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H.J.C. Huijberts
L. Colpier
P. Moreau

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Factorization and input-output decoupling by static output feedback for nonlinear control systems*

H.J.C. Huijberts†, L. Colpier**, P. Moreau**

* Department of Mathematics and Computing Science
  Eindhoven University of Technology
  P.O. Box 513
  5600 MB Eindhoven
  The Netherlands
  Email: hjch@bs.win.tue.nl

** Laboratoire d'Automatique de Nantes
  Ecole Centrale de Nantes/Université de Nantes
  U.A. CNRS 823
  1, Rue de la Noé
  44072 Nantes Cedex 03
  France

Abstract

In this paper we study the strong input-output decoupling problem via regular static output feedback for nonlinear control systems (SIODPof). It turns out that the solvability of the problem is equivalent to the solvability of a factorization problem for a set of functions with respect to certain codistributions. Checkable conditions for the solvability of this factorization problem are given.

1 Introduction

Since the beginning of the 80’s, a lot of progress has been made in the solution of nonlinear synthesis problems via static and dynamic state feedback (see the textbooks [7],[11] and the monograph [6] for an overview). To our best knowledge, there have hardly been any papers that tackle nonlinear synthesis problems via (static or dynamic) output feedback. Exceptions are [9] that gives conditions for controlled invariance of distributions via static output feedback, and [10], that studies the strong input-output decoupling problem via structure preserving static state feedback for Hamiltonian systems. (In the last reference, static output feedback appears on the stage since for Hamiltonian systems a structure preserving static state feedback is necessarily a static output feedback.)

†Research was partly performed while the second and third author were visiting Eindhoven University of Technology.

*Author to whom all correspondence should be sent.
In this paper we study the strong input-output decoupling problem via regular static output feedback (SIODPof) for square affine nonlinear control systems. It turns out that the question of solvability of the problem boils down to the question of solvability of a factorization problem of a set of functions with respect to certain codistributions. More specifically, consider a square affine nonlinear control system with \( m \) inputs and \( m \) outputs. Let \( \Pi \) be the codistribution spanned by the differentials of the output functions. Assuming that all relative degrees of the system are finite and the decoupling matrix of the system is invertible (which are necessary conditions for the solvability of the strong input-output decoupling via regular static state feedback, cf. [11],[7]), one may define certain codistributions \( \Omega_i \) \( (i = 1, \ldots, m) \) that characterize the dynamics of a strongly input-output decoupled system. In fact, it turns out that these codistributions are the annihilators of certain controllability distributions ([8]) for the system. For the \( i \)-th output \( (i = 1, \ldots, m) \) of the system one may then define certain functions \( a_i, b_{i1}, \ldots, b_{im} \). Then, the SIODPof is solvable if and only if for every \( i \in \{1, \ldots, m\} \) one can simultaneously factorize these functions with respect to \( \Omega_i \) and \( \Pi \) in the following way: there should exist functions \( \beta_{ij}, \gamma_i, \phi_i, \psi_i \) \( (i, j = 1, \ldots, m) \) such that \( d\beta_{ij}, d\gamma \in \Pi \), \( d\phi_i, d\psi_i \in \Omega_i \), and

\[
\beta_{ij} = \beta_{ij} \phi_i \\
a_i = \psi_i + \gamma_i \phi_i
\]

If this factorization is possible, then a regular static output feedback solving the SIODPof will depend on the functions \( \beta_{ij}, \gamma_i \), while the dynamics that remain after the problem has been solved are described by the functions \( \phi_i, \psi_i \).

The organization of the paper is as follows. In Section 2 we briefly review some concepts and results from the theory of differential forms that will be used in the rest of the paper. Next, in Section 3 we treat the structure of strongly input-output decoupled systems and introduce the codistributions \( \Omega_i \). Further, we discuss some of the properties of these codistributions. In Section 4 we define the SIODPof, and we reduce the problem to the factorization problem indicated above. In Section 5 we consider the solvability of this problem. In Section 6, we draw some conclusions.

## 2 Differential forms

In this section we give an overview of differential forms. For details, we refer to [1],[2],[3],[4],[13].

Let \( V \) be an \( r \)-dimensional vector space over \( \mathbb{R} \). A \( k \)-form \( \omega \) on \( V \) is a \( k \)-linear completely antisymmetric mapping \( \omega : V \times \cdots \times V \rightarrow \mathbb{R} \), i.e.,

\[
(\forall \alpha, \beta \in \mathbb{R})(\forall v_1, v'_1, v_2, \ldots, v_r \in V)\\
(\omega(\alpha v_1 + \beta v'_1, v_2, \ldots, v_r) = \alpha \omega(v_1, \ldots, v_r) + \beta \omega(v'_1, v_2, \ldots, v_r))
\]

\[
(\forall v_1, \ldots, v_r \in V)(\omega(v_1, \ldots, v_r) = -\omega(v_1, \ldots, v_r, v_{r-1} v_{r+1}, v_{r+2}, \ldots, v_r))
\]

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Note that a one-form on $V$ is just an element of $V^*$, the dual of $V$. The space of all $k$-forms on $V$ is denoted by $\Lambda^k(V^*)$. It is easily checked that the $k$-linearity and anti-symmetry of a $k$-form on $V$ implies that all $k$-forms are zero for $k > r$. We define

$$\Lambda(V^*) := \Lambda^0(V^*) \oplus \Lambda^1(V^*) \oplus \cdots \oplus \Lambda^r(V^*)$$

(3)

where

$$\Lambda^0(V^*) := \mathbb{R}$$

We call $\Lambda(V^*)$ the exterior algebra over $V^*$. An element $\omega \in \Lambda(V^*)$ is called a form on $V$ and may be written in a unique way as

$$\omega = \omega^0 + \omega^1 + \cdots + \omega^r$$

(4)

where $\omega^i \in \Lambda^i(V^*)$ ($i = 0, \ldots, r$).

We next define a product on $\Lambda(V^*)$, the so called wedge product (or exterior product). This product will be denoted by $\wedge$. First, let $\eta \in \Lambda^p(V^*)$, $\omega \in \Lambda^q(V^*)$. Then $\eta \wedge \omega \in \Lambda^{p+q}(V^*)$ is defined by

$$(\eta \wedge \omega)(v_1, \ldots, v_{p+q}) = \sum_{\pi} (\text{sign}(\pi)) \eta(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \omega(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)})$$

(5)

where the sum is over all permutations of $1, \ldots, p+1$, and $v_1, \ldots, v_{p+q} \in V$. For $\eta = \eta^0 + \cdots + \eta^r$, $\omega = \omega^0 + \cdots + \omega^r$, with $\eta^i, \omega^j \in \Lambda^i(V^*)$ ($i = 0, \ldots, r$), we define

$$\eta \wedge \omega = \sum_{i,j=0}^r \eta^i \wedge \omega^j$$

(6)

Note that the wedge-product is associative and distributive, but not commutative. Instead, it satisfies

$$\eta \wedge \omega = (-1)^{pq} \omega \wedge \eta, \quad \eta \in \Lambda^p(V^*), \omega \in \Lambda^q(V^*)$$

(7)

Let $v \in V$ be given. Then the interior product $v \lrcorner : \Lambda(V^*) \to \Lambda(V^*)$ is defined in the following way. First consider $\omega \in \Lambda^k(V^*)$. Then $v \lrcorner \omega \in \Lambda^{k-1}(V^*)$ is given by

$$(v \lrcorner \omega)(v_1, \ldots, v_{p-1}) = \omega(v, v_1, \ldots, v_{p-1})$$

(8)

where $v_1, \ldots, v_{p-1} \in V$. If $\omega = \omega^0 + \cdots + \omega^r$, with $\omega^i \in \Lambda^i(V^*)$ ($i = 0, \ldots, r$), then

$$(v \lrcorner \omega) = \sum_{i=0}^r (v \lrcorner \omega^i)$$

(9)

Next, consider an $n$-dimensional manifold $M$. Let $T^*_x M$ denote the cotangent space at $x \in M$ and let $T^* M$ denote the cotangent bundle of $M$. Since $T^*_x M$ is an $n$-dimensional vector space over $\mathbb{R}$, we may define $\Lambda^k(T^*_x M)$ for all $x \in M$, $k = 0, \ldots, n$, as well as

$$\Lambda(T^*_x M) := \sum_{k=0}^n \Lambda^k(T^*_x M)$$
We then define the bundles $\Lambda^k(T^*M)$, $\Lambda(T^*M)$ over $M$ by

$\Lambda^k(T^*M) := \bigcup_{x \in M} \Lambda^k(T_x^*M)$, $\quad (k = 0, \cdots, n)$ \hspace{1cm} (10)

$\Lambda(T^*M) := \sum_{k=0}^n \Lambda^k(T^*M)$ \hspace{1cm} (11)

A differential form on $M$ is now defined to be a smooth section of the bundle $\Lambda(T^*M)$, while a differential $k$-form on $M$ is defined to be a smooth section of the bundle $\Lambda^k(T^*M)$. So, roughly speaking, a differential $(k-)form on M is a "prescription" that assigns a $(k-)form \omega_x on T^*_xM to every x \in M in a smooth way. Note that by this definition a differential 0-form on $M$ is just a smooth function on $M$. When no confusion arises, we will simply call a differential $(k-)form on $M a $(k-)form on $M in the sequel.

The wedge product of the forms $\eta, \omega$ on $M$ is defined to be the form $(\eta \wedge \omega)$ satisfying

$$(\eta \wedge \omega)_x = \eta_x \wedge \omega_x, \quad (\forall x \in M)$$ \hspace{1cm} (12)

Let $\tau$ be a smooth vector field on $M$, and let $\omega$ be a form on $M$. Then the interior product $(\tau \lrcorner \omega)$ is defined to be the form satisfying

$$(\tau \lrcorner \omega)_x = \tau_x \lrcorner \omega_x, \quad (\forall x \in M)$$ \hspace{1cm} (13)

The exterior differential operator $d$ maps a $k$-form $\omega$ into a $(k+1)$-form $d\omega$, called the exterior derivative of $\omega$. The operator $d$ is uniquely defined by the following properties:

1. $d$ is linear:

$$(\forall \alpha, \beta \in \mathbb{R})(d(\alpha \eta + \beta \omega) = \alpha d\eta + \beta d\omega)$$

2. If $\eta$ is a $k$-form, then

$$d(\eta \wedge \omega) = d\eta \wedge \omega + (-1)^k \eta \wedge d\omega$$

3. $d^2 = 0$.

4. If $f$ is a 0-form, then $df$ is the ordinary differential $df$ of $f$.

5. $d$ is local: if $\eta$ and $\omega$ coincide on an open set $U$, then $d\eta = d\omega$ on $U$.

A $k$-form $\omega$ is called closed if $d\omega = 0$; it is called exact if there exists a $(k - 1)$-form $\eta$ such that $\omega = d\eta$. Note that, since $d^2 = 0$, an exact $k$-form is closed. The converse does not need to hold globally. However, it does hold locally, as is reflected by the following theorem.

Theorem 2.1 (Poincaré’s Lemma) If $M$ is smoothly contractible to a point $x_0 \in M$, then every closed form $\omega$ on $M$ is exact.
Let $\tau$ be a smooth vector field on $M$. The Lie-derivative $\mathcal{L}_\tau$ maps a $k$-form $\omega$ into a $k$-form $\mathcal{L}_\tau \omega$. $\mathcal{L}_\tau$ is uniquely defined by the following properties:

1. If $f$ is a 0-form on $M$, then
   $$\mathcal{L}_\tau f = \tau \lrcorner df$$

2. $\mathcal{L}_\tau$ is a derivation:
   $$\mathcal{L}_\tau (\eta \wedge \omega) = \eta \wedge \mathcal{L}_\tau \omega + \mathcal{L}_\tau \eta \wedge \omega$$

3. $\mathcal{L}_\tau$ commutes with $d$:
   $$\mathcal{L}_\tau (d\omega) = d(\mathcal{L}_\tau \omega)$$

From the definitions of interior product, exterior derivative, and Lie-derivative one may derive the following identities that will be frequently used in the sequel (here $\sigma, \tau$ denote smooth vector fields on $M$, and $\omega$ denotes a $k$-form on $M$).

$$\mathcal{L}_\tau \omega = d(\tau \lrcorner \omega) + \tau \lrcorner d\omega$$ \hspace{1cm} (14)

$$[\sigma, \tau] \lrcorner \omega = \mathcal{L}_\sigma (\tau \lrcorner \omega) - \tau \lrcorner \mathcal{L}_\sigma \omega \quad \text{(Leibniz formula)}$$ \hspace{1cm} (15)

$$\mathcal{L}_{[\sigma, \tau]} \omega = \mathcal{L}_\sigma \mathcal{L}_\tau \omega - \mathcal{L}_\tau \mathcal{L}_\sigma \omega$$ \hspace{1cm} (16)

Denote by $\Omega^1(M)$ the set of all one-forms on $M$. Then $\Omega^1(M)$ has the structure of a finitely generated module over the ring of smooth functions on $M$. A finitely generated submodule of $\Omega^1(M)$ is called a codistribution on $M$. A codistribution $\Omega$ on $M$ is called integrable if it has a set of generators consisting of exact one-forms.

**Theorem 2.2 (Frobenius Theorem)** Consider a codistribution $\Omega$ on $M$. Let $\{\omega^1, \cdots, \omega^d\}$ be a set of generators of $\Omega$ and let $\omega^{d+1}, \cdots, \omega^n$ be such that $\{\omega^1, \cdots, \omega^n\}$ generate $\Omega^1(M)$. Then $\Omega$ is integrable if and only if there exist smooth functions $\Gamma^k_{ij}$ ($i, k = 1, \cdots, d; j = i + 1, \cdots, n$) on $M$ such that

$$d\omega^k = \sum_{i=1}^d \sum_{j=i+1}^n \Gamma^k_{ij} \omega^i \wedge \omega^j \quad (k = 1, \cdots, d)$$ \hspace{1cm} (17)
3 Strongly input-output decoupled systems

3.1 Structure of strongly input-output decoupled systems

We consider a square nonlinear control system \( \Sigma \) of the form

\[
\begin{align*}
\dot{x} &= f(x) + \sum_{i=1}^{m} u_i g_i(x) =: f(x) + g(x)u \\
y &= h(x)
\end{align*}
\]

where \( x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n \) are local coordinates for the state space manifold \( M \), \( u \in \mathbb{R}^m \) denotes the controls and \( y \in \mathbb{R}^m \) denotes the outputs. We will assume throughout that the vector fields \( f, g_1, \cdots, g_m \) and the output mapping \( h : M \rightarrow \mathbb{R}^m \) are meromorphic.

Let \( \mathcal{K}_u \) denote the field of meromorphic functions of \( \{x, \{u^{(k)}| k \geq 0\}\} \) and define the vector space

\[
\mathcal{E} := \text{span}_{\mathcal{K}_u}\{d\xi | \xi \in \mathcal{K}_u\}
\]

For \( \Sigma \) we define in a natural way

\[
\begin{align*}
\dot{y} &= y(x, u) = \frac{\partial h}{\partial x}[f(x) + g(x)u] \\
y^{(k+1)} &= y(x, \dot{u}, \cdots, u^{(k)}) = \\
&\quad \frac{\partial y^{(k)}}{\partial x}[f(x) + g(x)u] + \sum_{i=0}^{k-1} \frac{\partial y^{(k)}}{\partial u^{(i)}} u^{(i+1)}
\end{align*}
\]

We define the relative degrees \( r_i \) (\( i = 1, \cdots, m \)) of \( \Sigma \) by ([5])

\[
r_i = \min\{k \in \mathbb{N} | dy^{(k)} \notin \text{span}_{\mathcal{K}_u}\{dx\}\}
\]

If all relative degrees are finite, we define the decoupling matrix \( B(x) \) of \( \Sigma \) as the \((m, m)\)-matrix with entries

\[
b_{ij}(x) = \left( \frac{\partial y^{(r_i)}}{\partial u_j} \right)(x) \quad (i, j = 1, \cdots, m)
\]

A system \( \Sigma \) is said to be input-output decoupled if each of its inputs influences one and only one of its outputs. The system is said to be strongly input-output decoupled if all relative degrees are finite, its decoupling matrix is an invertible diagonal matrix, and

\[
\left( \frac{\partial y^{(k)}}{\partial u_j} \right) = 0 \quad (i, j = 1, \cdots, m; j \neq i; k \geq r_i + 1)
\]

Remark 3.3 Note that a strongly input-output decoupled system is input-output decoupled. The converse does not need to hold (see [11] for details).
For $\Sigma$ we define the following subspaces of $\mathcal{E}$:

$$
\Omega_i := \{ \omega \in \text{span}_{K_n} \{ dx \} \mid \forall k \in \mathbb{N} : \omega^{(k)} \in \text{span}_{K_n} \{ dx, dy_i^{(r_1)}, \ldots, dy_i^{(r_i+k-1)} \} \} \quad (24)
$$

The subspaces $\Omega_i$ have the following interpretation.

**Proposition 3.4** Consider a square nonlinear system $\Sigma$ of the form (18), and assume that all its relative degrees are finite. Let $x_0 \in M$ be given. Define

$$
a_i(x) := \mathcal{L}^r_j h_i(x) \quad (i = 1, \ldots, m) \quad (25)
$$

$$
b_{ij}(x) = \mathcal{L}_j \mathcal{L}^{r_i-1} h_i(x) \quad (i, j = 1, \ldots, m) \quad (26)
$$

Then $\Sigma$ is strongly input-output decoupled around $x_0$ if and only if there exists a neighborhood $U \subset M$ of $x_0$ such that for all $x \in U$ we have:

$$
b_{ij}(x) = 0 \quad (i, j = 1, \ldots, m; i \neq j) \quad (27)
$$

$$
b_{ii}(x) \neq 0 \quad (i = 1, \ldots, m) \quad (28)
$$

$$
da_i(x), db_{ij}(x) \in \Omega_i(x) \quad (i = 1, \ldots, m) \quad (29)
$$

**Proof** See Appendix. \[\blacksquare\]

### 3.2 Properties of subspaces $\Omega_i$

In this subsection we will derive some further properties of the subspaces $\Omega_i$. Consider a system $\Sigma$ of the form (18) and assume that all relative degrees are finite. Let $x_0 \in M$ be given and assume that there exists a neighborhood $U \subset M$ of $x_0$ such that $B(x)$ is invertible for all $x \in U$. Let $\tilde{K}$ denote the subfield of $K_n$ consisting of the meromorphic functions of $x$.

Define the following subspaces for $i = 1, \ldots, m$:

$$
\tilde{\Omega}_i := \{ \omega \in \text{span}_{\tilde{K}} \{ dx \} \mid \forall k \in \mathbb{N} : \omega^{(k)} \in \text{span}_{\tilde{K}} \{ dx, dy_i^{(r_1)}, \ldots, dy_i^{(r_i+k-1)} \} \} \quad (30)
$$

Consider the sequence of subspaces $\tilde{\Omega}_i^k$ defined by

$$
\tilde{\Omega}_i^0 := \text{span}_{\tilde{K}} \{ dx \} \quad (31)
$$

$$
\tilde{\Omega}_i^{k+1} := \{ \omega \in \tilde{\Omega}_i^k \mid \omega \in \tilde{\Omega}_i^k + \text{span}_{\tilde{K}} \{ dy_i^{(r_1)} \} \} \quad (k \in \mathbb{N})
$$

**Proposition 3.5** There exists an integer $k^* \in \mathbb{N}$ such that

(i) $\forall k \geq k^* + 1: \tilde{\Omega}_i^k = \tilde{\Omega}_i^{k^*}$.

(ii) $\tilde{\Omega}_i = \tilde{\Omega}_i^{k^*}$.
Proof See Appendix. 

Since $\Omega_i \subset \text{span}_K\{dx\}$, it may be identified with a codistribution on $M$. It then turns out that the codistribution obtained in this way is the dual of a controllability distribution for $\Sigma$, as is reflected by the following proposition.

**Proposition 3.6** Consider a square nonlinear control system $\Sigma$ of the form (18) and assume that all its relative degrees are finite. Assume that the decoupling matrix of $\Sigma$ is invertible. Let $R_i^*$ denote the supremal controllability distribution for $\Sigma$ that is contained in

$$\Delta_i := \bigwedge_{k=0}^{r_i-1} \text{Ker } dy_i^{(k)}$$

Then

$$R_i^* = \Omega_i \quad (i = 1, \ldots, m)$$

**Proof** See Appendix. 

**Corollary 3.7** The codistributions $\Omega_i$ are integrable and invariant under regular static state feedback.

**Proof** Follows immediately from Proposition 3.6 and the fact that the distributions $R_i^*$ are involutive and invariant under regular static state feedback (cf. [8],[7]).

4 Strong input-output decoupling problem via regular static output feedback

4.1 Definition of the problem

**Definition 4.8** Consider a square nonlinear system $\Sigma$ of the form (18) and let $x_0 \in M$ be given. Then the strong input-output decoupling problem via regular static output feedback (SIODPof) is said to be solvable around $x_0$ if there exist a neighborhood $U \subset M$ of $x_0$ and mappings $\alpha : h(U) \rightarrow \mathbb{R}^m$, $\beta : h(U) \rightarrow \mathbb{R}^{m \times m}$ satisfying

$$|\beta \circ h(x)| \neq 0 \quad (\forall x \in U)$$

such that $\Sigma$, together with the output feedback

$$u = \alpha \circ h(x) + \beta \circ h(x)v$$

is input-output decoupled, when restricted to $U$.

**Remark 4.9** To simplify notation in the sequel, we will simply write $u = \alpha(y) + \beta(y)v$ rather than (35) for a static output feedback.
4.2 Reduction to factorization problem

In this section we reduce the SIODPof to a factorization problem for a set of functions with respect to certain codistributions.

**Theorem 4.10** Consider a nonlinear control system $\Sigma$ of the form (18) and assume that all its relative degrees are finite. Let $x_0 \in M$ be given. Define $a_i(x) (i = 1, \cdots, m)$ and $b_{ij}(x) (i, j = 1, \cdots, m)$ as in (25), (26) respectively. Furthermore, define the codistribution $\Pi := \text{span}\{dh_1, \cdots, dh_m\}$. Then the SIODPof is solvable around $x_0$ if and only if there exist a neighborhood $U \subset M$ of $x_0$ and functions $\beta_{ij}, \phi_i, \psi_i, \sigma_i : M \rightarrow \mathbb{R} (i, j = 1, \cdots, m)$ such that on $U$ we have

\[
\begin{align*}
    d\beta_{ij}, d\sigma_i &\in \Pi \quad (i, j = 1, \cdots, m) \\
    d\phi_i, d\psi_i &\in \bar{\Omega}_i \quad (i = 1, \cdots, m) \\
    b_{ij} &= \beta_{ij} \phi_i \quad (i, j = 1, \cdots, m) \\
    a_i &= \psi_i + \sigma_i \phi_i \quad (i = 1, \cdots, m)
\end{align*}
\]

**Proof** (sufficiency) Assume that $\beta_{ij}, \phi_i, \psi_i, \sigma_i$ satisfying (36),···,(39) exist. Define the matrices

\[
\begin{align*}
    \tilde{B}(y) &:= (\beta_{ij}(y))_{i,j=1,\cdots,m} \\
    \Phi(x) &:= \text{diag}(\phi_1(x), \cdots, \phi_m(x)) \\
    \Psi(x) &:= \text{col}(\psi_1(x), \cdots, \psi_m(x)) \\
    S(y) &:= \text{col}(\sigma_1(y), \cdots, \sigma_m(y))
\end{align*}
\]

and the vectors

\[
\begin{align*}
    y &= \{y_1^{(r_1)}, \cdots, y_m^{(r_m)}\}
\end{align*}
\]

We then have

\[
\begin{align*}
    \begin{pmatrix}
        y_1^{(r_1)} \\
        \vdots \\
        y_m^{(r_m)}
    \end{pmatrix} = \Psi(x) + \Phi(x)(S(y) + \tilde{B}(y)u)
\end{align*}
\]

Since $B(x)$ is invertible on some neighborhood $U \subset M$ of $x_0$, also $\tilde{B}(y)$ is invertible on this neighborhood. Apply the following regular static output feedback to $\Sigma$:

\[
    u = \tilde{B}(y)^{-1}(v - S(y))
\]
Then for (18,45) we have
\[
\begin{pmatrix}
  y_1^{(r_1)} \\
  \vdots \\
  y_m^{(r_m)}
\end{pmatrix}
= \Psi(x) + \Phi(x)v
\]  
(46)

From (37) and Proposition 3.4 it then follows that (18,45) is strongly input-output decoupled, and hence the SIODP of is solvable for \( \Sigma \) around \( x_0 \).

(necessity) Assume that the SIODP of is solvable for \( \Sigma \) around \( x_0 \), by means of the regular static output feedback
\[
u = \tilde{A}(y) + \tilde{B}(y)v
\]  
(47)

We then have that \( B(x)\tilde{B}(y) \) is an invertible diagonal matrix, say
\[
B(x)\tilde{B}(y) = \Phi(x) = \text{diag}(\phi_1(x), \ldots, \phi_m(x))
\]  
(48)

where \( d\phi_i \in \Omega_i \) \( (i = 1, \ldots, m) \). Denote the entries of \( \tilde{B}(y)^{-1} \) by \( \beta_{ij}(y) \) \( (i, j = 1, \ldots, m) \). By (48), \( B(x) = \Phi(x)\tilde{B}(y)^{-1} \) and hence
\[
\beta_{ij} = \beta_{ij}\phi_i \quad (i, j = 1, \ldots, m)
\]  
(49)

This establishes (38). Further, define
\[
\Psi(x) = \text{col}(\psi_1(x), \ldots, \psi_m(x)) := a(x) + B(x)\tilde{A}(y)
\]  
(50)

We then have that \( d\psi_i \in \Omega_i \) and
\[
a(x) = \Psi(x) - B(x)\tilde{A}(y) = \Psi(x) - \Phi(x)\tilde{B}(y)^{-1}\tilde{A}(y)
\]  
(51)

Denoting the entries of \( \tilde{B}(y)^{-1}\tilde{A}(y) \) by \( -\sigma_i(y) \) \( (i = 1, \ldots, m) \), we then obtain from (51):
\[
a_i = \psi_i + \sigma_i\phi_i \quad (i = 1, \ldots, m)
\]  
(52)

which establishes (39).

Motivated by the above theorem, we formulate the following factorization problems:

**Factorization Problem 1** Consider codistributions \( \Omega, \Pi \) on an \( n \)-dimensional manifold \( M \) and a function \( b : M \to \mathbb{R} \). Under what conditions do there locally exist functions \( \beta, \phi : M \to \mathbb{R} \) satisfying

1. \( d\beta \in \Pi \).
2. \( d\phi \in \Omega \).
3. \( b = \beta \phi \).

**Factorization Problem 2** Consider codistributions \( \Omega, \Pi \) on an \( n \)-dimensional manifold \( M \) and functions \( a, b : M \to \mathbb{R} \), where Factorization Problem 1 is solvable for \( b \). Under what conditions do there locally exist functions \( \sigma, \phi, \psi : M \to \mathbb{R} \) satisfying
1. \( d\sigma \in \Pi \).

2. \( d\phi \in \Omega \).

3. \( d\psi \in \Omega \).

4. \( a = \psi + \sigma b \).

**Factorization Problem 3** Consider codistributions \( \Omega, \Pi \) on an \( n \)-dimensional manifold \( M \) and functions \( a, b_1, \ldots, b_\ell : M \to \mathbb{R} \), where \( b_1, \ldots, b_\ell \) are not all identically zero. Under what conditions do there locally exist functions \( \beta_1, \ldots, \beta_\ell, \sigma, \phi, \psi : M \to \mathbb{R} \) satisfying

1. \( d\beta_1, \ldots, d\beta_\ell, d\sigma \in \Pi \).

2. \( d\phi \in \Omega \).

3. \( d\psi \in \Omega \).

4. \( b_i = \beta_i \phi \) (\( i = 1, \ldots, m \))

5. \( a = \psi + \sigma \phi \).

We then have:

**Corollary 4.11** Consider a nonlinear control system \( \Sigma \) of the form (18) and assume that all its relative degrees are finite. Let \( x_0 \in M \) be given. Define \( a_i(x) \) (\( i = 1, \ldots, m \)) and \( b_{ij}(x) \) (\( i, j = 1, \ldots, m \)) as in (25),(26) respectively. Furthermore, define the codistribution \( \Pi := \text{span}\{dh_1, \ldots, dh_m\} \). Then the SIODPof is solvable around \( x_0 \) if and only if there exists a neighborhood \( U \subset M \) of \( x_0 \) such that

(i) \( B(x) \) is invertible for all \( x \in U \).

(ii) For all \( i \in \{1, \ldots, m\} \) we have that Factorization Problem 3 is solvable on \( U \) for \( a_i, b_{i1}, \ldots, b_{im} \) with respect to \( \Omega_i \) and \( \Pi \).

**Proof** Follows immediately from Theorem 4.10 and the definition of Factorization Problem 3.

5 Solvability conditions for factorization problems

In the following subsections we will consider the three factorization problems. Rather than solving the problems in their full generality, we will solve them for codistributions \( \Omega \) and \( \Pi \) with a special structure that is imposed by the considerations in the previous section (except for part of the solution of Factorization Problem 1).

First of all, the codistributions \( \Omega_i \) and \( \Pi \) in the previous section are integrable, while we moreover have a basis of exact one-forms for \( \Pi \) in hand (namely \( dy_1, \ldots, dy_m \)). Secondly, the invertibility of the decoupling matrix of \( \Sigma \) implies that \( dy_j \notin \Omega_i \) (\( i, j = 1, \ldots, m; \ i \neq j \)).
Furthermore, we clearly have that $dy_i \in \Omega_i$ ($i = 1, \ldots, m$). Hence $\Omega_i \cap \Pi = \text{span}\{dy_i\}$ ($i = 1, \ldots, m$).

Based on the above considerations, we will solve the three factorization problems for integrable codistributions $\Omega$ and $\Pi$ having the following form:

$$\Omega = \text{span}\{\omega^1, \ldots, \omega^d, \omega^n\}$$  \hfill (53)

$$\Pi = \text{span}\{\omega^{n-m+1}, \ldots, \omega^n\}, \quad \omega^{n-m+1}, \ldots, \omega^n \text{ exact}$$  \hfill (54)

Throughout, we will be interested in local results. In particular, this means by Poincaré's Lemma that a one-form is exact if and only if it is closed.

### 5.1 Factorization Problem 1

We will first give necessary and sufficient conditions for solvability of Factorization Problem 1, and afterwards we will give a method to check these conditions for $\Omega$ and $\Pi$ satisfying (53),(54).

**Proposition 5.12** Consider integrable codistributions $\Omega$ and $\Pi$. Let a function $b : M \rightarrow \mathbb{R}$ be given. Then Factorization Problem 1 is locally solvable for $b$ if and only if there exists a one-form $\pi \in \Pi$ such that

(i) $db - \pi \in \Omega$.

(ii) $b^{-1}\pi$ is exact.

**Proof** (necessity) Assume that $\beta, \phi : M \rightarrow \mathbb{R}$ are such that $d\beta \in \Pi$, $d\phi \in \Omega$, and $b = \beta \phi$. Define $\pi := \phi d\beta$. Then obviously

$$db - \pi = \beta d\phi \in \Omega$$  \hfill (55)

Furthermore,

$$d(b^{-1}\pi) = -b^{-2}db \wedge \pi + b^{-1}d\pi = -b^{-2}db \wedge \pi + b^{-1}d\phi \wedge d\beta =$$

$$-b^{-2}db \wedge \pi + b^{-2}(\beta d\phi \wedge (\phi d\beta)) =$$

$$-b^{-2}(db \wedge \pi + (db - \pi) \wedge \pi) = 0$$  \hfill (56)

and hence $b^{-1}\pi$ is exact.

(sufficiency) Assume that $\pi \in \Pi$ satisfying (i) and (ii) exists. Let $\beta : M \rightarrow \mathbb{R}$ be such that

$$b^{-1}\pi = d\beta$$  \hfill (57)

Since $b^{-1}\pi$ is exact, we have

$$0 = d(b^{-1}\pi) = -b^{-2}db \wedge \pi + b^{-1}\pi$$  \hfill (58)
Define

\[ W := db - \pi \in \Omega \]

Then we have \( d\omega = -d\pi \) and hence

\[
d(b^{-1}\omega) = -b^{-2}(db \wedge \omega) + b^{-1}d\omega =
\]

\[
- b^{-2}(db \wedge \omega) - b^{-1}d\pi = b^{-2}(db \wedge \pi) - b^{-2}(db \wedge \pi) = 0
\]

and hence there exists a \( \phi : M \to \mathbb{R} \) such that

\[ b^{-1}\omega = d\phi \]

We then have

\[ db = \omega + \pi = bd(\tilde{\beta} + \hat{\phi}) \]

which yields

\[ b = ce^{(\tilde{\beta} + \hat{\phi})}, \ c \in \mathbb{R} \setminus \{0\} \]

Defining \( \beta := c \exp(\tilde{\beta}), \phi := \exp(\hat{\phi}) \), we obtain

\[ b = \beta \phi \]

which establishes our claim.

We next turn to the question in what way the conditions of Proposition 5.12 may be checked for integrable codistributions \( \Omega, \Pi \) satisfying (53),(54). Choose \( \omega^d, \cdots, \omega^{n-m} \) such that \( \omega^1, \cdots, \omega^n \) are linearly independent, and let \( \tau_1, \cdots, \tau_n \) be vector fields that are dual to \( \omega^1, \cdots, \omega^n \), i.e.,

\[ \tau_i \wedge \omega_j = \delta_{ij} \quad (i, j = 1, \cdots, n) \]

The following lemma gives some properties of the Lie-brackets \([\tau_i, \tau_j] (i, j = 1, \cdots, n)\) that will be used in the sequel.
Lemma 5.13  

(i) \((\forall i, j \in \{1, \cdots, n\})([\tau_i, \tau_j] \in \text{span}\{\tau_1, \cdots, \tau_{n-m}\})\).

(ii) \((\forall i, j \in \{d + 1, \cdots, n - 1\})([\tau_i, \tau_j] \in \text{span}\{\tau_{d+1}, \cdots, \tau_{n-m}\})\).

Proof

(i) Let \(i, j \in \{1, \cdots, n\}\). We have

\[
[\tau_i, \tau_j] = \sum_{k=1}^{n} ([\tau_i, \tau_j] J \omega^k) \tau_k
\]

(66)

By the Leibniz-formula,

\[
[\tau_i, \tau_j] J \omega^k = \mathcal{L}_{\tau_i}(\tau_j J \omega^k) - \tau_j J \mathcal{L}_{\tau_i} \omega^k =
\]

\[
\mathcal{L}_{\tau_i} \delta_{jk} - \tau_j J (d(\tau_i J \omega^k) + \tau_i J d\omega^k) =
\]

\[
-\tau_j J d\delta_{ik} - \tau_j J \tau_i J d\omega^k = \tau_i J \tau_j J d\omega^k
\]

(67)

Since \(\omega^{n-m+1}, \cdots, \omega^n\) are assumed to be exact, (67) yields in particular:

\[
[\tau_i, \tau_j] J \omega^k = 0 \quad (i, j = 1, \cdots, n; \ k = n - m + 1, \cdots, n)
\]

(68)

From (66) and (68) we then have

\[
[\tau_i, \tau_j] = \sum_{k=1}^{n-m} ([\tau_i, \tau_j] J \omega^k) \tau_k \in \text{span}\{\tau_1, \cdots, \tau_{n-m}\}
\]

(69)

(ii) Since \(\Omega\) is assumed to be integrable, by the Frobenius Theorem there exist \(\Gamma^k_{rs}\) such that

\[
d\omega^k = \sum_{r=1}^{d} \sum_{s=r+1}^{n} \Gamma^k_{rs} \omega^r \wedge \omega^s + \sum_{s=d+1}^{n-1} \Gamma^k_{ns} \omega^n \wedge \omega^s
\]

(70)

Now let \(i, j \in \{d + 1, \cdots, n - 1\}\). Then we have for \(k = 1, \cdots, d\):

\[
[\tau_i, \tau_j] J \omega^k = \tau_i J \tau_j J (\sum_{r=1}^{d} \sum_{s=r+1}^{n} \Gamma^k_{rs} \omega^r \wedge \omega^s + \sum_{s=d+1}^{n-1} \Gamma^k_{ns} \omega^n \wedge \omega^s) =
\]

\[
\tau_i J (- \sum_{r=1}^{d} \Gamma^k_{pj} \omega^r - \Gamma^k_{nj} \omega^n) = 0
\]

(71)

Combining (69) and (71), we establish (ii).

We now have the following result.
**Theorem 5.14** Consider integrable codistributions $\Omega$, $\Pi$ satisfying (53), (54). Choose one-forms $\omega^{d+1}, \ldots, \omega^{n-m}$ such that $\omega^1, \ldots, \omega^n$ are independent, and let $\tau_1, \ldots, \tau_n$ be vector fields that are dual to $\omega^1, \ldots, \omega^n$. Consider a function $b : M \rightarrow \mathbb{R}$. Then Factorization Problem 1 is solvable for $b$ if and only if

$$\mathcal{L}_{\tau_i} b = 0 \quad (i = d + 1, \ldots, n - m)$$

$$\mathcal{L}_{\tau_i}(b^{-1}\mathcal{L}_{\tau_j} b) = 0 \quad (i = 1, \ldots, d; \ j = n - m + 1, \ldots, n - 1)$$

In the sufficiency-part of the proof of Theorem 5.14, we use the following lemma.

**Lemma 5.15** Assume that $b$ satisfies (72), (73). Then there exists a function $\gamma : M \rightarrow \mathbb{R}$ satisfying

$$\mathcal{L}_{\tau_i} \gamma = 0 \quad (i = 1, \ldots, n - m)$$

$$\mathcal{L}_{\tau_i} \gamma = \mathcal{L}_{\tau_i}(b^{-1}\mathcal{L}_{\tau_j} b) \quad (i = n - m + 1, \ldots, n - 1)$$

**Proof** Equalities (74), (75) give a set of $(n - 1)$ PDE’s for $\gamma$. We need to check whether the integrability conditions are satisfied. In the sequel of this proof, we indicate application of Lemma 5.13. (i) by (i) and application of Lemma 5.13. (ii) by (ii).

First, let $i, j \in \{1, \ldots, n - m\}$. Then we have on the one hand

$$\mathcal{L}_{[\tau_i, \tau_j]} \gamma \overset{(74)}{=} 0$$

and on the other hand

$$\mathcal{L}_{[\tau_i, \tau_j]} \gamma = \mathcal{L}_{\tau_i} \mathcal{L}_{\tau_j} \gamma - \mathcal{L}_{\tau_j} \mathcal{L}_{\tau_i} \gamma \overset{(74)}{=} 0$$

Next, let $i, j \in \{n - m + 1, \ldots, n - m\}$. Then we have on the one hand

$$\mathcal{L}_{[\tau_i, \tau_j]} \gamma \overset{(75)}{=} 0$$

and on the other hand

$$\mathcal{L}_{[\tau_i, \tau_j]} \gamma = \mathcal{L}_{\tau_i} \mathcal{L}_{\tau_j} \gamma - \mathcal{L}_{\tau_j} \mathcal{L}_{\tau_i} \gamma \overset{(75)}{=} 0$$

$$= \mathcal{L}_{\tau_i} \mathcal{L}_{\tau_n}(b^{-1} \mathcal{L}_{\tau_j} b) - \mathcal{L}_{\tau_j} \mathcal{L}_{\tau_n}(b^{-1} \mathcal{L}_{\tau_i} b) =$$

$$\mathcal{L}_{[\tau_i, \tau_n]}(b^{-1} \mathcal{L}_{\tau_j} b) + \mathcal{L}_{\tau_n} \mathcal{L}_{\tau_i}(b^{-1} \mathcal{L}_{\tau_j} b) - \mathcal{L}_{[\tau_j, \tau_n]}(b^{-1} \mathcal{L}_{\tau_i} b) =$$

$$\mathcal{L}_{\tau_n} \mathcal{L}_{\tau_j}(b^{-1} \mathcal{L}_{\tau_i} b) \overset{(73)}{=} 0$$

$$= \mathcal{L}_{\tau_n}(b^{-1} \mathcal{L}_{[\tau_i, \tau_j]} b) - \mathcal{L}_{\tau_j}(b^{-1} \mathcal{L}_{[\tau_i, \tau_n]} b) = \cdots =$$

$$\mathcal{L}_{\tau_n}(b^{-1} \mathcal{L}_{[\tau_i, \tau_j]} b) \overset{(72)}{=} 0$$
Last, we consider $i \in \{1, \ldots, n - m\}$, $j \in \{n - m + 1, \ldots, n - 1\}$. Note that for $r \in \{d + 1, \ldots, n - m\}$, $s \in \{n - m + 1, \ldots, n - 1\}$ we have

$$L_{r_{(i)}}(b^{-1}L_{r_{(j)}}b) \overset{(72)}{=} b^{-1}L_{r_{(i)}}L_{r_{(j)}}b = b^{-1}(L_{[r_{(i)}, r_{(j)}]}b + L_{r_{(i)}}L_{r_{(j)}}b) \overset{(ii) \overset{(72)}{=}}{=} 0$$

(80)

We now have on the one hand

$$L_{[r_{(i)}, r_{(j)}]} \gamma \overset{(74)}{=} 0$$

(81)

and on the other hand

$$L_{[r_{(i)}, r_{(j)}]}(b^{-1}L_{r_{(j)}}b) + L_{r_{(i)}}L_{r_{(j)}}(b^{-1}L_{r_{(j)}}b) \overset{(ii)\overset{(73,80)}{=}}{=} 0$$

(82)

From (76,77,78,79,81,82) we conclude that the integrability conditions are satisfied and hence there exists a $\gamma : M - \mathbb{R}$ satisfying (74),(75).

**Proof of Theorem 5.14**

In this proof, we indicate application of Lemma 5.13.(i) by (i) and application of Lemma 5.13.(ii) by (ii).

*(necessity)* Assume that there exist $\beta, \phi : M - \mathbb{R}$ such that $d\beta \in \Pi$, $d\phi \in \Omega$ and $b = \beta \phi$. Note that

$$\Omega^\perp = \operatorname{span}\{\tau_{d+1}, \ldots, \tau_{n-1}\}$$

(83)

$$\Pi^\perp = \operatorname{span}\{\tau_1, \ldots, \tau_{n-m}\}$$

(84)

$$(\Omega + \Pi)^\perp = \Omega^\perp \cap \Pi^\perp = \operatorname{span}\{\tau_{d+1}, \ldots, \tau_{n-m}\}$$

(85)

Let $i \in \{d + 1, \ldots, n - m\}$. then

$$L_{r_{(i)}}b = L_{r_{(i)}}(\beta \phi) = \beta L_{r_{(i)}}\phi + \phi L_{r_{(i)}}\beta \overset{(85)}{=} 0$$

which establishes (72). Next, let $i \in \{1, \ldots, d\}$, $j \in \{n - m + 1, \ldots, n - 1\}$. Then

$$L_{r_{(i)}}(b^{-1}L_{r_{(j)}}b) = L_{r_{(i)}}(\beta^{-1}L_{r_{(j)}}(\beta \phi)) \overset{(83)}{=} L_{r_{(i)}}(\beta^{-1}L_{r_{(j)}}(\beta \phi)) \overset{(84)}{=}$$

$$\beta^{-1}L_{r_{(i)}}L_{r_{(j)}}\beta = \beta^{-1}(L_{[r_{(i)}, r_{(j)}]}\beta + L_{r_{(i)}}L_{r_{(j)}}\beta) \overset{(i)\overset{(84)}{=}}{=} 0$$

which establishes (73).

*(sufficiency)* Assume that $b$ satisfies (72),(73). According to Lemma 5.15, there exists a $\gamma : M - \mathbb{R}$ satisfying (74),(75). Define $\pi \in \Pi$ by

$$\pi = \sum_{j=n-m+1}^{n-1} (L_{r_{(j)}}b)\omega^j + b\gamma \omega^n$$

(86)
Then
\[ db - \pi = \sum_{i=1}^{n} (\mathcal{L}_{\tau_i} b) \omega^i - \sum_{j=n-m+1}^{n-1} (\mathcal{L}_{\tau_j} b) \omega^j - b\gamma \omega^n \] (87)

Furthermore,
\[ d(b^{-1} \pi) = d\left( \sum_{j=n-m+1}^{n-1} b^{-1}(\mathcal{L}_{\tau_j} b) \omega^j + \gamma \omega^n \right) = \]
\[ \sum_{j=n-m+1}^{n-1} d(b^{-1}\mathcal{L}_{\tau_j} b) \wedge \omega^j + \gamma \wedge \omega^n = \]
\[ \sum_{j=n-m+1}^{n-1} \sum_{i=1}^{n} (\mathcal{L}_{\tau_i}(b^{-1}\mathcal{L}_{\tau_j} b)) \omega^i \wedge \omega^j + \sum_{i=1}^{n-1} (\mathcal{L}_{\tau_i} \gamma) \omega^i \wedge \omega^n \] (73,74,75,80)

\[ + \sum_{j=n-m+1}^{n-1} \mathcal{L}_{\tau_n}(b^{-1}\mathcal{L}_{\tau_j} b) \omega^n \wedge \omega^j + \]
\[ \sum_{i=n-m+1}^{n-1} (\mathcal{L}_{\tau_i}(b^{-1}\mathcal{L}_{\tau_j} b)) \omega_i \wedge \omega^n = \]
\[ \sum_{j=n-m+1}^{n-2} \sum_{i=j+1}^{n-1} (\mathcal{L}_{\tau_i}(b^{-1}\mathcal{L}_{\tau_j} b) - \mathcal{L}_{\tau_j}(b^{-1}\mathcal{L}_{\tau_i} b)) \omega^i \wedge \omega^j = \cdots = \]
\[ \sum_{j=n-m+1}^{n-2} \sum_{i=j+1}^{n-1} (b^{-1}\mathcal{L}_{[\tau_i,\tau_j]} b) \omega^i \wedge \omega^j \] (72)

and hence \( b^{-1} \pi \) is exact. Thus, there exists a one-form \( \pi \in \Omega \) such that \( db - \pi \in \Pi \) and \( b^{-1} \pi \) is exact. By Proposition 5.12 this implies that Factorization Problem 1 is solvable for \( b \).

5.2 Factorization Problem 2

We next turn to Factorization Problem 2 for the special case we are interested in. To recapitulate, we are given two integrable codistributions \( \Omega \) and \( \Pi \) satisfying (53),(54), and functions \( a, b : M \to \mathbb{R} \), where \( b \) satisfies (83),(84). We are interested in the question under what conditions there locally exist functions \( \sigma, \phi : M \to \mathbb{R} \) such that
\[ d\sigma \in \Pi \]
\[ d\psi \in \Omega \]
\[ a = \psi + \sigma b \] (89)

The answer is given by the following theorem.
Theorem 5.16

Consider integrable codistributions $\Omega$ and $\Pi$ satisfying (53),(54). Choose one-forms $\omega^{d+1}, \ldots, \omega^{n-m}$ such that $\omega^1, \ldots, \omega^n$ are independent, and let $\tau_1, \ldots, \tau_n$ be vector fields that are dual to $\omega^1, \ldots, \omega^n$. Consider functions $a, b : M \to \mathbb{R}$, where Factorization Problem 1 is solvable for $b$, i.e., $b$ satisfies (83),(84). Then Factorization Problem 2 is solvable for $a$ if and only if

$$\mathcal{L}_{\tau_i} a = 0 \quad (i = d + 1, \ldots, n - m)$$

$$\mathcal{L}_{\tau_i}(b^{-1}\mathcal{L}_{\tau_i} a) = 0 \quad (i = 1, \ldots, n - m; \ j = n - m + 1, \ldots, n - 1)$$

Proof

As before, in this proof we indicate application of Lemma 5.13.(i) by (i) and application of Lemma 5.13.(ii) by (ii).

(necessity) Assume that Factorization Problem 2 is solvable for $a$. Let $\sigma, \psi$ be such that $d\sigma \in \Pi$, $d\psi \in \Omega$ and

$$a = \psi + \sigma b$$

Let $i \in \{d + 1, \ldots, n - m\}$. Then

$$\mathcal{L}_{\tau_i} a = \mathcal{L}_{\tau_i}(\psi + \sigma b) = \mathcal{L}_{\tau_i} \psi + \sigma \mathcal{L}_{\tau_i} b + b \mathcal{L}_{\tau_i} \sigma = 0$$

which establishes the necessity of (90). Next, let $i \in \{1, \ldots, n - m\}$, $j \in \{n - m + 1, \ldots, n - 1\}$. Then

$$\mathcal{L}_{\tau_i}(b^{-1}\mathcal{L}_{\tau_j} a) = \mathcal{L}_{\tau_i}(b^{-1}\mathcal{L}_{\tau_j}(\psi + \sigma b)) =$$

$$\mathcal{L}_{\tau_i}(b^{-1}\mathcal{L}_{\tau_j} \psi) + \mathcal{L}_{\tau_i} \mathcal{L}_{\tau_j} \sigma + \mathcal{L}_{\tau_i}(b^{-1}\sigma \mathcal{L}_{\tau_j} b) =$$

$$\mathcal{L}_{[\tau_i, \tau_j]} \sigma + \mathcal{L}_{\tau_i} \mathcal{L}_{\tau_j} \sigma + (b^{-1}\mathcal{L}_{\tau_j} b)(\mathcal{L}_{\tau_i} \sigma + \sigma \mathcal{L}_{\tau_i}(b^{-1}\mathcal{L}_{\tau_j} b) = 0$$

where the last equality follows from Lemma 5.13.(i) and (73,80,84). This establishes the necessity of (91).

(sufficiency) Assume that $a$ satisfies (90) and (91). Let $k \in \{n - m + 1, \ldots, n - 1\}$ and define

$$\hat{a}_k := \mathcal{L}_{\tau_k} a$$

By (91) we have for $i = 1, \ldots, n - m$:

$$b \mathcal{L}_{\tau_i} \hat{a}_k = \hat{a}_k \mathcal{L}_{\tau_i} b$$

Furthermore, we have by Lemma 5.13.(ii) and (90) for $i = d + 1, \ldots, n - m$:

$$\mathcal{L}_{\tau_i} \hat{a}_k = \mathcal{L}_{\tau_i} \mathcal{L}_{\tau_k} a = \mathcal{L}_{[\tau_i, \tau_k]} a + \mathcal{L}_{\tau_k} \mathcal{L}_{\tau_i} a = 0$$
Now let \( i \in \{1, \ldots, n-m\}, j \in \{n-m+1, \ldots, n-1\} \). Then

\[
0 = \mathcal{L}_{r_j} \mathcal{L}_{r_i}(b^{-1} \hat{a}_k) =
\]

\[
\mathcal{L}_{[r_j, r_i]}(b^{-1} \hat{a}_k) + \mathcal{L}_{r_i} \mathcal{L}_{r_j}(b^{-1} \hat{a}_k) =
\]

\[
\mathcal{L}_{r_i}(-b^{-2}(\mathcal{L}_{r_j} b) \hat{a}_k + b^{-1} \mathcal{L}_{r_j} \hat{a}_k) =
\]

\[
- b^{-1} \hat{a}_k \mathcal{L}_{r_i}(b^{-1} \mathcal{L}_{r_j} b) - b^{-1}(\mathcal{L}_{r_j} b)(\mathcal{L}_{r_i}(b^{-1} \hat{a}_k) + \mathcal{L}_{r_i}(b^{-1} \mathcal{L}_{r_j} \hat{a}_k)) \quad (84, 91)
\]

\[
\mathcal{L}_{r_i}(b^{-1} \mathcal{L}_{r_j} \hat{a}_k) = - b^{-2}(\mathcal{L}_{r_j} b)(\mathcal{L}_{r_i} \hat{a}_k) + b^{-1} \mathcal{L}_{r_i} \mathcal{L}_{r_j} \hat{a}_k \quad (96)
\]

\[
b^{-1}(-\mathcal{L}_{r_i} \hat{a}_k) \mathcal{L}_{r_i}(b^{-1} \hat{a}_k) + \mathcal{L}_{r_i} \mathcal{L}_{r_j} \hat{a}_k) = b^{-1} \hat{a}_k \mathcal{L}_{r_i}(b^{-1} \mathcal{L}_{r_j} \hat{a}_k)
\]

and hence

\[
\mathcal{L}_{r_i}(b^{-1} \hat{a}_k) = 0 \quad (i = 1, \ldots, n-m; \ j, k = n-m+1, \ldots, n-1) \quad (99)
\]

By Theorem 5.14, we obtain from (97) and (99) that for \( k = n-m+1, \ldots, n-1 \) there exist \( \gamma_k, \delta_k \) with \( d\gamma_k, d\delta_k \in \Omega \), such that

\[
\mathcal{L}_{r_k} a = \gamma_k \delta_k \quad (100)
\]

Furthermore, we know that there exist \( \beta, \phi \) with \( d\beta, d\phi \in \Omega \) such that

\[
b = \beta \phi \quad (101)
\]

and from (91) we know that

\[
d(b^{-1} \mathcal{L}_{r_k} a) \in \Pi \quad (102)
\]

Now

\[
d(b^{-1} \mathcal{L}_{r_k} a) = - b^{-2}(\mathcal{L}_{r_k} a) d\beta + b^{-1} d\mathcal{L}_{r_k} a =
\]

\[
b^{-1}(\delta_k(-\beta^{-1} \gamma_k d\beta + d\gamma_k) + \gamma_k(\phi^{-1} \delta_k d\phi + d\delta_k)) \quad (103)
\]

Combining (102) and (103) we obtain

\[
\phi d\delta_k = \beta_k d\phi \quad (104)
\]

which gives

\[
\delta_k = c\phi, \ c \in \mathbb{H} \quad (k = n-m+1, \ldots, n-1) \quad (105)
\]

It is easily checked that there exists \( \gamma : \mathcal{M} \to \mathbb{H} \) such that \( d\gamma \in \Pi \) and

\[
\mathcal{L}_{r_k} \gamma = \gamma_k \quad (k = n-m+1, \ldots, n-1) \quad (106)
\]

Define

\[
\tilde{\gamma} := \gamma - \frac{a}{c\phi} \quad (107)
\]
Then
\[ L_{\tau k} \tilde{\psi} = 0 \quad (k = n - m + 1, \ldots, n - 1) \]  \hfill (108)
and hence \( d\tilde{\psi} \in \Omega \). We then have
\[ a = -c\phi\tilde{\psi} + c\phi\gamma = -c\phi\tilde{\psi} + \frac{c\gamma}{\beta} \beta\phi = \psi + \sigma b \]  \hfill (109)
where \( \psi := -c\phi\tilde{\psi} \), \( \sigma := (c\gamma/\beta) \). Clearly, we have \( d\psi \in \Omega \), \( d\sigma \in \Pi \). Hence we have established our claim. \[ \]

5.3 Factorization Problem 3

We now consider integrable codistributions \( \Omega \) and \( \Pi \) of the form (53),(54) and functions \( a, b_1, \ldots, b_\ell : M \rightarrow \mathbb{R} \) where \( b_1, \ldots, b_\ell \) are not all identically zero. We are interested in the question under what conditions there locally exist functions \( \beta_1, \ldots, \beta_\ell, \sigma, \phi, \psi : M \rightarrow \mathbb{R} \) such that
\[
d\beta_1, \ldots, d\beta_\ell, \sigma \in \Pi \\
d\phi, d\psi \in \Omega \\
b_i = \beta_i\phi \quad (i = 1, \ldots, \ell) \\
a = \psi + \sigma \phi 
\]  \hfill (110)
The solution of this factorization problem is given in the following theorem.

**Theorem 5.17** Consider integrable codistributions \( \Omega \) and \( \Pi \) satisfying (53),(54) and functions \( a, b_1, \ldots, b_\ell : M \rightarrow \mathbb{R} \), where \( b_1, \ldots, b_\ell \) are not all identically zero. Choose \( k \in \{1, \ldots, \ell\} \) such that \( b_k \neq 0 \). Then Factorization Problem 3 is solvable if and only if

(i) Factorization Problem 1 is solvable for \( b_k \).

(ii) Factorization Problem 2 is solvable for \( a, b_k \).

(iii) \( \forall i \in \{1, \ldots, \ell\} : d(b_i/b_k) \in \Pi \).

**Proof** (necessity) Let \( \beta_1, \ldots, \beta_\ell, \sigma, \phi, \psi : M \rightarrow \mathbb{R} \) be such that (110) holds. Then clearly Factorization Problem 1 is solvable for \( b_k \). Furthermore, we have
\[ a = \psi + \sigma \phi = \psi + \frac{\sigma}{\beta_k} b_k \]  \hfill (111)
Since \( d(\sigma/\beta_k) \in \Pi \), we see that Factorization Problem 2 is solvable for \( a, b_k \). Finally, we have for \( i \in \{1, \ldots, \ell\} : \\
d\left(\frac{b_i}{b_k}\right) = d\left(\frac{\beta_i\phi}{\beta_k\phi}\right) = d\left(\frac{\beta_i}{\beta_k}\right) \in \Pi 
\]  \hfill (112)
\[ \]

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(sufficiency) Assume that (i), (ii), (iii) are satisfied. Let $\beta_k, \phi, \psi, \tilde{\sigma}, \rho_1, \ldots, \rho_{k-1}, \rho_{k+1}, \ldots, \rho_{\ell}$ be such that

$$b_k = \beta_k \phi, \quad d\beta_k \in \Pi, \quad d\phi \in \Omega$$

$$a = \psi + \tilde{\sigma} b_k, \quad d\psi \in \Omega$$

$$\frac{b_i}{b_k} = \rho_i \quad (i = 1, \ldots, \ell; \ i \neq k)$$

where by (iii) we have that $d\rho_i \in \Pi$. Define $\beta_i = \rho_i \beta_k (i = 1, \ldots, \ell; \ i \neq k)$, $\sigma = \tilde{\sigma} \beta_k$. Then clearly $d\beta_i, d\sigma \in \Pi$. Furthermore,

$$b_i = b_k \rho_i = \beta_k \rho_i \phi = \beta_i \phi \quad (i = 1, \ldots, \ell; \ i \neq k)$$

$$a = \psi + \tilde{\sigma} b_k = \psi + \tilde{\sigma} \beta_k \phi = \psi + \sigma \phi$$

Hence Factorization Problem 3 is solvable for $a, b_1, \ldots, b_\ell$.

6 Conclusions

In this paper we have given conditions for the solvability of the strong input-output decoupling problem via regular static output feedback (SIODPof). It turned out that the solvability of the SIODPof is equivalent to the solvability of a factorization problem for a set of functions with respect to certain codistributions. A constructive method to check the solvability of this factorization problem was given.

Although the conditions for solvability of the SIODPof that were presented in this paper can be checked, the paper does not give a procedure to obtain a static output feedback that solves the problem. The question of how to obtain such a feedback remains a topic for future research.

An extension of the SIODPof is the strong input-output decoupling problem via regular measurement feedback (SIODPmf). In this problem, one looks for a regular static measurement feedback

$$Q_{mf}: \ u = \alpha(z) + \beta(z)v, \quad |\beta(z)| \neq 0$$

where $z = k(x) \in M^p$ denotes the measurements that can be made of the system, such that the system $\Sigma \circ Q_{mf}$ with controls $v$ and outputs $y = h(x)$ is strongly input-output decoupled. Analogous to Corollary 4.11 one may prove the following result.

**Theorem 6.18** Consider a nonlinear control system $\Sigma$ of the form (18) with measurements $z = k(x) \in M^p$ and assume that the relative degrees $r_i$ of the outputs $y_i$ are all finite. Let $x_0 \in M$ be given. Define $a_i(x)$ ($i = 1, \ldots, m$) and $b_{ij}(x)$ ($i, j = 1, \ldots, m$) as in (25), (26) respectively. Then the SIODPmf is solvable around $x_0$ if and only if there exists a neighborhood $U \subset M$ of $x_0$ such that

(i) $B(x)$ is invertible for all $x \in U$. 

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(ii) For all \( i \in \{1, \ldots, m\} \) we have that Factorization Problem 3 is solvable on \( U \) for \( a_i, b_{i1}, \ldots, b_{im} \) with respect to \( \tilde{\Omega} \) and \( \tilde{\Pi} := \text{span}\{dk_1, \ldots, dk_p\} \).

The question that arises here is in what way one may translate condition (ii) of the above theorem into checkable conditions analogous to the ones obtained in Section 5. The problem is that one cannot use the special structure that was employed in Section 5 any more. The solution of this problem also remains a topic for future research.

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References


Appendix: Proofs of results from Section 3

Proof of Proposition 3.4

It is easily checked that

$$y_i^{(r_i)} = a_i(x) + \sum_{j=1}^{m} b_{ij}(x)u_j \quad (i = 1, \cdots, m)$$  \hspace{1cm} (113)

Hence the requirement that the decoupling matrix is an invertible diagonal matrix is equivalent to (27), (28). In the rest of the proof we assume that (27), (28) hold. We then have

$$y_i^{(r_i)} = a_i(x) + b_{ii}(x)u_i$$  \hspace{1cm} (114)

We now prove that

$$dy_i^{(r_i+k)} \in \text{span}_{K_u} \{ dx, du_i, \cdots, du_i^{(r_i+k)} \} \quad (\forall k \in \mathbb{N})$$  \hspace{1cm} (115)

if and only if

$$da_i^{(k)}, db_i^{(k)} \in \text{span}_{K_u} \{ dx, dy_i^{(r_i)}, \cdots, dy_i^{(r_i+k-1)} \} \quad (\forall k \in \mathbb{N})$$  \hspace{1cm} (116)

Note that (115) is equivalent to (23), while (116) is equivalent to (29). By (114) we have for $k \in \mathbb{N}$:

$$y_i^{(r_i+k)} = a_i^{(k)} + \sum_{r=0}^{k} \binom{k}{r} b_{ii}^{(r)} u_i^{(k-r)}$$  \hspace{1cm} (117)

and hence

$$dy_i^{(r_i+k)} = da_i^{(k)} + \sum_{r=0}^{k} \binom{k}{r} \left[ u_i^{(k-r)} dy_i^{(r)} b_{ii}^{(r)} + b_{ii}^{(r)} du_i^{(k-r)} \right]$$  \hspace{1cm} (118)

First assume that (116) holds. We prove by induction that this implies (115). By (118), (115) clearly holds for $k = 0$. Next assume that (115) holds for $k = 0, \cdots, \ell - 1$. Then

$$dy_i^{(r_i+\ell)} = da_i^{(\ell)} + \sum_{r=0}^{\ell} \binom{\ell}{r} \left[ u_i^{(\ell-r)} dy_i^{(r)} b_{ii}^{(r)} + b_{ii}^{(r)} du_i^{(\ell-r)} \right]$$  \hspace{1cm} (116) \hspace{1cm} (119)

$$\in \text{span}_{K_u} \{ dx, dy_i^{(r_i)}, \cdots, dy_i^{(r_i+\ell-1)}, du_i, \cdots, du_i^{(\ell)} \} = \text{span}_{K_u} \{ dx, du_i, \cdots, du_i^{(\ell)} \}$$  \hspace{1cm} (116) \hspace{1cm} (119)

which implies that (115) holds for all $k \in \mathbb{N}$.

Next, assume that (115) holds. We show by induction that this implies (116). Clearly, (116) holds for $k = 0$. Next, assume that (116) holds for $k = 0, \cdots, \ell - 1$. We have

$$dy_i^{(r_i+\ell)} = \omega + (da_i^{(\ell)} + u_i db_i^{(\ell)})$$  \hspace{1cm} (120)
where by the induction hypothesis and (115)
\[
\omega := \sum_{r=0}^{\ell-1} \binom{\ell}{r} [u_i^{(\ell-r)}db_{ii}^{(r)} + b_{ii}^{(r)}du_i^{(\ell-r)}] + b_{ii}^{(0)}du_i \in \\text{span}_K \{dx, dy_i^{(r)}, \ldots, dy_i^{(r+i-2)}, du_i, \ldots, du_i^{(\ell)}\} = \\text{span}_K \{dx, du_i, \ldots, du_i^{(\ell)}\}
\]
Then (115),(120),(121) imply that we have
\[
(da_i^{(\ell)} + u_i db_{ii}^{(\ell)}) \in \text{span}_K \{dx, du_i, \ldots, du_i^{(\ell)}\}
\]
or, equivalently,
\[
d(a_i^{(\ell)} + u_i db_{ii}^{(\ell)}) \in \text{span}_K \{dx, du_i, \ldots, du_i^{(\ell)}\}
\]
This implies that for all \(j \in \{1, \ldots, i-1, i+1, \ldots, m\}\) we have
\[
\frac{\partial}{\partial u_j} \left( a_i^{(\ell)} + u_i db_{ii}^{(\ell)} \right) = 0
\]
This gives
\[
0 = \frac{\partial}{\partial u_i} \left( \frac{\partial}{\partial u_j} \left( a_i^{(\ell)} + u_i db_{ii}^{(\ell)} \right) \right) = \frac{\partial}{\partial u_j} \left( \frac{\partial a_i^{(\ell)}}{\partial u_i} \right) + \left( \frac{\partial b_{ii}^{(\ell)}}{\partial u_i} \right) + u_i \frac{\partial}{\partial u_j} \left( \frac{\partial b_{ii}^{(\ell)}}{\partial u_i} \right)
\]
Now
\[
\frac{\partial^2 a_i^{(\ell)}}{\partial u_j \partial u_i} = \frac{\partial^2}{\partial u_j \partial u_i} \left( \frac{\partial a_i^{(\ell-1)}}{\partial x} [f(x) + g(x)u] + \sum_{r=0}^{\ell-2} \frac{\partial a_i^{(\ell-1)}}{\partial u_i^{(r)}} u_i^{(r+1)} \right) = \frac{\partial^2}{\partial u_j \partial u_i} \left( \frac{\partial a_i^{(\ell-1)}}{\partial x} \right) f(x) + g(x)u + \sum_{r=0}^{\ell-2} \frac{\partial^2}{\partial u_j \partial u_i} \left( \frac{\partial a_i^{(\ell-1)}}{\partial x} u_i^{(r+1)} \right) = 0
\]
Similarly, we have
\[
\frac{\partial^2 b_{ii}^{(\ell)}}{\partial u_j \partial u_i} = 0
\]
Thus, (125),(126),(127) yield
\[
\frac{\partial b_{ii}^{(\ell)}}{\partial u_j} = 0
\]
which, together with (124), gives
\[
\frac{\partial u_i^{(\ell)}}{\partial u_j} = 0 \tag{129}
\]
Then from (123), (128), (129) we obtain
\[
da_i^{(\ell)}, db_i^{(\ell)} \in \text{span}_\mathcal{K}_u \{dx, du_i, \ldots, du_i^{(\ell)}\} \tag{130}
\]
What remains to be shown is that, if (115) holds, we have for all \(k \in \mathbb{N}\):
\[
\text{span}_\mathcal{K}_u \{dx, dy_i^{(r)}, \ldots, dy_i^{(r+k)}\} = \text{span}_\mathcal{K}_u \{dx, du_i, \ldots, du_i^{(k)}\} \tag{131}
\]
First consider the case \(k = 0\). Then,
\[
\text{span}_\mathcal{K}_u \{dx, dy_i^{(r)}\} = \text{span}_\mathcal{K}_u \{dx, da_i + u_i db_i\} = \text{span}_\mathcal{K}_u \{dx, du_i\} \tag{132}
\]
and hence (131) holds for \(k = 0\). Next assume that (131) holds for \(k = 0, \ldots, \ell - 1\). By (115) and (118) we have that \(dy_i^{(r+\ell)}\) has the form
\[
dy_i^{(r+\ell)} = \omega + b_{ij} du_i^{(\ell)} \tag{133}
\]
where \(\omega \in \text{span}_\mathcal{K}_u \{dx, du_i, \ldots, du_i^{(\ell - 1)}\}\). By the induction hypothesis this implies
\[
\text{span}_\mathcal{K}_u \{dx, dy_i^{(r)}\} = \text{span}_\mathcal{K}_u \{dx, dy_i^{(r)}, \ldots, dy_i^{(r+\ell - 1)}, \omega + b_{ij} du_i^{(\ell)}\} = \tag{134}
\]
and hence (131) holds for all \(k \in \mathbb{N}\). Combining (130) and (131), we obtain that (116) holds for all \(k \in \mathbb{N}\). 

**Proof of Proposition 3.5**

(i) Follows from the fact that \(\tilde{\Omega}_i^k\) is a non-increasing sequence of subspaces with \(\text{dim}(\tilde{\Omega}_i^k) \leq n\).

(ii) Consider the sequence of subspaces defined by
\[
\tilde{\Omega}_i^0 := \text{span}_\mathcal{K}\{dx\}
\]
\[
\tilde{\Omega}_i^{k+1} := \{\omega \in \tilde{\Omega}_i^k \mid \omega^{(k+1)} \in \text{span}_\mathcal{K}_u \{dx, dy_i^{(r)}, \ldots, dy_i^{(r+k)}\}\} \tag{135}
\]
\((k \in \mathbb{N})\)

Clearly, \(\tilde{\Omega}_i^k\) is a non-increasing sequence of subspaces with \(\text{dim}(\tilde{\Omega}_i^k) \leq n\). Hence there exists a \(k\) such that
\[
\forall k \geq \tilde{k} + 1 : \tilde{\Omega}_i^k = \tilde{\Omega}_i^{\tilde{k}} \tag{136}
\]
It is easily checked that

\[ \tilde{\Omega}_i = \tilde{\Omega}_i^k \tag{137} \]

We will now show by induction that for all \( k \in \mathbb{N} \) we have

\[ \tilde{\Omega}_i^k = \tilde{\Omega}_i^k \tag{138} \]

and hence that (ii) holds, with \( \tilde{k} = k^* \). By (31),(135) we have that (138) holds for \( k = 0 \). Now assume that (138) holds for \( k = 0, \cdots, \ell \).

Let \( \omega \in \tilde{\Omega}_i^{\ell+1} \), i.e.

\[ \omega \in \tilde{\Omega}_i^\ell = \tilde{\Omega}_i^\ell \tag{139} \]

and

\[ \tilde{\omega} \in \tilde{\Omega}_i^\ell + \text{span}_{\mathcal{K}_n}(\{d y_i^{(r_1)}\}) = \tilde{\Omega}_i^\ell + \text{span}_{\mathcal{K}_n}(\{d y_i^{(r_1)}\}) \tag{140} \]

By (140) there exist \( \tilde{\omega} \in \tilde{\Omega}_i^\ell, \pi \in \text{span}_{\mathcal{K}_n}(\{d y_i^{(r_1)}\}) \) such that \( \tilde{\omega} = \omega + \pi \). Then

\[ \omega^{(\ell+1)} = \omega^{(\ell)} + \pi^{(\ell)} \in \text{span}_{\mathcal{K}_n}(\{d x, d y_i^{(r_1)}, \cdots, d y_i^{(r_1+\ell)}\}) \tag{141} \]

by definition of \( \tilde{\Omega}_i^\ell \). Thus we have \( \omega \in \tilde{\Omega}_i^{\ell+1} \). Hence we have proved that

\[ \tilde{\Omega}_i^{\ell+1} \subset \tilde{\Omega}_i^{\ell+1} \tag{142} \]

Next, let \( \omega \in \tilde{\Omega}_i^{\ell+1} \), i.e.,

\[ \omega \in \tilde{\Omega}_i^\ell = \tilde{\Omega}_i^\ell \tag{143} \]

and

\[ \omega^{(\ell+1)} \in \text{span}_{\mathcal{K}_n}(\{d x, d y_i^{(r_1)}, \cdots, d y_i^{(r_1+\ell)}\}) \tag{144} \]

By (143) we have that

\[ \omega = \omega_x + \pi \tag{145} \]

where \( \omega_x \in \tilde{\Omega}_i^{\ell-1} = \tilde{\Omega}_i^{\ell-1} \) and \( \pi \in \text{span}_{\mathcal{K}_n}(\{d y_i^{(r_1)}\}) \). Then by (144):

\[ \omega_x^{(\ell)} = \omega^{(\ell+1)} - \pi^{(\ell)} \in \text{span}_{\mathcal{K}_n}(\{d x, d y_i^{(r_1)}, \cdots, d y_i^{(r_1+\ell-1)}\}) \tag{146} \]

and hence

\[ \omega_x \in \tilde{\Omega}_i^\ell = \tilde{\Omega}_i^\ell \tag{147} \]

By (143),(145),(147) we then have \( \omega \in \tilde{\Omega}_i^\ell \) and thus

\[ \tilde{\Omega}_i^{\ell+1} \subset \tilde{\Omega}_i^{\ell+1} \tag{148} \]

From (142) and (148) we obtain that \( \tilde{\Omega}_i^{\ell+1} = \tilde{\Omega}_i^{\ell+1} \), and hence (138) holds for \( k = \ell + 1 \), which establishes our claim.

\[ \blacksquare \]
Proof of Proposition 3.6

Consider the following algorithm ([7]) to calculate $R_i^*$:

$$S'_0 := \{0\}$$

$$S'_k := \Delta_i \cap ([k, S'_k] + \sum_{j=1}^{m} [g_j, S'_j] + \mathcal{G}) + S'_k \quad (k \in \mathbb{N})$$

where $\mathcal{G} = \text{span}\{g_1, \ldots, g_m\}$, together with algorithm (31). We are going to prove by induction that for all $k \in \mathbb{N}$ we have

$$S'^k_k = \Omega^k_i$$

Clearly, (150) holds for $k = 0$. Assume that (150) holds for $k = 0, \ldots, \ell$. We have from (149):

$$S'^{k+1}_{k+1} = (\Delta_i^k + [f, S'^k_i] \cap (\bigcap_{j=1}^{m} [g_j, S'^k_j] \cap \mathcal{G}) \cap S'^k_i =$$

$$\{\omega \in \Omega^k_i \mid (\exists \varpi \in \Pi_i)(\exists \varphi \in \mathcal{G}) (\forall \tau \in S'^k_i)(\forall \sigma \in \{f, g_1, \ldots, g_m\})([\sigma, \tau]|_\omega = 0))\}$$

where

$$\Pi_i := \Delta_i^k = \text{span}\{dy_i, \ldots, dy_i^{(r_i-1)}\}$$

Assume that $\omega \in S'^{k+1}_{k+1}$, say $\omega = \hat{\omega} + \bar{\omega}$, where $\hat{\omega} \in \Pi_i$ and $\bar{\omega} \in \mathcal{G}$ satisfies

$$(\forall \tau \in S'^k_i)(\forall \sigma \in \{f, g_1, \ldots, g_m\})([\sigma, \tau]|_\omega = 0)$$

It is straightforwardly checked that for all $k \in \mathbb{N}$ we have that $\Pi_i \subset \Omega^k_i$. Hence

$$\hat{\omega} \in \text{span}_{\mathcal{K}_i}\{dy_i, \ldots, dy_i^{(r_i)}\} \subset \Omega^k_i + \text{span}_{\mathcal{K}_i}\{dy_i^{(r_i)}\}$$

Furthermore, we have (with some abuse of notation):

$$\hat{\omega} = \mathcal{L}_{f+g} \hat{\omega} = \mathcal{L}_f \hat{\omega} + \sum_{j=1}^{m} \mathcal{L}_{g_j} \hat{\omega} =$$

$$\mathcal{L}_f \hat{\omega} + \sum_{j=1}^{m} (u_j \mathcal{L}_{g_j} \hat{\omega} + (g_j \downarrow \hat{\omega}) du_j) =$$

$$\mathcal{L}_f \hat{\omega} + \sum_{j=1}^{m} u_j \mathcal{L}_{g_j} \hat{\omega}$$

Now let $\tau \in S'^k_i$, $\sigma \in \{f, g_1, \ldots, g_m\}$. Then we have by the Leibniz formula:

$$\tau \downarrow \mathcal{L}_\sigma \hat{\omega} = \mathcal{L}_\sigma(\tau \downarrow \hat{\omega}) = [\sigma, \tau]|_\hat{\omega} = \mathcal{L}_\sigma(\tau \downarrow \hat{\omega}) =$$

$$\mathcal{L}_\sigma(\tau \downarrow (\omega - \hat{\omega})) = \mathcal{L}_\sigma(\tau \downarrow \hat{\omega}) = 0$$

$$27$$
where the one-but-last equality follows from the fact that \( \omega \in S^i_{\ell+1} \subset S^i_{\ell} \) and the last equality from the fact that \( \tilde{\omega} \in \Pi_i \subset S^i_\ell \). By (154),(155),(156) we then have
\[
\tilde{\omega} \in S^i_\ell + \operatorname{span}_{\mathcal{K}} \{ d y_i^{(r_i)} \} = \bar{\Omega}^i_\ell + \operatorname{span}_{\mathcal{K}} \{ d y_i^{(r_i)} \}
\] (157)

Hence \( \omega \in \bar{\Omega}^{i+1}_\ell \) and thus
\[
S^i_{\ell+1} \subset \bar{\Omega}^{i+1}_\ell
\] (158)

Next, assume that \( \omega \in \bar{\Omega}^{i+1}_\ell \), i.e.,
\[
\omega \in \bar{\Omega}^i_\ell = S^i_\ell
\] (159)
and
\[
\tilde{\omega} \in \bar{\Omega}^i_\ell + \operatorname{span}_{\mathcal{K}} \{ d y_i^{(r_i)} \}
\] (160)

By (159) we have that there exist \( \bar{\omega} \in \Pi_i, \tilde{\omega} \in \mathcal{G}^i_\ell \) such that \( \omega = \bar{\omega} + \tilde{\omega} \). Clearly, \( \tilde{\omega} \in \operatorname{span}_{\mathcal{K}_n} \{ d y_i, \cdots, d y^{r_i}_i \} \subset \bar{\Omega}^i_\ell + \operatorname{span}_{\mathcal{K}_n} \{ d y_i^{(r_i)} \} \). Moreover, since \( \tilde{\omega} \in \mathcal{G}^i_\ell \), we have that \( \tilde{\omega} \in \operatorname{span}_{\mathcal{K}} \{ d x \} \). With (160) this yields:
\[
\tilde{\omega} \in \bar{\Omega}^i_\ell = S^i_\ell
\] (161)

Hence we have for all \( \tau \in S^i_\ell \):
\[
0 = \tau \int \tilde{\omega} = \tau \int (\mathcal{L}_f \tilde{\omega} + \sum_{j=1}^m (u_j \mathcal{L}_{g_j} \tilde{\omega} + (g_j \mathcal{L}_j \tilde{\omega})d u_j ) =
\]
\[
(\tau \int \mathcal{L}_f \tilde{\omega} + \sum_{j=1}^m u_j (\tau \int \mathcal{L}_{g_j} \tilde{\omega}) =
\]
\[
\mathcal{L}_f (\tau \int \tilde{\omega}) - [f, \tau] \int \tilde{\omega} + \sum_{j=1}^m u_j (\mathcal{L}_{g_j} (\tau \int \tilde{\omega}) - [g_j, \tau] \int \tilde{\omega} =
\]
\[
- [f, \tau] \int \tilde{\omega} - \sum_{j=1}^m u_j [g_j, \tau] \int \tilde{\omega}
\]
which gives that
\[
(\forall \tau \in S^i_\ell) (\forall \sigma \in \{f, g_1, \cdots, g_m\}) ([\sigma, \tau] \int \tilde{\omega} = 0)
\] (163)

This implies that \( \omega \in S^i_{\ell+1} \) and hence
\[
\bar{\Omega}^{i+1}_\ell \subset S^i_{\ell+1}
\] (164)

From (158),(164) we obtain that \( \bar{\Omega}^{i+1}_\ell = S^i_{\ell+1} \) and hence (150) holds for \( k = \ell + 1 \). This establishes our claim.