

## Solution to Problem 63-6 : Asymptotic distribution of lattice points in a random rectangle

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## PROBLEMS AND SOLUTIONS

EDITED BY MURRAY S. KLAMKIN

All problems and solutions should be sent to Murray S. Klamkin, Department of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, and should be submitted in accordance with the instructions given on the inside front cover. An asterisk placed beside a problem number indicates that the problem was submitted without solution. Proposers and solvers whose solution is published will receive 10 reprints of the corresponding problem section. Other solvers will receive just one reprint.

### SOLUTIONS†

Late solutions:

Problems 62-12, 63-1, and 63-3 were also solved by SIDNEY SPITAL (California State Polytechnic College).

Problem 62-7 was also solved by J. K. MACKENZIE (Chemical Research Laboratories, Melbourne, Victoria), who gives the more complete numerical result

$$P(r)/2\pi = 1.275659 - 0.12500 |r| - 0.039306r^2 + 0.003906 |r|^3 \\ + 0.000914r^4 - 0.000072 |r|^5 - \dots,$$

for  $|r| < 2$ .

Problem 63-6, *Asymptotic Distribution of Lattice Points in a Random Rectangle*, by WALTER WEISSBLUM (AVCO Corporation).

An  $n \times n^{-1}$  rectangle is thrown at random on the plane with angle uniformly distributed in  $0 \leq \theta \leq 2\pi$  and center of rectangle also uniformly distributed in  $0 \leq x \leq 1, 0 \leq y \leq 1$ . Find the limit as  $n \rightarrow \infty$  of the distribution of the number of lattice points contained in the rectangle.

Solution by N. G. DE BRUIJN (Technological University, Eindhoven, Netherlands).

Let  $p_j(n)$  be the probability of catching  $j$  lattice points;  $p_j$  denotes its limit as  $n \rightarrow \infty$ . We have

$$(1) \quad \sum_{j=0}^{\infty} p_j(n) = 1, \quad \sum_{j=0}^{\infty} j p_j(n) = 1,$$

since the expectation of  $j$  equals the area of the rectangle. We shall show that

$$(2) \quad p_j = 3\pi^{-2}((j-1)^{-2} - 2j^{-2} + (j+1)^{-2}), \quad j = 2, 3, \dots,$$

and then (1) produces the values of  $p_0$  and  $p_1$ :

$$p_0 = 3\pi^{-2}, \quad p_1 = 1 - 21/(4\pi^2).$$

If three lattice points form a proper triangle, then its area is  $\geq \frac{1}{2}$ . So if our rectangle catches all three, two of them lie in vertices of the triangle. As this

† In order to decrease the large backlog of solutions, this issue only contains solutions. Proposals will be resumed in the next issue.

happens with probability zero, we may assume that if more than two lattice points are caught, then they are all on a line.

Let  $\mathbf{v}$  be a nonzero vector with integral components. To each lattice point  $P$  we let correspond a "needle", viz., the line segment  $(P, P + \mathbf{v})$ . The probability that our rectangle catches a needle is easily shown to be  $K(|\mathbf{v}|)$ , where  $|\mathbf{v}|$  is the length of  $\mathbf{v}$ , and

$$K(d) = 0 \quad \text{if } d > n,$$

$$K(d) = 2\pi^{-1} \int_0^{\arcsin(1/nd)} (n^{-1} - d \sin \phi) (n - d \cos \phi) d\phi \quad \text{if } 0 < d \leq n.$$

Let  $t_j(n; \mathbf{v})$  be the probability that the rectangle catches exactly  $j$  lattice points ( $j \geq 2$ ) with difference vector  $\mathbf{v}$  (that is, the probability that there exists a lattice point  $P$  such that  $P + \mathbf{v}, \dots, P + j\mathbf{v}$  are inside, but  $P$  and  $P + (j + 1)\mathbf{v}$  outside). The probability for catching a needle corresponding to a vector  $k\mathbf{v}$  can now be expressed as follows:

$$K(k|\mathbf{v}|) = t_{k+1}(n, \mathbf{v}) + 2t_{k+2}(n, \mathbf{v}) + 3t_{k+3}(n, \mathbf{v}) + \dots,$$

whence, for  $j = 2, 3, \dots$ ,

$$t_j(n, \mathbf{v}) = K((j + 1)|\mathbf{v}|) - 2K(j|\mathbf{v}|) + K((j - 1)|\mathbf{v}|).$$

So for  $j = 2, 3, \dots$ , we have  $p_j(n) = s_{j+1}(n) - 2s_j(n) + s_{j-1}(n)$ , where for  $k = 1, 2, \dots$ ,

$$(3) \quad s_k(n) = \frac{1}{2} \sum_{\mathbf{v}}^* K(k|\mathbf{v}|),$$

where  $*$  denotes that the summation is taken over all primitive vectors with integral components ("primitive" means that the components have g.c.d. 1; in other words, that the vector is not a multiple of a smaller integral vector). The factor  $\frac{1}{2}$  arises from the fact that if a sequence of lattice points can be described by a vector  $\mathbf{v}$ , then it can also be described by  $-\mathbf{v}$ .

It remains to show that  $s_k(n) \rightarrow 3\pi^{-2}k^{-2}$  as  $n \rightarrow \infty$ . We easily evaluate

$$K(\rho) = \pi^{-1}n^{-2}\rho^{-1}\{n - \rho + O((\rho + 1)^{-1})\}.$$

Taking into account that the probability of a lattice vector being primitive equals  $6\pi^{-2}$ , we obtain

$$\frac{1}{2} \sum_{\mathbf{v}}^* K(k|\mathbf{v}|) \sim \frac{1}{2} \cdot 6\pi^{-2} \int_0^{2\pi} d\phi \int_0^{n/k} \rho K(k\rho) d\rho \sim \frac{1}{2} \cdot 6\pi^{-2} \cdot 2\pi \cdot \pi^{-1}n^{-2} \cdot \frac{1}{2}n^2k^{-2} = 3\pi^{-2}k^{-2}.$$

We remark that it follows from (2) that  $\sum_0^\infty j(j - 1)p_j = 1$ , whence the expectation of the square of the number of lattice points in the rectangle equals 2.

Also, if  $b$  is a constant,  $0 < b \leq 1$ , and if we throw a rectangle  $n \times bn^{-1}$ , then we obtain  $p_0 = 1 - b + 3b^2/\pi^2$ ,  $p_1 = b - 21b^2/4\pi^2$ ,  $p_j = 3\pi^{-2}b^{-2}((j - 1)^{-2} - 2j^{-2} + (j + 1)^{-2})$ . If  $b > 1$ , however, the problem gets more difficult, for then there is a positive probability to catch nontrivial triangles.

Also solved by the proposer.

*Problem 63-7, On Commutative Rotations*, by JOEL BRENNER (Stanford Research Institute).

Show that if two nontrivial proper rotations in  $E_3$  are commutative, then they are rotations about the same axis or else rotations through  $180^\circ$  about two mutually perpendicular axes.

Solution by THEODORE KATSANIS (NASA, Lewis Research Center).

Let  $A$  and  $B$  denote two nontrivial commutative rotations, and let  $x_A$  and  $x_B$  denote their respective axes. Let  $P$  be a plane containing  $x_A$  and parallel to  $x_B$ . If  $x_A$  is parallel to  $x_B$ , let  $O$  be any point on  $x_A$ ; otherwise, let  $O$  be the intersection of the projection on  $P$  of  $x_B$  with  $x_A$ . In either case  $A(O) = O$ , so that  $B(O) = BA(O) = AB(O)$ . Since  $B(O)$  is left fixed by  $A$ , it must lie on  $x_A$ . But if  $x_B$  does not intersect  $x_A$ ,  $B(O)$  cannot lie on  $x_A$ . Hence  $x_B$  intersects  $x_A$  and  $B(O) = O$ .

Consider now a point  $S$  on  $x_A$ ,  $S \neq O$ . Then  $A(S) = S$ , so that  $B(S) = BA(S) = AB(S)$ , and  $B(S)$  is left fixed by  $A$ . Hence  $B(S)$  lies on  $x_A$ . This means that either  $x_A = x_B$ , or else  $x_B$  is perpendicular to  $x_A$ , in which case  $B$  must also be a  $180^\circ$  rotation.

Solution by K. A. POST (Technological University, Eindhoven, Netherlands).  
Suppose  $A$  has the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}$$

where  $c$  and  $s$  stand for  $\cos \phi$  and  $\sin \phi$  respectively, i.e.,  $A$  represents a rotation about the  $x$ -axis through an angle  $\phi$ . Let  $B = (b_{ij})$  represent a rotation such that  $AB = BA$ . Equating both sides we get six equations

$$\begin{cases} (1-c)b_{12} + sb_{13} = 0, & sb_{31} + (c-1)b_{21} = 0, \\ sb_{12} + (c-1)b_{13} = 0, & (1-c)b_{31} + sb_{21} = 0, \\ s(b_{32} + b_{23}) = 0, & s(b_{22} - b_{33}) = 0. \end{cases}$$

As  $s^2 + (c-1)^2 = 2 - 2c \neq 0$  ( $A$  is nontrivial), we obtain  $b_{12} = b_{13} = b_{21} = b_{31} = 0$ . Now there are two cases.

$$(1) \quad b_{22} = b_{33} = c', \quad b_{32} = -b_{23} = s', \quad b_{11} = 1.$$

(Rotations about the same axis).

$$(2) \quad s = 0, c = -1, \quad b_{11} = -1, \quad b_{22} = -b_{33} = c', \quad b_{32} = b_{23} = s'.$$

(Rotations through  $180^\circ$  about mutually perpendicular axes).

Also solved by W. F. EBERLEIN (University of Rochester) by means of a coordinate free spinor analysis, A. MAYER (Reeves Instrument Corp.) in two ways and by the proposer in two ways.