Topological existence proof for a non-linear two-point boundary value problem

Philips J. Res. **36**, 229-238, 1981 R 1042

TOPOLOGICAL EXISTENCE PROOF FOR A NON-LINEAR TWO-POINT BOUNDARY VALUE PROBLEM

by N. G. DE BRUIJN

Abstract

The paper gives an existence proof for the solution of a boundary value problem that arose in work on gas discharges. If A and B are positive constants, it is shown that the differential equations $u'' + x^{-1}u' + (A+Bv)u = 0$, $v'' + x^{-1}v' - uv = 0$ have a solution with $u(x) > 0$ $(0 \leqslant x < 1)$, $u(1) = 0$, $u'(0) = 0$, $v'(0) = 0$, $v(1) = 1$ if and only if $A^{\frac{1}{2}} < \gamma < (A+B)^{\frac{1}{2}}$, where γ is the first zero of the Bessel function J_0. The proof uses the idea of the winding number of a closed curve in a plane with respect to a given point.

1. Introduction

The following boundary value problem arose in work by Dr M. J. C. van Gemert on gas discharges.

A and B are positive constants. The functions u and v of the independent variable x have to satisfy

$$u'' + x^{-1}u' + (A+Bv)u = 0,$$

$$v'' + x^{-1}v' - uv = 0,$$

$$u(x) > 0 \quad (0 \leqslant x < 1), \quad u(1) = 0, \quad u'(0) = 0, \quad v'(0) = 0, \quad v(1) = 1.$$

The primes denote differentiation with respect to x.

In this paper the question of existence of a solution will be settled. The present author got the question from Dr J. K. M. Jansen (Eindhoven University of Technology) who proved (essentially by Sturm's method, ref. 2, chap. XI.3) that a necessary condition for existence is that $A^{\frac{1}{2}} < \gamma < (A+B)^{\frac{1}{2}}$ where γ is the first zero of the Bessel function J_0 ($\gamma = 2.405$). Dr Jansen asked whether this condition is sufficient. In this paper it will be shown that this is indeed the case.

We get rid of the parameters in the differential equations by the transformation $x = x^* A^{\frac{1}{2}}$, $u = Bu^*$, $v = AB^{-1}v^*$. Omitting the asterisks again, we get

$$u'' + x^{-1} u' + (1 + v) u = 0, \tag{1.1}$$

$$v'' + x^{-1} v' - uv = 0, \tag{1.2}$$

$$u(x) > 0 \ (0 \leqslant x < \xi_0), \quad u(\xi_0) = 0, \quad u'(0) = 0, \tag{1.3}$$

$$v'(0) = 0, \quad v(\xi_0) = \eta_0, \tag{1.4}$$

with the particular values $\xi_0 = A^{\frac{1}{2}}, \eta_0 = B/A$. For general positive values of ξ_0 and η_0 the condition for existence will be shown to be (sec. 5)

$$\gamma(1 + \eta_0)^{-\frac{1}{2}} < \xi_0 < \gamma, \tag{1.5}$$

which specializes to $A^{\frac{1}{2}} < \gamma < (A + B)^{\frac{1}{2}}$ if $\xi_0 = A^{\frac{1}{2}}, \eta_0 = B/A$.

2. The topological method

Let p and q be real numbers, $p > 0, q \geqslant 0$. If we observe the solution of the system (1.1), (1.2) with initial values

$$u(0) = p, \quad v(0) = q, \quad u'(0) = 0, \quad v'(0) = 0 \tag{2.1}$$

for x increasing from 0 onwards as long as $u > 0$, we see that $(xv')' > 0$ and that v increases (except for the case that v is identically zero), and u decreases as long as $u > 0, v > 0$. As long as this is the case we have $(xu')' < 0$, so there is a finite ξ such that $u(x) = 0$ for the first time at $x = \xi$ (actually it follows from lemma 2 that $\xi < \gamma q^{-\frac{1}{2}}$). We define η by $\eta = v(\xi)$. The numbers ξ and η are uniquely determined by p and q, and the region $p > 0, q \geqslant 0$ is mapped continuously into the region $\xi > 0, \eta \geqslant 0$. These things follow by routine methods from general principles of the theory of differential equations, and we do not bother to prove them here.

Let us denote this continuous mapping by φ. This φ is vector-valued: $\varphi(p,q) = (\xi,\eta)$.

The point $(\gamma, 0)$ clearly belongs to the range of φ: For every $p > 0$ the pair $(pJ_0, 0)$ satisfies (1.1), (1.2) and (2.1). For all other solutions we have $\eta > 0$.

Let ξ_0, η_0 be positive numbers satisfying (1.3). We shall show that (ξ_0, η_0) belongs to the range of φ by means of the winding number method. We produce a closed curve in the (p,q)–domain and show that its image under φ is a curve that encircles the point (ξ_0, η_0). For a simple exposition of this principle we refer to ref. 1; various applications in analysis can be found in ref. 3.

It should be noted that this principle enables us to prove existence, not uniqueness. The mapping φ can be visualized as a process of spreading the (p,q)–plane over the (ξ,η)–plane. If this happens without any folding, we have uniqueness. It seems to be hard, however, to prove that there are no folds.

The contour in the (p,q)-plane will be the rectangle shown in fig. 1. The number η_1 can be any number $> \eta_0$. The number σ has to be small, P has to be large. The image in the (ξ, η)-plane is as shown schematically in fig. 2. The image of (I) is the single point $(\gamma, 0)$. The image of (II) closely resembles a part of the curve $\xi^2(1 + \eta) = \gamma$, at least if σ is small. The image of (III) is safely above the line $\eta = \eta_1$. The real difficulty lies in studying the image of (IV). If ξ_1 is any number $< \gamma$ (if $\xi_0 < \gamma$ we can take care that $\xi_0 < \xi_1 < \gamma$) we can show

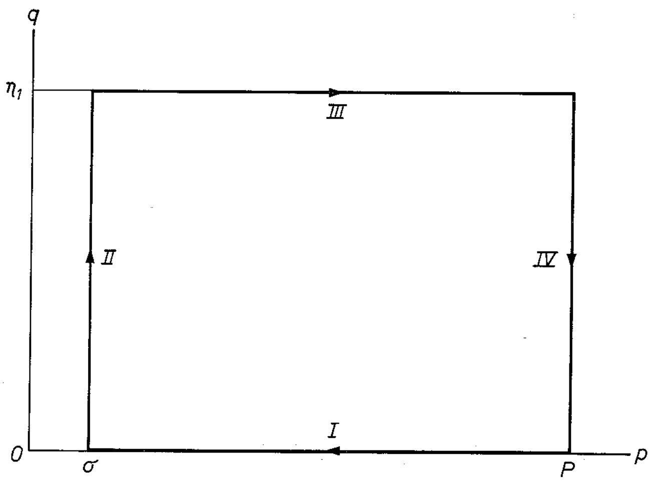

Fig. 1.

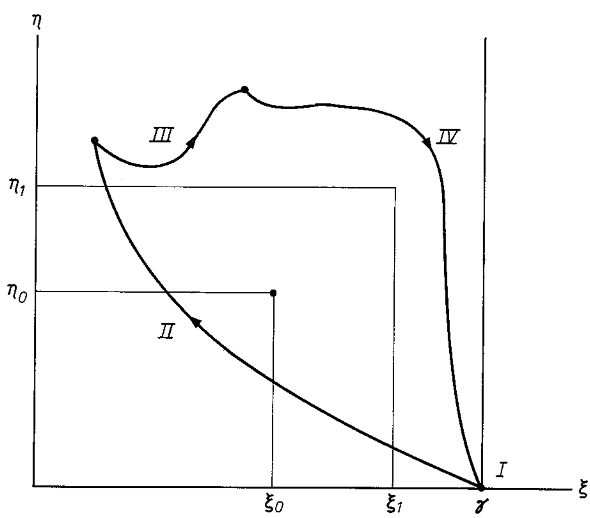

Fig. 2.

that P can be taken so large that the image of (IV) stays outside the rectangle with vertices $(0,0), (0,\eta_1), (\xi_1,0), (\xi_1,\eta_1)$.

In fig. 2 we have drawn (ξ_0,η_0) such that $0<\xi_0<\gamma$, $\xi_0^2(1+\eta_0)>\gamma$. It will be clear that the winding number of the curve about this point is not zero, and therefore the winding number principle produces the existence proof we wanted to have.

In sec. 3 we show some lemmas needed for the final conclusions in sec. 4.

3. Some lemmas

Lemma 1.

Let c_1, c, ε be real numbers, $0<c_1<c$, $\varepsilon>0$. We abbreviate

$$Q=(2\,cc_1^{-1}+4\log\varepsilon^{-1})^2\,(c-c_1)^{-2}. \tag{3.1}$$

Let u be a continuous function such that $u(x)\geqslant Q$ $(0\leqslant x\leqslant c)$ and let v be a solution of (1.2) with $v'(0)=0, v(0)>0$. Then we have

$$0<v(x)<\varepsilon\, v(c)\ \ (0<x\leqslant c_1). \tag{3.2}$$

Proof.

The functions v and v' are positive throughout $0\leqslant x\leqslant c$ (see the beginning of sec. 2). Putting $v'/v=y$ we have $y'+x^{-1}y+y^2\geqslant Q$, $y\geqslant 0$. If $y=\frac{1}{2}Q^{\frac{1}{2}}$ for some x we have $(xy')'>0$ at that point. So once y has reached that level it cannot get under that level for larger values of x. Therefore we can have $y\leqslant\frac{1}{2}Q^{\frac{1}{2}}$ at most on an interval $c_1\leqslant x\leqslant c_2$ with $c_2\leqslant c$. On that interval we have $(xy)'\geqslant\frac{1}{2}xQ, 0\leqslant xy\leqslant\frac{1}{2}cQ^{\frac{1}{2}}$. Hence the length of the interval cannot exceed $cc_1^{-1}Q^{-\frac{1}{2}}$. As this is at most $\frac{1}{2}(c-c_1)$ we have $y>\frac{1}{2}Q^{\frac{1}{2}}$ at least on $\frac{1}{2}(c+c_1)\leqslant x<c$. Hence

$$\int_{c_1}^{c} y(x)\,\mathrm{d}x\geqslant\tfrac{1}{2}(c-c_1)\cdot\tfrac{1}{2}Q^{\frac{1}{2}}\geqslant\log\varepsilon^{-1},$$

and it follows that $v(c_1)\leqslant\varepsilon v(c)$.

Lemma 2.

Let b be a real number, $b>1$. Let w be a continuous function on $0\leqslant x\leqslant\gamma$ satisfying $1\leqslant 1+w(x)\leqslant b$. Let u be a solution of the equation

$$u''+x^{-1}u'+(1+w)\,u=0, \tag{3.3}$$

with $u'(0)=0$. Then there is a number ξ with

$$\gamma b^{-\frac{1}{2}}\leqslant\xi\leqslant\gamma, \tag{3.4}$$

such that u is positive for $0\leqslant x<\xi$ and zero at ξ. Moreover, if x_0 is any number with $0 \leqslant x_0 < \xi$, and if $-u'(x_0)/u(x_0)$ is abbreviated to r, then we have $r > 0$ and

$$x_0 + (2b^{\frac{1}{2}} + 2r)^{-1} \leqslant \xi < x_0 + b^{\frac{1}{2}}\, k/r, \tag{3.5}$$

where k is an absolute constant (see (3.8)).

Proof.

Putting $y = u'/u$ we get instead of (3.3)

$$y' + y^2 + x^{-1} y + (1 + w) = 0, \quad y(0) = 0. \tag{3.6}$$

Consider the functions u_1 and u_2, defined by $u_1(x) = J_0(xb^{\frac{1}{2}})$, $u_2(x) = J_0(x)$. On the curves $y = u_1'/u_1$ and $y = u_2'/u_2$ (with vertical asymptotes $x = \gamma b^{-\frac{1}{2}}$ and $x = \gamma$, respectively) the line elements of the differential equation (3.6) are as drawn in fig. 3, due to the condition $1 < 1 + v < b$.

It follows that the solution of (3.6) stays between these two curves. Therefore it has a vertical asymptote $x = \xi$ with some ξ satisfying (3.4). It follows that $u(x) > 0$ on $0 \leqslant x < \xi$ and $u(\xi) = 0$.

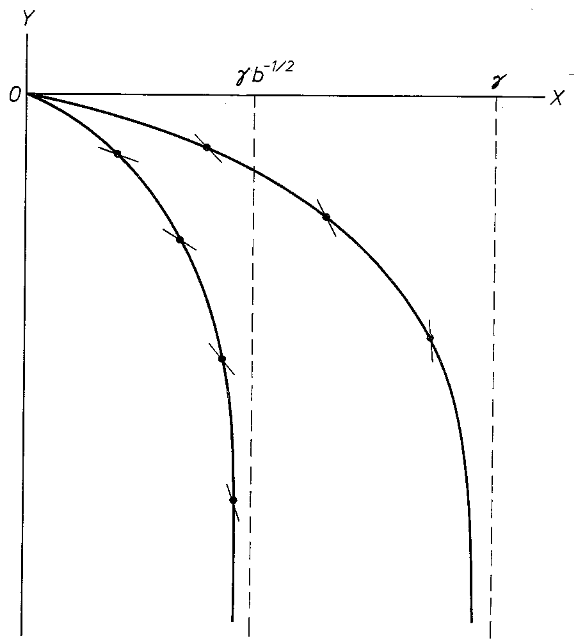

Fig. 3.

Next we take an x_0 with $0 < x_0 < \xi$, we put $r = -u'(x_0)/u(x_0)$ and we take a number s with $s > r$. Through the point $(x_0, -s)$ we draw the curve

$$y = s(\lambda - x_0)/(x - \lambda), \quad x_0 \leqslant x < \lambda, \tag{3.7}$$

where $\lambda = x_0 + s/(b + s^2)$. The derivative of this y equals

$$-y^2 - b(\lambda - x_0)^2(\lambda - x)^{-2},$$

and that is $\leqslant -y^2 - b$, and a fortiori $\leqslant -y^2 - x^{-1}y - (1 + w)$. Hence the line elements of (3.6) cut the curve (3.7) as shown in fig. 4, and we infer that our solution of (3.6), which passes through the point $(x_0, -r)$, stays to the right of that curve. Hence $\lambda \leqslant \xi$.

Thus far we have shown that $\xi \geqslant x_0 + s/(b + s^2)$ for all $s \geqslant r$. The function $s/(b + s^2)$, considered as a function of s, has its maximum at $s = b^{\frac{1}{2}}$, whence $\xi \geqslant x_0 + \frac{1}{2}b^{-\frac{1}{2}}$ if $r \leqslant b^{\frac{1}{2}}$. If $r > b^{\frac{1}{2}}$ we still have $\xi \geqslant x_0 + r/(b + r^2)$. We easily infer that $\xi \geqslant x_0 + (2b^{\frac{1}{2}} + 2r)^{-1}$ holds in both cases.

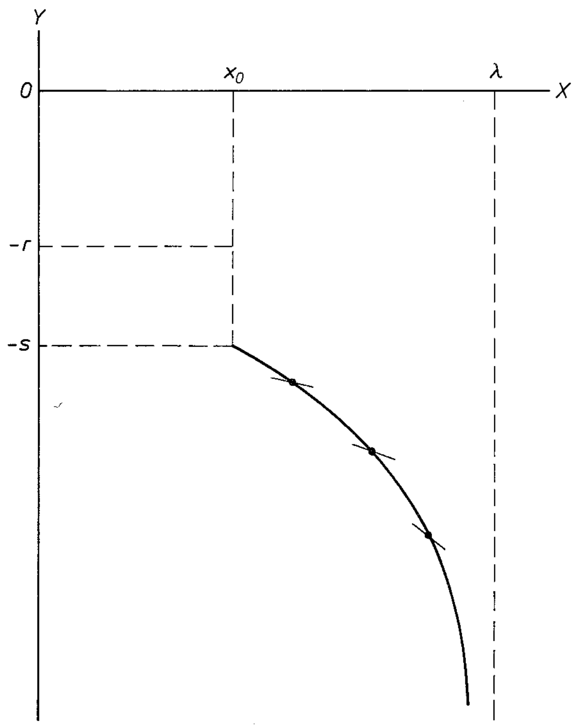

Fig. 4.

In order to prove the upper estimate of (3.5) we transform the equation (3.6), putting $xy(x) = z(x)$:

$$z'' + x^{-1} z^2 + (1 + w) x = 0.$$

It follows that $(z^{-1})' > x^{-1}$. As $z < 0$ for $x_0 \leqslant x < \xi$, and $z^{-1} \to 0$ as $x \to \xi$, we infer that

$$(x_0 r)^{-1} = \int_{x_0}^{\xi} (z^{-1})' \, \mathrm{d}x \geqslant \int_{x_0}^{\xi} x^{-1} \, \mathrm{d}x > \int_{x_0}^{\xi} \xi^{-1} \, \mathrm{d}x = (\xi - x_0)/\xi,$$

hence $(\xi - x_0) r < \xi/x_0$. This inequality is useless if x_0 is small, but we know (cf. fig. 3) that for $0 \leqslant x_0 \leqslant \frac{1}{2}\gamma b^{-\frac{1}{2}}$ we have $r \leqslant |u_1'(x_0)/u_1(x_0)|$. Hence

$$(\xi - x_0) r \leqslant \xi b^{\frac{1}{2}} \, |J_0'(\tfrac{1}{2}\gamma)/J_0(\tfrac{1}{2}\gamma)|$$

if $x_0 \leqslant \frac{1}{2}\gamma b^{-\frac{1}{2}}$. If $x_0 > \frac{1}{2}\gamma b^{-\frac{1}{2}}$ we have $(\xi - x_0) r < \xi/x_0 \leqslant 2\xi\gamma^{-1} b^{\frac{1}{2}}$. So if we take

$$k = \max (2, \gamma \, |J_0'(\tfrac{1}{2}\gamma)/J_0(\tfrac{1}{2}\gamma)|) \tag{3.8}$$

and we use $\xi \leqslant \gamma$ (see (3.3)), we get the upper estimate of (3.5).

Lemma 3.

Let ξ_1 and η_1 be real numbers, and assume $0 < \xi_1 < \gamma$, $\eta_1 > 0$. Then there is a positive number P with the following property: If ξ is a real number with $0 < \xi < \xi_1$, and if the functions u and v satisfy (1.1) and (1.2) for $0 \leqslant x \leqslant \xi$, with $u(0) \geqslant P$, $u(x) > 0$ $(0 \leqslant x < \xi)$, $u(\xi) = 0$, $u'(0) = v'(0) = 0$, then we have $v(\xi) \geqslant \eta_1$.

Proof.

On the interval $0 \leqslant x \leqslant \xi_1$ the function $u_2 = u(0) J_0$ satisfies

$$u_2'' + x^{-1} u_2' + u_2 = 0$$

and has initial values $u_2(0) = u(0), u_2(0) = 0$. Let M be the maximum of $|J_0'(x)/J_0(x)|$ for $0 \leqslant x \leqslant \xi_1$. If the "perturbation" v is small, the value of $u'(x)/u(x)$ does not deviate much from $u_2'(x)/u_2(x)$. In particular we can find a number $\delta > 0$ such that for all p with $0 \leqslant p < \xi_1$ the following is true: If $|v(x)| < \delta$ for $0 \leqslant x \leqslant p$ then

$$|u'(x)/u(x)| \leqslant 2M \qquad (0 \leqslant x \leqslant p).$$

We take k as in (3.8) and we define

$$\begin{aligned}
\varepsilon &= \delta/\eta_1, \\
\mu &= ((4 + 3\gamma^{-1})(1 + \eta_1)^{\frac{1}{2}} + 8M)^{-1}, \\
P &= \gamma(1 + \eta_1)^{\frac{1}{2}} k\mu^{-3}(2\gamma\mu^{-1} + 4 \log \varepsilon^{-1})^2, \\
c_1 &= \xi - 2\mu, \; c = \xi - \mu.
\end{aligned}$$

Note that δ, ε, μ, M and P depend on ξ_1 and η_1 only. The numbers c_1 and c, however, depend on ξ_1, η_1 and ξ.

We shall assume that $u(0) \leqslant P, 0 < \xi < \xi_1, v(\xi) < \eta_1$, and we shall derive a contradiction. As v is non-decreasing for $0 \leqslant x \leqslant \xi$ (see the beginning of sec. 2) we have $1 \leqslant 1 + v(x) \leqslant 1 + \eta_1$ for $0 \leqslant x \leqslant \xi$. We apply lemma 2, taking $w(x) = v(x)$ $(0 \leqslant x \leqslant \xi)$, $w(x) = v(\xi)$ $(\xi \leqslant x \leqslant \gamma)$. The property that u is positive for $0 \leqslant x < \xi$ and zero at ξ, determines ξ uniquely, and therefore the ξ of lemma 2 is the same as the one we have here. In particular (3.4) says that $\gamma(1 + \eta_1)^{-\frac{1}{2}} \leqslant \xi$. It follows that $\xi > 3\mu$ and therefore $c_1 > \mu$.

By (3.5) we have

$$|u'(x)/u(x)| \leqslant (1 + \eta_1)^{\frac{1}{2}} k(\xi - x)^{-1} \qquad (0 \leqslant x \leqslant c),$$

and it follows by integration that

$$|u(0)/u(x)| \leqslant c(1 + \eta_1)^{\frac{1}{2}} k/\mu \qquad (0 \leqslant x \leqslant c).$$

Since $u(0) \geqslant P$ we derive $u(x) \geqslant Q$ $(0 \leqslant x \leqslant c)$, with Q given by (3.1) (note that $c < \gamma, c_1 > \mu$ and $c - c_1 = \mu$). By lemma 1 we now have (3.2). From our assumption $v(\xi) < \eta_1$ and from the monotonicity of v we obtain

$$|v(x)| < \delta\,(0 \leqslant x < c_1).$$

Taking the p occurring at the beginning of this proof to be equal to c_1, we find

$$|u'(x)/u(x)| \leqslant 2M \qquad (0 \leqslant x \leqslant c_1).$$

Finally applying (3.5) with $x_0 = c_1$ we get

$$c_1 + (2\,(1 + \eta_1)^{\frac{1}{2}} + 4M)^{-1} \leqslant \xi$$

and this contradicts the definition of c_1.

4. Conclusion

Theorem 1.

Assume $\xi_0 > 0$ and assume that u and v satisfy (1.1) and (1.2) for $0 \leqslant x \leqslant \xi_0$, with $u(x) > 0$ $(0 \leqslant x < \xi_0)$, $u(\xi_0) = 0, v(0) > 0, v'(0) = 0$. Then we have (with $\eta_0 = v(\xi_0)$) $\gamma(1 + \eta_0)^{-\frac{1}{2}} < \xi_0 < \gamma$.

Proof.

From (3.4) we only get the weaker form $\gamma(1 + \eta_0)^{-\frac{1}{2}} \leqslant \xi \leqslant \gamma$. With the method we used for (3.4) it takes some trouble to argue that the equality signs cannot hold, and it seems to be easier to start anew, using Sturm's method in the way Dr Jansen did (see sec. 1). Put $u_0(x) = J_0(\gamma x/\xi_0)$, then

$$u_0'' + x^{-1} u_0' + u_0 \gamma^2 \xi_0^{-2} = 0,$$

and

$$\int_0^{\xi_0} x u u_0 \,[1 + v - (\gamma/\xi)^2]\, \mathrm{d}x = \int_0^{\xi_0} (x\,(u_0' u - u' u_0))'\, \mathrm{d}x = 0.$$

Now $1 + v(\xi_0) \leqslant (\gamma_m/\xi_0)^2$ is impossible, since $1 + v(x) < 1 + v(\xi_0)$ $(0 \leqslant x < \xi_0)$, and $1 \geqslant (\gamma/\xi_0)^2$ is impossible because of $v(x) > 0$ $(0 \leqslant x < \xi_0)$. Therefore $1 < (\gamma/\xi_0)^2 < 1 + v(\xi_0)$.

Theorem 2.

Assume $\xi_0 > 0, \eta_0 > 0, \gamma(1 + \eta_0)^{-\frac{1}{2}} < \xi_0 < \gamma$. Then there exist functions u and v that satisfy (1.1) and (1.2) for $0 \leqslant x \leqslant \xi_0$, with $u(x) > 0$ $(0 \leqslant x < \xi)$, $u(\xi) = 0, u'(0) = 0, v(0) > 0, v'(0) = 0, v(\xi_0) = \eta_0$.

Proof.

We choose numbers σ, ξ_1, η_1 with

$$0 < \sigma < 2\gamma^{-2}\,\eta_0^{-1}\,(1 + \eta_0 - \gamma^2\,\xi_0^{-2}),\ \xi_0 < \xi_1 < \gamma,\ \eta_1 > \eta_0 \tag{4.1}$$

and we take P according to lemma 3. With these values of η_1, σ and P we consider the rectangle of fig. 1 and its image under the mapping φ (see sec. 2).

If (p, q) lies on the lower edge (I), the solution for v is identically zero, whence $u = pJ_0$, and $\varphi(p, q)$ stays at the point $(0, \gamma)$.

Next take (p, q) on the edge (II), where $p = \sigma$, $0 \leqslant q \leqslant \eta_1$. Put $\varphi(p, q) = (\xi, \eta)$. We have $0 \leqslant u(x) \leqslant \sigma$ $(0 \leqslant x \leqslant \xi)$, and $v(x)$ is positive and increasing (see the beginning of sec. 2). Therefore $v''(x) \leqslant \sigma v(x)$, and, by integration

$$v(\gamma) = v(0) + \int_0^{\gamma} (\gamma - x)\, v''(x)\, \mathrm{d}x \leqslant v(0) + \tfrac{1}{2}\sigma\gamma^2\, v(\gamma),$$

whence

$$(1 - \tfrac{1}{2}\sigma\gamma^2)\,\eta \leqslant v(0).$$

If we apply lemma 2 to the function u^*, defined by $u^*(x) = u((1 + v(0))^{\frac{1}{2}}\, x)$, we get $\xi \leqslant (1 + v(0))^{-\frac{1}{2}}$.

It follows that

$$\xi^2(1 + (1 - \tfrac{1}{2}\sigma\gamma^2)\,\eta) \leqslant \gamma^2. \tag{4.2}$$

On the other hand we have, by (4.1), $\xi_0^2(1 + (1 - \frac{1}{2}\sigma\gamma^2)\,\eta_0) > \gamma^2$. So the image of the edge (II) does not get to the right of the curve $\xi^2(1 + (1 - \frac{1}{2}\sigma\gamma^2)\,\eta) = \gamma^2$ whereas the point (ξ_0, η_0) lies to the right.

The image of the edge (III) lies entirely above the level $\eta = \eta_1$, simply because $q = \eta_1$ implies $v(\xi) > v(0) = \eta_1$.

Finally, the image of (IV) is a curve that runs from some point above the level $\eta = \eta_1$ to the point $(\gamma, 0)$ without entering into the rectangle

$$0 \leqslant \xi \leqslant \xi_1, 0 \leqslant \eta \leqslant \eta_1.$$

This was proved in lemma 3 and depicted in fig. 2.

From this survey it follows that the image of our rectangular contour has non-zero winding number with respect to the point (ξ_0, η_0). It follows that at least one interior point of the rectangle is mapped onto (ξ_0, η_0).

Philips Research Laboratories *Eindhoven, May 1981*

REFERENCES

[1]) W. G. Chinn and N. E. Steenrod, First Concept of Topology, New Mathematical Library, **18**, Random House, 1966.
[2]) P. Hartman, Ordinary Differential Equations, J. Wiley and Sons, New York–London–Sydney, 1964.
[3]) M. A. Krasnosel'skiy, A. I. Perov, A. I. Povolotskiy and P. P. Zabreiko, Plane Vector Fields, Academic Press, New York/London, 1966.