Solution to problem 87-6* : The entropy of a Poisson distribution

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Finally, substituting for x from (11), we end up considering the simple unconstrained scalar optimization problem

$$(15)\qquad \max_z \frac{\sqrt{1+z^2}-1}{z}\left[1+\frac{z^2(\sqrt{1+z^2}-1)}{4\{2(1+\mu hz)\pm\sqrt{4-z^4-2(1+2\mu h^2)z^2+8\mu hz}\}}\right].$$

Also recall that z lies in the range (14) and the optimal z must satisfy (12), namely

$$(16)\qquad z^2 \leqq 1 \pm \tfrac{1}{2}\sqrt{4-z^4-2(1+2\mu h^2)z^2+8\mu hz} \leqq hz - z^2.$$

Finally, the optimal fluid level x is found by substituting the optimal z in (13).

The problem parameters are $\mu = 0.1132$ and $h = 3.4375$. For these parameters the quartic equation associated with (14) has a unique real positive solution $\bar{z} = 0.914$ and therefore (14) yields the range $0 < z \leqq \bar{z} = 0.914$. Furthermore, the maximum in (15) is then attained at the boundary $\bar{z}$ of the z-range, i.e., $z^* = \bar{z} = 0.914$ and thus $x^* = 1/\bar{z} = 1.094$. Finally, note that (12) is then satisfied since $0.914 < 1.094 < 2.523$.

The insight gained above affords us the possibility of ascertaining the range of validity (in parameter space) of our solution. In particular, if μ is sufficiently small, then $\bar{z} > 1$, $x^* = 1/\bar{z} < 1$ and the left-hand part of (12) is violated. Hence, the smallest possible value of the parameter μ for our analysis to be valid is that which makes $\bar{z} = 1$ the unique real, positive solution of the quartic solution

$$z^4 + 2(1+2\mu h^2)z^2 - 8\mu hz - 4 = 0.$$

In other words $\mu \geqq \bar{\mu} = (1/4h(h-2))$. Furthermore, it is required that $h > \bar{h} = 2$. Hence, the parameter space (μ, h) is partitioned by the curve $\bar{\mu}(h) = (1/4h(h-2))$, $h > \bar{h} = 2$.

The Entropy of a Poisson Distribution

Problem 87-6*, *by* C. ROBERT APPLEDORN (Indiana University).

The following problem arose during a study of data compression schemes for digitally encoded radiographic images. Specifically, Huffman optimum (minimum) codes are employed to reduce the storage requirements for high resolution gray-level images, for example, 2048 × 2048 pixels by 10 bits per pixel for a single image. Entropy estimates provide a lower bound for the average storage in terms of bits per pixel that can be achieved using these coding methods.

Due to the nature of the problem, the Poisson probability density function with mean value parameter m is of particular interest:

$$p(k) = \frac{m^k e^{-m}}{k!}.$$

The entropy H associated with this probability density function is given by

$$H = -\sum_{k=0}^{\infty} p(k) \log p(k).$$

The problem that develops during the entropy calculation is to determine a closed-form solution to the following summation:

$$\sum_{k=0}^{\infty} \log(k!) \frac{m^k e^{-m}}{k!}.$$

It is *conjectured* by the author that for large m (say, $m > 10$), the entropy H can be approximated by

$$H = \tfrac{1}{2}\log(2\pi e m).$$

Prove or disprove.

Solution by RONALD J. EVANS (University of California, San Diego).

Let $\varphi(x) = \log\Gamma(x+1)$. By definition of the entropy H,

$$H = \sum_{k=0}^{\infty} p(k)(\varphi(k) + m - k\log m) = m(1-\log m) + \varphi_\infty,$$

where

$$\varphi_\infty = \sum_{k=0}^{\infty} p(k)\varphi(k).$$

By [2, Chap. 3, Entry 10], [3, Entry 10], as (real) $m \to \infty$,

$$\varphi_\infty = \varphi(m) + \frac{m\varphi^{(2)}(m)}{2} + \frac{m\varphi^{(3)}(m)}{6} + \frac{m^2\varphi^{(4)}(m)}{8} + O(m^{-2}).$$

(Several minor errors in [2, Chap. 3, §10] and [3, §10] are corrected in [4].) Substituting in the well-known asymptotic expansions for the derivatives $\varphi^{(n)}(m)$ [1, (6.1.41), (6.3.18), (6.4.11)], we deduce that, as $m \to \infty$,

$$H = \frac{1}{2}\log(2\pi e m) - \frac{1}{12m} + O(m^{-2}).$$

REFERENCES

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[2] B. C. BERNDT, *Ramanujan's Notebooks, Part* I, Springer-Verlag, New York, 1985.

[3] B. C. BERNDT, R. J. EVANS, AND B. M. WILSON, *Chapter* 3 *of Ramanujan's Second Notebook*, Adv. in Math., 49 (1983), pp. 123–169.

[4] R. J. EVANS, *Ramanujan's Second Notebook: Asymptotic expansions for hypergeometric series and related functions*, in Proc. Ramanujan Centenary Conference, Academic Press, New York, 1988.

Solution by J. BOERSMA (Eindhoven University of Technology, Eindhoven, the Netherlands).

It is easily seen that the entropy H associated with the Poisson probability density function is given by

$$H = -\sum_{k=0}^{\infty} p(k)\log p(k) = m - m\log m + \sum_{k=0}^{\infty}\log(k!)\frac{m^k e^{-m}}{k!}.$$

In the latter series replace $\log(k!)$ by Malmstén's representation [1, Formula 1.9(1)]

$$\log(k!) = \int_0^\infty \left[k - \frac{1-e^{-kt}}{1-e^{-t}}\right]\frac{e^{-t}}{t}\,dt;$$

then it is found that

$$(*)\qquad \begin{aligned} H &= m - m\,log\,m + \int_0^\infty \left\{m - \frac{1-\exp[m(e^{-t}-1)]}{1-e^{-t}}\right\}\frac{e^{-t}}{t}\,dt \\ &= m - m\log m - \int_0^1 \left[m - \frac{1-e^{-ms}}{s}\right]\frac{ds}{\log(1-s)} \end{aligned}$$

by the substitution $1 - e^{-t} = s$.

The final integral in (*) is decomposed into

$$-m\int_0^1\left[\frac{1}{\log(1-s)}+\frac{1}{s}\right]ds+\int_0^1\left(\frac{m}{s}-\frac{1-e^{-ms}}{s^2}\right)ds+\frac{1}{2}\int_0^1\frac{1-e^{-ms}}{s}\,ds$$

$$+\int_0^1\frac{1}{s}\left[\frac{1}{\log(1-s)}+\frac{1}{s}-\frac{1}{2}\right]ds-\int_0^1\frac{e^{-ms}}{s}\left[\frac{1}{\log(1-s)}+\frac{1}{s}-\frac{1}{2}\right]ds,$$

whereby the successive integrals are shortly denoted by $I_1, I_2, \cdots, I_5$. By back substitution $s = 1 - e^{-t}$ it is found that

$$I_1=m\int_0^\infty\left(\frac{1}{t}-\frac{1}{1-e^{-t}}\right)e^{-t}\,dt=-m\gamma,$$

where γ denotes Euler's constant (cf. [1, Formula 1.7(19)]). Through an integration by parts the second integral reduces to

$$I_2=\left.\frac{1-e^{-ms}}{s}\right|_0^1+m\int_0^1\frac{1-e^{-ms}}{s}\,ds=1-e^{-m}-m+m\int_0^m\frac{1-e^{-t}}{t}\,dt$$

$$=1-e^{-m}-m+m[E_1(m)+\log m+\gamma],$$

in which $E_1(m)=\int_m^\infty e^{-t}t^{-1}\,dt$ stands for the exponential integral (cf. [1, Formulae 9.7(1), (5)]). In the same manner we have

$$I_3=\frac{1}{2}\int_0^m\frac{1-e^{-t}}{t}\,dt=\frac{1}{2}[E_1(m)+\log m+\gamma].$$

In the fourth integral we substitute $s = 1 - e^{-t}$, which yields

$$I_4=\int_0^\infty\left[\frac{e^{-t}}{(1-e^{-t})^2}-\frac{1+\frac{1}{2}t}{1-e^{-t}}\frac{e^{-t}}{t}\right]dt;$$

next, I_4 is decomposed into

$$I_4=\int_0^\infty\left[\frac{e^{-t}}{(1-e^{-t})^2}-\frac{1}{t^2}\right]dt+\int_0^\infty\left[\frac{1}{t^2}-\frac{1+\frac{1}{2}t}{1-e^{-t}}\frac{e^{-t}}{t}\right]dt$$

$$=\left.\left(\frac{-1}{1-e^{-t}}+\frac{1}{t}\right)\right|_0^\infty+\int_0^\infty\left[\frac{1}{t^2}-\left(1+\frac{1}{2}t\right)\frac{e^{-t}}{t}-\frac{1+\frac{1}{2}t}{e^t-1}\frac{e^{-t}}{t}\right]dt$$

$$=-\frac{1}{2}+\int_0^\infty\left[\frac{1-e^{-t}}{t^2}-\frac{e^{-t}}{t}-\frac{1}{2}e^{-t}\right]dt-\int_0^\infty\left[\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right]\frac{e^{-t}}{t}\,dt$$

$$-\frac{1}{2}\int_0^\infty\left(\frac{1}{e^t-1}-\frac{1}{t}\right)e^{-t}\,dt.$$

The latter integrals can be evaluated directly and by use of [1, Formulae 1.9(3), 1.7(19)], viz.

$$\int_0^\infty\left[\frac{1-e^{-t}}{t^2}-\frac{e^{-t}}{t}-\frac{1}{2}e^{-t}\right]dt=-\left.\frac{1-e^{-t}}{t}\right|_0^\infty-\frac{1}{2}=\frac{1}{2},$$

$$\int_0^\infty\left[\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right]\frac{e^{-t}}{t}\,dt=1-\frac{1}{2}\log(2\pi),$$

$$\int_0^\infty\left(\frac{1}{e^t-1}-\frac{1}{t}\right)e^{-t}\,dt=\int_0^\infty\left[-1+\frac{1}{1-e^{-t}}-\frac{1}{t}\right]e^{-t}\,dt=\gamma-1.$$

Thus we find

$$I_4 = -\tfrac{1}{2} + \tfrac{1}{2}\log(2\pi) - \tfrac{1}{2}\gamma.$$

By inserting the previous results into (*), we obtain the representation

$$H = \frac{1}{2}\log(2\pi e m) + \left(m + \frac{1}{2}\right)E_1(m) - e^{-m} - \int_0^1 \frac{e^{-ms}}{s}\left[\frac{1}{\log(1-s)} + \frac{1}{s} - \frac{1}{2}\right] ds.$$

It does not seem possible to evaluate the latter integral in closed form; however, an evaluation by numerical integration is certainly feasible. The asymptotics of the integral, as $m \to \infty$, is readily established by use of Watson's lemma. To that end we insert the Taylor series

$$\frac{1}{\log(1-s)} + \frac{1}{s} - \frac{1}{2} = \sum_{k=1}^{\infty} a_k s^k, \qquad |s| < 1$$

with coefficients $a_1 = \frac{1}{12}$, $a_2 = \frac{1}{24}$, $a_3 = \frac{19}{720}$, etc., and integrate term by term. Then the complete asymptotic expansion of the entropy H is found as

$$H \sim \frac{1}{2}\log(2\pi e m) - \sum_{k=1}^{\infty} a_k (k-1)! m^{-k} \qquad (m \to \infty)$$

apart from exponentially small terms of order $O(e^{-m})$.

REFERENCE

[1] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, AND F. G. TRICOMI, *Higher Transcendental Functions*, Vols. I, II, McGraw–Hill, New York, 1953.

A similar solution was obtained by PETER SMITH (Keele University, U.K.).

N. M. BLACHMAN (GTE Government Systems Corp., Mountain View, California) notes that since the Poisson distribution has variance m and becomes approximately normal for large m, its entropy becomes that part of the $N(m, n)$ distribution, viz., $\log\sqrt{2\pi e m}$.

A. A. JAGERS (Universiteit Twente, Enschede, the Netherlands) shows that

$$\tfrac{1}{2}\log(2\pi m) < H < \tfrac{1}{2}\log(2\pi m) + 1.$$

Also solved by W. B. JORDAN (Scotia, New York), J. VAN KAN, P. SONNENVELD (Delft University of Technology), T. KAUFFMAN (Houston, Texas), S. L. PAVERI-FONTANA (Virginia Polytechnic Institute and State University, Blacksburg, Virginia), T. J. OTT (Bell Communications Research, Morristown, New Jersey), P. R. PUDAITE (student, University of Illinois, Urbana), M. RENARDY (Virginia Polytechnic Institute and State University, Blacksburg, Virginia), and P. WAGNER (University of Innsbruck, Austria).

An Identity

Problem 87-8, *by* JOHN W. MOON (University of Alberta).

Show that

$$\sum_{n=1}^{\infty} \frac{56n^2 + 33n - 8}{(n+2)(n+1)} f_n^2 = 1$$