot{y}_t(t) = f(x(t)) + g(x_t(t))\omega(t) = f(x(t)) + g(x_t(t))\omega(x_t(t))$. Using (17), and recalling the definition of $e(t)$ in (14), we obtain $\dot{y}_t(t) = f(x(t)) + g(x_t(t))\omega(x_t(t))$. This is equivalent to the dynamics of system $\mathcal{P}$ for $t \geq a_0$, given by (7), with $\omega(t) = e(t)$ and the functions $f$ and $g$ defined by (17), i.e., for all $t \geq a_0$, we have $x = x_t$.

Additionally, from (C.1) and (16) we have, $y(t) = \frac{d}{dx}k(x(t)) = \frac{d}{dx}\left(\frac{f(x(t))}{f_t(x(t))}\right)\omega(x_t(t)).$ Once again, using (17) and (e(t) = o(t)), we have, $y(t) = h(x_t(t)) + h(x_t(t))o(t)$, which is the same as defined in (7), for $t \geq a_0$, since we have already shown $x = x_t$.

Therefore, we can see that system $\mathcal{P}$ can be expressed as the feedback interconnection $\Sigma - \Delta$, with the functions $f_0$, $h_0$, $f$, $g$, $\dot{h}$, and $I$ defined by (17).

Appendix D. Proof of Lemma 4

(1) For $t \in [0, a_0)$: Using the definition of $\Delta$ in (15), we have $(\Delta x)(t) = 0$, for all $t \in [0, a_0)$, and clearly (18) holds in this case since $S(t, x(t), (\Delta x)(t)) = -\gamma^2(\dot{z}(t))Rz(t) \leq 0$.

(2) For $t \geq a_0$: Let $w$ denote

$$w(t) = (\Delta x)(t) = \int_{t_k}^{t} z(\dot{\zeta})d\dot{\zeta}, \forall t \in [a_0, a_{1}], \forall k \in \mathbb{N}. \quad (D.1)$$

Using Jensen's inequality, we obtain

$$w^2(t)Rw(t) \leq (t - t_k) \int_{t_k}^{t} z^2(\dot{\zeta})Rz(\dot{\zeta})d\dot{\zeta} \leq (\bar{h} + \bar{r}) \int_{t_k}^{t} z^2(\dot{\zeta})Rz(\dot{\zeta})d\dot{\zeta}. \quad (D.2)$$

Using the change of variable $s = \zeta - t_k$, we obtain

$$w^2(t)Rw(t) \leq (h + r) \int_{t_k}^{t} z^2(s)Rz(s)ds. \quad (D.3)$$

Since the inner integral on the right-hand side of the inequality in (D.2) is always positive, we can upper bound the left-hand side in (D.2) using the limits of $s$ and obtain

$$w^2(t)Rw(t) \leq (h + r) \int_{t_k}^{t} z^2(s)Rz(s)ds.$$

Since the inner integral on the right-hand side of the inequality in (D.2) is always positive, we can upper bound the left-hand side in (D.2) using the limits of $s$ and obtain

$$w^2(t)Rw(t) \leq (h + r) \int_{t_k}^{t} z^2(s)Rz(s)ds.$$

Hence, using the definition of $w(t)$ in (D.1), we have

$$w^2(t)Rw(t) \leq (h + r) \int_{t_k}^{t} z^2(s)Rz(s)ds.$$

Using the inequality in (22), we proceed to prove that the assumptions introduced in Theorem 1 will hold for $V(x) = x^TPx$ and $S(t, x(t), w(t))$ defined by (19). For the LTI system $\mathcal{P}_1$, the functions given in (17) are given by $\dot{y}_t(t) = Ax(t), h_0(x(t)) = KAx(t), f(x(t)) = Ax(t), h(x_t(t)) = B, h_0(x(t)) = KAx(t)$, and $I(x(t)) := KB$, where $A = (A + BK)$.

(1) Satisfaction Assumption 1, i.e., (9): By virtue of Lemma 4, we can see that the supply function $S(t, x(t), w(t))$ defined by (19) satisfies assumptions (9) in Theorem 1, i.e., $\int_{0}^{T} S(t, \theta, \gamma(t), (\Delta \gamma)(t))d\theta \leq 0, \forall t \geq 0$.

(2) Satisfaction Assumption 2, i.e., (10): With $V(x) = x^TPx$, $P > 0$ and $x \in \mathbb{R}^n$, we have that $\delta_{min}(P)\|x\|^2 \leq x^TPx \leq \delta_{max}(P)\|x\|^2$, implying Assumption 2 is satisfied with $q = 2, c_1 = \delta_{min}(P)$ and $c_2 = \delta_{max}(P)$.

(3) Satisfaction Assumption 3, inequality (11): Consider the function $S(t, x(t), w(t))$ defined by (19). For all $t \in [0, a_0)$, with $\gamma(t) = h_0(x(t)) = KAx(t)$ and $\omega(t) = 0$, we have for all $t \in [0, a_0)$,

$$-S(t, \gamma(t), \omega(t)) = -\int_{0}^{T} S(t, \theta, \gamma(t), (\Delta \gamma)(t))d\theta \leq 0, \forall t \geq 0, \forall \theta \in [0, a_0).$$

(4) Satisfaction Assumption 3, inequality (12): We have $V(x) = x^TPx$ for all $t \geq 0$. For all $t \in [0, a_0)$, $\dot{x}_t(t) = \dot{x}_0(t) = Ax(t)$, $V(x(t)) = x^TPx$, and therefore, inequality (12) is satisfied for any $\lambda \leq \lambda_{max}(P)^{1/2} \lambda_{max}(P)^{-1/2}$. Hence, inequality (12) is satisfied.

(5) Satisfaction Assumption 3, inequality (13): Consider the function $W(t) = V(x(t)) + \alpha V(x(t)) = -e^{-\omega(t-a_0)}S(t, x(t), e(t))$, defined for all $t \geq a_0$ with $V(x) = x^TPx$, and the function $S(t, x(t), e(t))$ defined by (19). Clearly, inequality (13) in Assumption 3 holds if $W(t) \leq 0$, for all $t \geq a_0$. We have, $S(t, x(t), e(t)) = e^{-\omega(t-a_0)}S(t, x(t), e(t)) = e^{-\omega(t-a_0)}S(t, x(t), e(t))$, where

\[ N = \begin{bmatrix} \bar{K} & \bar{A} \\ 0 & I \end{bmatrix}, \quad \bar{Y} = \begin{bmatrix} -\gamma^2R & 0 \\ 0 & R \end{bmatrix} \quad \text{and} \quad \bar{Y} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \]

Therefore, we have that for all $t \geq a_0$, $W(t) = V(x(t)) + \alpha V(x(t)) = -e^{-\omega(t-a_0)}S(t, x(t), e(t)).$
which is guaranteed by \((22)\). Consequently, we have proved that inequality (13) in Assumption 3 is satisfied for the chosen storage and supply functions.

We have shown that all the assumptions of Theorem 1 hold for \(V(x) = x^T P x + S(t, y(t), e(t))\) defined by (19) and, therefore, the exponential stability of system \(\mathcal{P}\) is guaranteed with a decay rate greater than or equal to \(\alpha/2\).

**Appendix F. Proof of Lemma 7**

(1) **Expressing \(e\), using \(\Delta_0\):** Recalling the definition of \(e(t)\) in (24), and by using the operator definition for \(\Delta_0\) in (27), we can state using (16) that for all \(t \in [0, s_0]\), \(e(t) = 0 = (\Delta_0 u)(t)\). Similarly, for all \(t \in [s_0, s_0 + 1)\), \(e(t) = \kappa(x_s) - \kappa(x_s) = -\int_{s_0}^t \kappa(x_s) dx - \int_{s_0}^t u(t) dx = (\Delta_0 u)(t)\). Hence, we have \(e(t) = (\Delta_0 u)(t), \forall t \geq 0\).

(2) **Expressing \(e^2\):** Using \(\Delta_0\) in a similar manner, using the definition of \(e^2(t)\) in (25) and the operator definition for \(\Delta_0\) defined in (28), we have, for all \(t \in [0, a_0] \cup [a_{k-1}, s_k)\), \(e(t) = 0 = (\Delta_0 u)(t)\), where \(e(t)\) and \(e(t)\) given by (24) and (25), respectively. By virtue of Lemma 7, we have

\[
\begin{align*}
\dot{t} & = \frac{\kappa(x_s)}{\kappa(x_s)} e^{-\kappa(x_s)} f(x(t)) \text{ with } f(x(t)) \text{ defined by (25).}
\end{align*}
\]

with \(\Delta_0 \) and \(\Delta_0 \) given in (27) and (28), respectively. In order to establish the equivalence between system \(\mathcal{P}\) and the feedback interconnection \(\Sigma - \Delta\), we begin by reformulating the dynamics of system \(\mathcal{P}\) for all \(t \in [0, a_0], t \in [a_k, s_0)\), and \(t \in [s_0, a_0 + 1)\), respectively, for all \(t \geq 0\).

(1) **For all \(t \in [0, a_0]$:** Consider the system \(\mathcal{P}\), the notations \(y(t) = [y_1(t) \ y_2(t)]^T = [u_1(t) \ u_2(t)]^T\), with \(u_2\) defined by (16), and \(o(t) = [e(t) \ e(t)]^T\), with \(e(t)\) and \(e(t)\) given by (24) and (25), respectively. By virtue of Lemma 7, we have

\[
\begin{align*}
\dot{t} & = \frac{\kappa(x_s)}{\kappa(x_s)} e^{-\kappa(x_s)} f(x(t)) \text{ with } f(x(t)) \text{ defined by (25).}
\end{align*}
\]

with \(\Delta_0 \) and \(\Delta_0 \) defined by (27) and (28), respectively. In order to establish the equivalence between system \(\mathcal{P}\) and the feedback interconnection \(\Sigma - \Delta\), we begin by reformulating the dynamics of system \(\mathcal{P}\) for all \(t \in [0, a_0], t \in [a_k, s_0)\), and \(t \in [s_0, a_0 + 1)\), respectively, for all \(t \geq 0\).

(2) **For all \(t \in [a_k, s_0]$, \(k \in \mathbb{N}$:** The dynamics of system \(\mathcal{P}\) is given by \(x(t) = x(t) + g(x(t)) \dot{u}(t) = f(x(t)) + g(x(t)) \kappa(x_s) = f(x(t)) + g(x(t)) \kappa(x_s) + g(x(t)) \kappa(x_s)\), \(\kappa(x_s)\), and \(\kappa(x_s)\). Using the definitions of \(\Delta_0\) and \(\Delta_0\), we have \(e(t) = \kappa(x_s) - \kappa(x_s)\), \(\forall t \in [a_k, s_0], e(t) = \kappa(x_s) - \kappa(x_s)\), \(\forall t \in [a_k, s_0 + 1)\), and \(e(t) = 0 = (\Delta_0 u)(t)\), respectively,

\[
\begin{align*}
\dot{t} & = \frac{\kappa(x_s)}{\kappa(x_s)} e^{-\kappa(x_s)} f(x(t)) \text{ with } f(x(t)) \text{ defined by (25).}
\end{align*}
\]

with \(\Delta_0 \) and \(\Delta_0 \) defined by (27) and (28), respectively. In order to establish the equivalence between system \(\mathcal{P}\) and the feedback interconnection \(\Sigma - \Delta\), we begin by reformulating the dynamics of system \(\mathcal{P}\) for all \(t \in [a_k, s_0], x \in \mathbb{R}^n, e(t) = \kappa(x_s) - \kappa(x_s)\), \(\forall t \in [a_k, s_0 + 1)\), and \(e(t) = 0 = (\Delta_0 u)(t)\), respectively,

\[
\begin{align*}
\dot{t} & = \frac{\kappa(x_s)}{\kappa(x_s)} e^{-\kappa(x_s)} f(x(t)) \text{ with } f(x(t)) \text{ defined by (25).}
\end{align*}
\]

with \(\Delta_0 \) and \(\Delta_0 \) defined by (27) and (28), respectively. In order to establish the equivalence between system \(\mathcal{P}\) and the feedback interconnection \(\Sigma - \Delta\), we begin by reformulating the dynamics of system \(\mathcal{P}\) for all \(t \in [a_k, s_0], x \in \mathbb{R}^n, e(t) = \kappa(x_s) - \kappa(x_s)\), \(\forall t \in [a_k, s_0 + 1)\), and \(e(t) = 0 = (\Delta_0 u)(t)\), respectively,

\[
\begin{align*}
\dot{t} & = \frac{\kappa(x_s)}{\kappa(x_s)} e^{-\kappa(x_s)} f(x(t)) \text{ with } f(x(t)) \text{ defined by (25).}
\end{align*}
\]
Substituting (H.2) into inequality (H.1), we have for all $t \in [s_k, s_{k+1})$, $k \in \mathbb{N}$, $v(t) \in \mathcal{W}^{m,p}$, such that $\xi(\theta, (\bar{\Delta}t), (\bar{\Delta}v)) \neq \emptyset$, where we have used that $(t - s_k) \leq \bar{h}$ for all $t \in [s_k, s_{k+1})$. Now, for any $t \in [s_k, s_{k+1})$, since $(\Delta v)(t) = 0$, $\forall \theta \in [0, \sigma_0)$ (see (27)), we can state that $\int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta = \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta = 0$ (see (27)).

\[ \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta = 0. \]

(1.2)

We know that for $t \in [s_k, s_{k+1})$, $(\Delta_g)(t) = 0$. Additionally, using the upper bound in (1.2), we have that

\[ \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta \leq \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta. \]

\[ \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta \leq 0. \]

\[ \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta = 0. \]

(1.3)

Next, we simplify each of the integrals present on the right side of the inequality above. First, consider the term $\int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta$. Hence, $\int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta \leq \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta$. This leads to

\[ \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta \leq \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta. \]

\[ \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta \leq \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta. \]

(1.4)

Thus, by combining (1.5)–(1.6), we have that $\int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta \leq \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta$. This gives us

\[ \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta \leq \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta. \]

\[ \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta \leq \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta. \]

\[ \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta \leq \int_0^t e^{\theta(\bar{\Delta}t)}((\Delta v)(t)) \theta R(\theta, (\bar{\Delta}v)(t)) \theta d\theta. \]

(1.7)

Appendix J. Proof of Theorem 12

Let us recall the linear sampled-data system $P_t$ described in Section 4.3 by (20). The sampling-induced error is given by

\[ e_s(t) = \{K_s(K_t) - K_s(\bar{v})\} \theta, \forall \theta \in [0, \sigma_0), \]

where $\Delta$ is given by (27). Similarly, the delay-induced error is given by $e_d(t) = \{K_s(\bar{v}) - K_s(\bar{v})\} \theta, \forall \theta \in [0, \sigma_0)$. Therefore, we have

\[ e_s(t) = \{K_s(K_t) - K_s(\bar{v})\} \theta, \forall \theta \in [0, \sigma_0), \]

\[ e_d(t) = \{K_s(\bar{v}) - K_s(\bar{v})\} \theta, \forall \theta \in [0, \sigma_0). \]

(1.8)
\((\Delta_\delta(Kx))(t)\), where \(\Delta_\delta\) is given by (28). Additionally, the functions defined in (30) are given by

\[
\tilde{f}_0(x(t)) = Ax(t), \quad \tilde{h}_0(x(t)) = \begin{bmatrix} KAx(t) \\ KAX(t) \end{bmatrix},
\]

\[
f(x(t)) = \tilde{A}x(t), \quad \tilde{g}(x(t)) = \begin{bmatrix} B \\ B \end{bmatrix},
\]

\[
h(x(t)) = \begin{bmatrix} KAX(t) \\ KAX(t) \end{bmatrix}, \quad \tilde{h}(x(t)) = \begin{bmatrix} KB \\ KB \end{bmatrix}.
\]

Let us consider that condition (35) holds. Then, we proceed to prove that the assumptions introduced in Theorem 1 will hold for the storage function \(V(x) = x^T P x\) and the supply function \(S: \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}\) given by

\[
S(t, y(t), o(t)) = \begin{cases} 
S_1(t, \begin{bmatrix} 1 & 0 \end{bmatrix}y(t), \begin{bmatrix} 1 & 0 \end{bmatrix}o(t)) & \text{if } t \in [0, a_0), \\
S_2(t, \begin{bmatrix} 0 & 1 \end{bmatrix}y(t), \begin{bmatrix} 0 & 1 \end{bmatrix}o(t)) & \text{if } t \in [a_0, \infty).
\end{cases}
\]

Since \(y_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\) and \(o_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\) are defined by (32) and (34), respectively, with \(\beta = \alpha\). Additionally, based on the functions given in (J.1), we have \(y_1(t) = y_2(t) = Kx(t)\). Let us now show that the assumptions in Theorem 1 are validated.

1. (Satisfying Assumption 1, i.e., (9)): By virtue of Lemmas 9 and 10, we have that \(\int_{t_0}^{t} S_2(\theta, y(\theta), (\Delta_\delta)\theta) \, d\theta \leq 0\), \(\forall t \geq 0\), and \(\int_{t_0}^{t} S_2(\theta, y(\theta), (\Delta_\delta)\theta) \, d\theta \to 0\) as \(t \to 0\). Consequently, as per the definition of the supply function in (J.2), we obtain \(\int_{t_0}^{t} S(\theta, y(\theta), (\Delta_\delta)\theta) \, d\theta \leq 0\), \(\forall t \geq 0\).

2. (Satisfying Assumption 2, i.e., (10)): With \(V(x) = x^T P x\) and \(P \succ 0\) and \(x \in \mathbb{R}^n\), we have \(\delta_{\min}(P)|x|^2 \leq x^T P x \leq \delta_{\max}(P)|x|^2\), implying Assumption 2 is satisfied with \(q = 2\), \(c_1 = \delta_{\min}(P)\) and \(c_2 = \delta_{\max}(P)\).

3. (Satisfying Assumption 3, inequality (11)): Consider the function \(S(t, y(t), o(t))\) defined in (J.1). Then, we need to prove that \(-S(t, y(t), o(t)) \leq \rho V(x(t)), \forall t \in (0, a_0]\). We proceed to prove this inequality by considering the time intervals \([0, a_0)\) and \([a_0, \infty)\) separately.

For all \(t \in [0, a_0)\): Using the definition of system \(\Sigma\) in (7), (30), and the operator \(\Delta\) defined in (8), (28) we have that \(y(t) = \tilde{h}_0(x(t)) = \begin{bmatrix} KAX(t) \\ KAX(t) \end{bmatrix}\), and \(o(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\), \(\forall t \in [0, a_0)\).

Hence, for all \(t \in [0, a_0)\), \(-S(t, y(t), o(t)) = -S(t, \tilde{h}_0(x(t)), 0) = -(S_1(t, KA\tilde{x}(t), 0) + S_2(t, KA\tilde{x}(t), 0)) = e^{-(t-a_0)} \chi(t)(\Delta_\delta) (y^2_{x_1 R_1} + y^2_{x_2 R_2} + y_{x_1 R_1} R_1 + y_{x_2 R_2} R_2)\).

Therefore, \(-S(t, y(t), o(t)) \leq \rho_1 V(x(t))\),

\[
\rho_1 = \max_{\{0, a_0\}} \left[ \frac{\delta_{\min}(P)}{\delta_{\min}(P)} \right],
\]

where \(\chi_1 = \frac{\delta_{\min}(P)}{\delta_{\min}(P)}\) and \(\chi_2 = \frac{\delta_{\max}(P)}{\delta_{\min}(P)}\).

For all \(t \in [a_0, \infty)\): From (J.1), we have \(y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}\) with \(y_1(t) = y_2(t) = KA\tilde{x}(t)\),

\[
y(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = \begin{bmatrix} Kx_0 - Kx(t) \\ 0 \end{bmatrix}. \quad \text{Since the system is in open loop for all } t \in [a_0, \infty), e^{(t-a_0)} \chi(x(t)), \text{we therefore have that}
\]

\[
\begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = \begin{bmatrix} K \left[ e^{(t-a_0)} - I \right] x(t) \end{bmatrix}, \quad \forall t \in [a_0, \infty).
\]

Now, consider the function \(S_2\) defined in (32). Since we have already shown in Lemma 7 that \((\Delta_\delta(y)\tilde{y})(t) = e_{\delta(t)}\), we have that \(S_2(t, y_1(t), (\Delta_\delta)\tilde{y})(t) = S_1(t, y_1(t), e_{\delta(t)})\),

\[
e^{(t-a_0)} \begin{bmatrix} y_1(t) \\ e_1(t) \end{bmatrix} \begin{bmatrix} y^2_{x_1 R_1} + y^2_{x_2 R_2} + (1 - y^2_{x_1 R_1}) e_{\delta(t)} \end{bmatrix},
\]

and thus, from (J.5) and (J.6), we get \(S_1(t, y_1(t), e_{\delta(t)}) = S_1(t, KA\tilde{x}(t), K \left[ e^{(t-a_0)} - I \right] x(t)) = x^T(t) M x(t), \forall t \in [0, a_0)\), where \(M(t) = e^{(t-a_0)} \begin{bmatrix} KA \\ K \left[ e^{(t-a_0)} - I \right] \end{bmatrix} \left[ -y_{x_1 R_1} \begin{bmatrix} R_1 \\ R_1 \end{bmatrix} \begin{bmatrix} y_{x_1 R_1} R_1 \\ y_{x_2 R_2} R_2 \end{bmatrix} (1 - y_{x_1 R_1}^2) \right] e_{\delta(t)}\).

Similarly, considering the function \(S_2\) defined by (34), we have that

\[
S_2(t, y_2(t), (\Delta_\delta(y)\tilde{y})(t)) = S_2(t, y_2(t), e_{\delta(t)}) = e^{(t-a_0)} \begin{bmatrix} y_2(t) \\ e_2(t) \end{bmatrix} \begin{bmatrix} -y^2_{x_1 R_1} R_1 \\ 0 \\ R_1 \\ 0 \end{bmatrix}, \quad \forall t \in [0, a_0),
\]

and thus, from (J.5) and (J.6), \(S_2(t, y_2(t), e_{\delta(t)}) = S_2(t, KA\tilde{x}(t), 0) = x^T(t) M x(t), \forall t \in [0, a_0)\), with \(M(t) = -y^2_{x_1 R_1} e^{(t-a_0)} K\begin{bmatrix} R_1 \\ R_1 \end{bmatrix}\). Therefore, we have the total supply function \(S\) satisfying \(-S(t, y(t), o(t)) = -S_1(t, y_1(t), e_{\delta(t)}) - S_2(t, y_2(t), e_{\delta(t)}) = x^T(t) M x(t), \forall t \in [0, a_0), \text{where } M(t) = -M(t) - N(t)\). Hence, for all \(t \in [0, a_0)\), we can state that

\[-S(t, y(t), o(t)) \leq \rho_2 V(x(t))\],

where \(\rho_2 = \max_{\{0, a_0\}} \left[ \frac{\delta_{\min}(M(t))}{\delta_{\max}(P)} \right]\).
\[
\begin{pmatrix}
x(t) \\
e(t)
\end{pmatrix}
\begin{bmatrix}
\phi \\
\psi
\end{bmatrix}
\begin{pmatrix}
\xi(t) \\
\eta(t)
\end{pmatrix} = \begin{pmatrix}
\bar{A}'P + P\bar{A} + \alpha P
\end{pmatrix} + \begin{pmatrix}
\bar{B}'P
\end{pmatrix}, \quad \text{with} \quad \Gamma = \begin{pmatrix}
\bar{A}'P + P\bar{A} + \alpha P \\
\bar{B}'P
\end{pmatrix} + \begin{pmatrix}
\bar{B}'P
\end{pmatrix}
\]


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