Robust stabilization of systems with multiplicative perturbations

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Abstract

In this paper we discuss robust stabilization of systems subject to multiplicative perturbations. For a given ball of perturbed systems around our nominal model, we find necessary and sufficient conditions under which we can find a controller which stabilizes each system in this ball. We give an explicit formula for a number γ^* with the property: the radius of the ball is less than γ^* if and only if there exists one controller which stabilizes all systems in the ball. Therefore γ^* is the best we can do. Moreover, under an extra assumption and if they exist, we derive an explicit formula for a controller which stabilizes all systems in the given ball.

Keywords: H_∞ control, algebraic Riccati equation, quadratic matrix inequality.

1 Introduction

Any model we make of a plant will contain inaccuracies. Hence, if we have a certain goal for our plant then, while designing a controller on the basis of our model, we have to keep in mind that our model is inaccurate. In this paper we want to stabilize our model in such a way that we can guarantee closed loop stability when we apply this controller to our plant. In literature one distinguishes three main cases:

- Additive perturbations
- Multiplicative perturbations
- Normalized co-prime factor perturbations

For each of these cases we want to find a controller which stabilizes all perturbed models which satisfy:

(i) the perturbed systems lie in a ball around the nominal model with respect to the L_∞ norm.

(ii) the perturbed model and the nominal model have the same number of unstable poles.

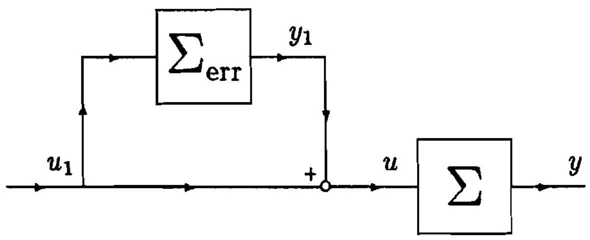
Clearly one wants to do this for a ball which is as large as possible. Using the small gain theorem one can show (see [10, 15]) that for each one of the above classes of perturbations the problem can be reduced to the problem of finding an internally stabilizing controller for the nominal system which satisfies an H_∞ norm bound on a related closed loop system.
The results for additive perturbations are known (see [5]): the radius of the largest ball of perturbed systems for which one controller exists which stabilizes all systems in this ball, is equal to the smallest Hankel singular value of the nominal model. Moreover, explicit formulas are available to derive such a controller.
Also for the case of normalized co-prime factor perturbations (see [10]) results are available. One investigates perturbations of a normalized co-prime factorization of our nominal model. We want to find the radius of the largest ball of perturbed models (in fact a ball of factorizations of perturbed models), which we can stabilize by one controller. This maximal radius can be expressed in terms of the Hankel norm of the co-prime factors. Again an explicit method is available to derive such a controller.
However, for multiplicative perturbations there was no explicit formula available for the radius of the largest ball around our nominal model for which one controller exists which stabilizes all systems in this ball. As we noted before all three cases can be reduced to an H_∞ control problem. The problem with multiplicative perturbations is that in general we find a singular two-block H_∞ control problem. Contrary to this, additive perturbations and normalized co-prime factor perturbations yield regular one-block H_∞ control problems. The regular one-block H_∞ control problem was the first H_∞ problem to be solved since it turned out to be equivalent to the so-called Nehari problem (see [2, 4]). Contrary to this, the singular H_∞ control problem was solved only very recently (see [12, 13]).
In this paper we investigate multiplicative perturbations. We will show that the results on the singular H_∞ control problem give us a rather straightforward method to derive explicitly the supremum of the radius of all balls around our nominal model for which there exists a controller which stabilizes each system in this ball. However, contrary to the other cases, this supremum will in general not be attained.

2 Problem formulation and results

We assume that we have a continuous time system Σ, being an imperfect model of a certain plant P, of the form:

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du. \end{cases} \tag{2.1}$$

We assume that the error is multiplicative, i.e. we assume that the plant P is exactly described by the following interconnection:



(2.2)

Here Σ_{err} is some arbitrary system such that the interconnection (2.2) has the same number of unstable poles as Σ. In other words, we assume that the plant is described by the system Σ interconnected as in diagram (2.2) with another system Σ_{err}. The system Σ_{err} represents the uncertainty. Our goal is to find conditions under which there exists a controller Σ_F of the form:

$$\Sigma_F : \begin{cases} \dot{p} = Kp + Ly, \\ u_1 = Mp + Ny. \end{cases} \tag{2.3}$$

such that the interconnection (2.2) is stabilized by this controller for all systems Σ_{err} which do not change the number of unstable poles of the interconnection (2.2) and which have $\mathcal{L}_\infty$ norm less than *or equal to* some, a priori given, number γ. The $\mathcal{L}_\infty$ norm is defined by:

$$\|G\|_\infty := \sup_{\omega \in \mathcal{R}} \|G(i\omega)\|$$

where $\|\cdot\|$ denotes the largest singular value. In [15] there is a result for multiplicative perturbations:

Lemma 2.1 : *Let a system Σ and a controller Σ_F of the form (2.1) and (2.3) respectively be given. The following conditions are equivalent:*

(*i*) *If we apply the controller Σ_F from y to u_1 to the interconnection (2.2), then the closed loop system is well-posed and internally stable for every system Σ_{err} such that*

- *Σ_{err} has $\mathcal{L}_\infty$ norm less than or equal to γ,*
- *The interconnection (2.2) and Σ have the same number of unstable poles.*

(*ii*) *Σ_F internally stabilizes Σ and if G and G_F denote the transfer matrices of Σ and Σ_F, respectively, then $I - GG_F$ is invertible as a proper rational matrix and*

$$\|GG_F(I - GG_F)^{-1}\|_\infty < \gamma^{-1}.$$

□

Next, we will state a version of the result from [13, 14] which is our main tool to derive results for the problem of robust stabilization with multiplicative perturbations. We first need some definitions. We consider the linear, time-invariant, finite-dimensional system:

$$\Sigma_e : \begin{cases} \dot{x} = Ax + Bu + Ew, \\ y = C_1x + D_{11}u + D_{12}w, \\ z = C_2x + D_{21}u, \end{cases} \tag{2.4}$$

A central role will be played by the quadratic matrix inequality. For any $\gamma > 0$ and matrix $Q \in \mathcal{R}^{n\times n}$ we define the following matrix:

$$G_\gamma(Q) := \begin{pmatrix} AQ + QA^{\mathrm{T}} + EE^{\mathrm{T}} + \gamma^2 QC_2^{\mathrm{T}}C_2Q & QC_1^{\mathrm{T}} + ED_{12}^{\mathrm{T}} \\ C_1Q + D_{12}E^{\mathrm{T}} & D_{12}D_{12}^{\mathrm{T}} \end{pmatrix}.$$

If $G_\gamma(Q) \geq 0$, we say that Q is a solution of the quadratic matrix inequality at γ. In addition to this matrix, we define a matrix pencil:

$$M_\gamma(Q,s) := \begin{pmatrix} sI - A - \gamma^2 QC_2^{\mathrm{T}}C_2 \\ -C_1 \end{pmatrix}.$$

We note that $M_\gamma(Q,s)$ is the observability pencil associated with the system:

$$\begin{cases} \dot{x} = (A + \gamma^2 QC_2^{\mathrm{T}}C_2)\, x, \\ y = \qquad -C_1 x. \end{cases}$$

We define the following transfer matrix:

$$G_{di}(s) := C_1\,(sI - A)^{-1}\,E + D_{12},$$

In the formulation of the result from [13, 14] we also require the concept of *invariant zero* of the system $\Sigma = (A, B, C, D)$. These are all $s \in C$ such that

$$\operatorname{rank} \begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix} < \operatorname{normrank} \begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix}.$$

Here "*normrank*" denotes the rank of a matrix as a matrix with entries in the field of rational functions. Moreover, let C^+ (C^0) denote all $s \in C$ such that Re $s > 0$ (Re $s = 0$). Finally, let $\rho(M)$ denote the spectral radius of the matrix M. We are now in a position to formulate the result from [13, 14] which we shall use:

Theorem 2.2 : *Consider the system (2.4). Let $\gamma > 0$ be given. Assume that both the system (A, B, C_2, D_{21}) as well as the system (A, E, C_1, D_{12}) have no invariant zeros on the imaginary axis. Finally, assume that D_{21} is injective. Then the following two statements are equivalent:*

(*i*) *For the system (2.4) there exists a time-invariant, finite-dimensional dynamic compensator Σ_F of the form (2.3) such that the resulting closed loop system, with transfer matrix G_{cl}, is internally stable and has H_∞ norm less than γ^{-1}, i.e. $\|G_{cl}\|_\infty < \gamma^{-1}$.*

(*ii*) *There exist positive semi-definite matrices P and Q such that the following conditions are satisfied:*

(*a*) $0 = A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2 + \gamma^2 PEE^{\mathrm{T}}P - (PB + C_2^{\mathrm{T}}D_{21})(D_{21}^{\mathrm{T}}D_{21})^{-1}(B^{\mathrm{T}}P + D_{21}^{\mathrm{T}}C)$

(*b*) *The matrix A_{cl} is asymptotically stable where:*

$$A_{cl} := A - B\,(D_{21}^{\mathrm{T}}D_{21})^{-1}\,(B^{\mathrm{T}}P + D_{21}^{\mathrm{T}}C) + \gamma^2 EE^{\mathrm{T}}P.$$

(*c*) $G_\gamma(Q) \geq 0$,

(*d*) *rank* $G_\gamma(Q) = normrank\, G_{di}$,

(*e*) *rank* $\begin{pmatrix} M_\gamma(Q,s) & G_\gamma(Q) \end{pmatrix} = n + normrank\, G_{di} \qquad \forall\, s \in C^0 \cup C^+$,

(*f*) $\rho(PQ) < \gamma^{-2}$. □

We define a new system:

$$\Sigma_{nm}: \begin{cases} \dot{x} = Ax + Bu + Bw \\ y = Cx + Du + Dw, \\ z = u. \end{cases} \tag{2.5}$$

This system is of interest as it is easily checked that a given controller Σ_F of the form (2.3) stabilizes Σ given by (2.1) if and only if Σ_F applied from y to u internally stabilizes Σ_{nm}. Moreover, the transfer matrix mentioned in condition (ii) of lemma 2.1 is equal to the closed loop transfer matrix from w to z of $\Sigma_{nm} \times \Sigma_F$. We can therefore rephrase our problem formulation in terms of our new system Σ_{nm} with the help of lemma 2.1. To this end, we investigate how G, M and G_{di} as defined above look like for the system (2.5):

$$\begin{aligned} G(Q) &:= \begin{pmatrix} AQ + QA^{\mathrm{T}} + BB^{\mathrm{T}} & QC^{\mathrm{T}} + BD^{\mathrm{T}} \\ CQ + DB^{\mathrm{T}} & DD^{\mathrm{T}} \end{pmatrix} \\ M(Q,s) &:= \begin{pmatrix} sI - A \\ -C \end{pmatrix} \\ G_{di}(s) &:= C(sI-A)^{-1}B + D \end{aligned}$$

We assume that (A, B) is stabilizable and (C, A) is detectable, i.e. we assume that our nominal model is stabilizable. We define P_m and Q_m as the unique positive semidefinite matrices satisfying the following conditions:

(*i*)
- $A^{\mathrm{T}} P_m + P_m A = P_m B B^{\mathrm{T}} P_m$,
- $A - BB^{\mathrm{T}} P_m$ is asymptotically stable.

(*ii*)
- $G(Q_m) \geq 0$,
- rank $G(Q_m) = \mathrm{rank}_{\mathcal{R}(s)}\, G_{di}$,
- rank $\begin{pmatrix} M(Q_m, s) & G(Q_m) \end{pmatrix} = n + \mathrm{rank}_{\mathcal{R}(s)}\, G_{di}$.

The existence and uniqueness of such a P_m has been shown in [8].
The first two conditions on Q_m require that Q_m is a rank-minimizing solution of a linear matrix inequality. The same linear matrix inequality also appears in the *singular* filtering problem (see [11], this problem is dual to the singular linear quadratic control problem). It is shown in [12] that rank minimizing solutions of this linear matrix inequality have a 1-1 relationship with solutions of a given reduced order Riccati equation. Moreover, using the results in that paper it is straightforward to show that the largest solution of the linear matrix inequality, whose existence is guaranteed since (C,A) is detectable (dualize the results in [3, 16]), satisfies all the requirements on Q_m. This shows existence of Q_m. Uniqueness of Q_m was shown in [12].
The above enables us to formulate our main result:

Theorem 2.3 : *Let a system Σ be given with realization (A, B, C, D) and state space $\mathcal{R}^n$. Assume that (A, B) is stabilizable and that (C, A) is detectable. Moreover, let $\gamma > 0$ be given. Assume that A has no eigenvalues on the imaginary axis and assume that (A, B, C, D) has no invariant zeros on the imaginary axis. We define the auxiliary system Σ_{nm} by (2.5). Under the above assumptions the following three conditions are equivalent:*

(*i*) *There exists a controller Σ_F from y to u of the form (2.3) which, when applied to the interconnection (2.2), yields a closed loop system that is well-posed and internally stable for all systems Σ_{err} such that:*

- *Σ_{err} has $\mathcal{L}_\infty$ norm less than or equal to γ,*
- *The interconnection (2.2) and Σ have the same number of unstable poles.*

(*ii*) *There exists a controller Σ_F from y to u of the form (2.3) which, when applied to the system Σ_{nm}, yields a closed loop system that is well-posed, internally stable and has H_∞ norm less than γ^{-1}.*

(*iii*) *Either A is stable or $1 + \rho(P_m Q_m) < \gamma^{-2}$.* □

Proof : The equivalence between (i) and (ii) is a direct result from lemma 2.1. To show the equivalence between (ii) and (iii) we have to investigate four cases:

- *A is stable:* the matrices $P := 0$ and $Q := Q_m$ satisfy condition (iii) of theorem 2.2.
- *A is not stable and $\gamma < 1$:* we define

$$P := \frac{P_m}{1 - \gamma^2}, \qquad Q := Q_m \tag{2.6}$$

  Then it is straightforward to check that P and Q are the unique matrices satisfying requirements (a)-(e) of condition (iii) of theorem 2.2 for Σ_{nm}. Moreover, P and Q satisfy condition (iii) of theorem 2.3 if and only if P and Q satisfy the final requirement (f) of condition (iii) of theorem 2.2.

- *A is not stable and $\gamma = 1$:* the stability requirement for P reduces to the requirement that A is stable which, by assumption, is not true.
- *A is not stable and $\gamma > 1$:* the unique matrices P and Q satisfying conditions (a)-(e) of part (iii) of theorem 2.2 are given by (2.6). Note that $P \leq 0$.

The equivalence between (ii) and (iii) is then for each of the above cases a direct result from theorem 2.2. ■

Remarks:

(*i*) If $\gamma < 1$ then our class of perturbations does include all stable systems Σ_{err} with H_∞ norm less than or equal to γ. The restriction $\gamma < 1$ does not matter since we know that condition (iii) of the above theorem is only satisfied if $\gamma < 1$ (or A is stable but in that case Σ_{err} should be stable anyway because of the assumption on the number of unstable poles).

(*ii*) We have an explicit bound for the allowable size of perturbations: part (iii) shows that for every γ smaller than the bound $[1+\rho(P_m Q_m)]^{-1/2}$ we can find a suitable controller satisfying part (i).

(*iii*) For additive perturbations (see [5]) it has been shown that the upper bound only depends on the antistable part of Σ. It should be noted that this is *not* true for the bound $[1+\rho(P_m Q_m)]^{-1/2}$ which we obtained for multiplicative perturbations.

(*iv*) Note that because this is, in general, a singular problem we have not been able to find an explicit formula for a controller satisfying part (i). Note that we know that a controller satisfies part (i) if and only if this controller satisfies part (ii).

It would be nice to have explicit formulas available for controllers satisfying part (i) of theorem 2.3. It turns out that if we make one extra assumption then we can complete our results:

Theorem 2.4 : *Let a system Σ with stabilizable and detectable realization (A, B, C, D) be given. Moreover, let $\gamma > 0$ be given. Assume that A is not stable and has no eigenvalues on the imaginary axis and assume that (A, B, C, D) has no invariant zeros on the imaginary axis. Finally, assume that D is surjective. We define the auxiliary system Σ_{nm} by (2.5). Let $\gamma < [1+\rho(P_m Q_m)]^{-1/2}$. Then the compensator, described by the following state space realization satisfies part (i) of the previous theorem.*

$$\left\{ \begin{array}{l} \dot{p} = \quad Ap + K(Cp + Du - y), \\ u = -B^{\mathrm{T}} P_m p \end{array} \right. \tag{2.7}$$

where

$$K := \gamma^2 \left((\gamma^{-2} - 1)I - Q_m P_m \right)^{-1} (BD^{\mathrm{T}} + Q_m C^{\mathrm{T}})(DD^{\mathrm{T}})^{-1} . \qquad \square$$

Proof : This theorem is a direct consequence of [1]. ■

Remarks:

(*i*) Note that the structure of the controller (2.7) is the standard structure of a state observer coupled with a state feedback which we also encounter in linear quadratic Gaussian control theory (see [9]). *The only difference* with the optimal controller from LQG theory, is that the Kalman gain $(BD^{\mathrm{T}} + Q_m C^{\mathrm{T}})(DD^{\mathrm{T}})^{-1}$ is replaced by the matrix K. Note that for $\gamma \downarrow 0$ the matrix K converges to the Kalman gain.

(*ii*) If D is not surjective, then techniques to find suitable controllers are discussed in [13, 14]. Even for the suboptimal case no explicit formulas for suitable controllers are available.

(*iii*) If we could find an internally stabilizing controller for Σ_{nm} which makes the H_∞ norm of the closed loop system equal to $[1+\rho(P_m Q_m)]^{-1/2}$, than this would clearly be the optimally robust controller in this setting. In [6] an explicit method is given to find such a controller in case D is surjective. In that paper, for the case that the direct feedthrough matrix from u to y is unequal to zero, no explicit formulas for such a controller are given. This is related to the well-posed condition we have to impose on our controller. Unfortunately, this direct feedthrough matrix is equal to D for Σ_{nm} and hence unequal to 0.

This theorem completes the picture if D is surjective. If D is not surjective then it can be shown that in general there exists no *proper* controller satisfying part (i) of theorem 2.3 for the boundary value for γ.

Acknowledgement: I would like to thank professor Ruth Curtain for suggesting the problem as a nice application of the singular H_∞ control problem. Moreover, I would like to thank Harry Trentelman for spending two hours of his valuable time by reading a preliminary version of this paper.

3 Conclusion

In this paper we have derived results for robust stabilization of systems subject to unstructured multiplicative perturbations. Though some open questions remain a rather complete picture is given. We think that this paper nicely complements the results on additive and normalized co-prime factor perturbations.
From a practical point of view it is certainly desirable to derive the results of this paper in case we have frequency-weighted multiplicative perturbations. The techniques in [6, 13, 14] enable us to treat this case in a similar way. However no explicit upper bound can be given, but only a test whether for a given γ we can find a suitable controller.
An important open question for future research is the investigation of the unnatural assumption on the number of unstable poles. Using normalized co-prime factor perturbations one is not restraining the number of unstable poles. On the other hand, this method has the disadvantage that, since the co-prime factors are only theoretical tools, it is unclear from a practical point of view when two systems are close using this distance concept. Using the ideas of the gap/graph metric some of this intuition is available but it is still not very satisfying. More work needs to be done to find perturbation classes we can treat (e.g. using H_∞ control) and are intuitive enough for the engineer to work with.

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