

Asymptotic expansion of a class of Fermi-Dirac integrals

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ASYMPTOTIC EXPANSION OF A CLASS OF FERMI–DIRAC INTEGRALS*

J. BOERSMA† AND M. L. GLASSER‡

Abstract. A procedure is presented for obtaining the complete asymptotic expansion of a class of fractional integrals (of Riemann–Liouville type), in which the integrand contains the product of two derivatives of the Fermi–Dirac integral. The procedure uses two-sided Laplace transforms and Abelian asymptotics of the inverse Laplace transform. The fractional integrals considered arise in various problems from statistical mechanics and solid state physics.

Key words. Fermi–Dirac integral, asymptotic expansion, Riemann–Liouville fractional integral, Laplace transform, Abelian asymptotics

AMS(MOS) subject classifications. 41A60, 33A70, 44A10, 26A33, 82

1. Introduction. This paper is concerned with the asymptotic expansion, as $\eta \rightarrow \infty$, of the class of integrals

$$(1.1) \quad G_{\mu,p}^{(m,n)}(\eta) = \frac{1}{\Gamma(\mu)} \int_{-\infty}^{\eta} (\eta - t)^{\mu-1} F_p^{(m)}(t) F_p^{(n)}(t) dt \quad (\mu > 0).$$

Here, m and n are nonnegative integers, and $F_p^{(m)}(t)$ denotes the m th derivative of the Fermi–Dirac integral $F_p(t)$ defined by [1]

$$(1.2) \quad F_p(t) = \frac{1}{\Gamma(p+1)} \int_0^{\infty} \frac{x^p dx}{1 + e^{x-t}} \quad (p > -1).$$

The class of Riemann–Liouville fractional integrals (1.1) is important in a number of areas of statistical mechanics and solid state physics. Two examples are the exchange energy of a d -dimensional electron gas [6] ($\mu = (d - 1)/2$, $p = -\frac{1}{2}$, $m = n = 0$) and the temperature-dependent gradient expansion coefficients for the interaction functional of an inhomogeneous electron gas [5] ($\mu = 2$, $p = -\frac{1}{2}$, $m = n = 2$).

In the special case $p = -\frac{1}{2}$, $m = n = 0$, the asymptotics of the integral (1.1) has been treated by Glasser and Boersma [6]. Their procedure, which uses the two-sided Laplace transform, is generalized in the present paper to accommodate the additional parameters p , m , and n . The Laplace transform method is explained in § 2, where it is also shown that the asymptotic analysis may be restricted to the case $m = n$. Let the Laplace transform of $[F_p^{(m)}(t)]^2$ be denoted by $g(s)$ (with transform variable s), then the asymptotic expansion of $G(\eta)$ as $\eta \rightarrow \infty$ can be found by applying Abelian asymptotics to the series expansion of $g(s)$ around $s = 0$. By starting from a suitable integral representation for $g(s)$ as derived in § 3, the expansion of $g(s)$ around $s = 0$ is determined in §§ 4 and 5. In the final section, § 6, the corresponding complete asymptotic expansion of $G(\eta)$, as given by (1.1), is presented.

2. Laplace transform method. Following the procedure of § 3 of [6], we first determine the two-sided Laplace transform of (1.1):

$$(2.1) \quad \bar{G}_{\mu,p}^{(m,n)}(s) = \int_{-\infty}^{\infty} e^{-s\eta} G_{\mu,p}^{(m,n)}(\eta) d\eta = s^{-\mu} g_p^{(m,n)}(s)$$

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† Department of Mathematics and Computing Science, Eindhoven University of Technology, Eindhoven, the Netherlands.

‡ School of Science, Clarkson University, Potsdam, New York 13676.

where

$$(2.2) \quad g_p^{(m,n)}(s) = \int_{-\infty}^{\infty} e^{-st} F_p^{(m)}(t) F_p^{(n)}(t) dt.$$

From the known asymptotic behaviour [1]

$$(2.3) \quad F_p(t) = O(e^t), \quad t \rightarrow -\infty, \quad F_p(t) = O(t^{p+1}), \quad t \rightarrow \infty,$$

it follows that the Laplace transforms $\bar{G}_{\mu,p}^{(m,n)}(s)$ and $g_p^{(m,n)}(s)$ are defined in the strip $0 < \operatorname{Re} s < 2$.

Assuming that $m \leq n$ in (2.2) and integrating by parts, we establish the recurrence relations

$$(2.4) \quad g_p^{(m,m+1)}(s) = \frac{1}{2} s g_p^{(m,m)}(s),$$

$$(2.5) \quad g_p^{(m,n)}(s) = s g_p^{(m,n-1)}(s) - g_p^{(m+1,n-1)}(s), \quad n \geq m+2.$$

By repeated application of these relations we are led to

$$(2.6) \quad \begin{aligned} g_p^{(m,m+2)}(s) &= \frac{1}{2} s^2 g_p^{(m,m)}(s) - g_p^{(m+1,m+1)}(s), \\ g_p^{(m,m+3)}(s) &= \frac{1}{2} s^3 g_p^{(m,m)}(s) - \frac{3}{2} s g_p^{(m+1,m+1)}(s), \\ g_p^{(m,m+4)}(s) &= \frac{1}{2} s^4 g_p^{(m,m)}(s) - 2s^2 g_p^{(m+1,m+1)}(s) + g_p^{(m+2,m+2)}(s). \end{aligned}$$

The coefficients in (2.4) and (2.6) are now used to form the polynomials

$$(2.7) \quad \begin{aligned} p_0(s) &= 1, \quad p_1(s) = \frac{1}{2}s, \quad p_2(s) = \frac{1}{2}s^2 - 1, \\ p_3(s) &= \frac{1}{2}s^3 - \frac{3}{2}s, \quad p_4(s) = \frac{1}{2}s^4 - 2s^2 + 1, \dots, \end{aligned}$$

which, by (2.5), satisfy the recurrence relation

$$(2.8) \quad p_k(s) = s p_{k-1}(s) - p_{k-2}(s), \quad k \geq 2.$$

The latter recurrence relation is identical to that of the Chebyshev polynomials $T_k(s/2)$ (cf. [3, § 10.11]). Thus we find

$$(2.9) \quad p_k(s) = T_k\left(\frac{s}{2}\right) = \frac{1}{2} k \sum_{l=0}^{[k/2]} \frac{(-1)^l (k-l-1)!}{l!(k-2l)!} s^{k-2l}, \quad k \geq 1,$$

whereupon the results in (2.6) generalize to

$$(2.10) \quad g_p^{(m,m+k)}(s) = \frac{1}{2} k \sum_{l=0}^{[k/2]} \frac{(-1)^l (k-l-1)!}{l!(k-2l)!} s^{k-2l} g_p^{(m+l,m+l)}(s), \quad k \geq 1.$$

Consequently, without loss of generality we can restrict our further asymptotic analysis to the case $m = n$. Accordingly, we simplify the notation by setting $G_{\mu,p}^{(m,m)}(\eta) \equiv G_{\mu,p}^{(m)}(\eta)$ and $g_p^{(m,m)}(s) \equiv g_p^{(m)}(s)$.

In the Laplace transform method the asymptotic expansion of $G_{\mu,p}^{(m)}(\eta)$ as $\eta \rightarrow \infty$ is obtained by applying Abelian asymptotics [2, Kap. 7] to the series expansion of $g_p^{(m)}(s)$ around $s = 0$. To determine the latter expansion, we rewrite the integral (2.2) in a more convenient form by means of Parseval's formula:

$$(2.11) \quad g_p^{(m)}(s) = \int_{-\infty}^{\infty} e^{-st} [F_p^{(m)}(t)]^2 dt = \int_{-\infty}^{\infty} f(u) f(-u) du$$

where

$$(2.12) \quad f(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-st/2} F_p^{(m)}(t) e^{iut} dt = \frac{(s/2 - iu)^m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-st/2} F_p(t) e^{iut} dt.$$

The second integral in (2.12) is evaluated by inserting the integral representation (1.2) for $F_p(t)$, interchanging the order of integration, and applying the substitution $y = e^{x-t}$ in the t -integral:

$$(2.13) \quad \int_{-\infty}^{\infty} e^{-st/2} F_p(t) e^{iut} dt = \frac{1}{\Gamma(p+1)} \int_0^{\infty} x^p e^{-(s/2-iu)x} dx \int_0^{\infty} \frac{y^{s/2-iu-1}}{1+y} dy \\ = \frac{\pi(s/2-iu)^{-p-1}}{\sin[\pi(s/2-iu)]}.$$

The result for $f(u)$ thus found is inserted into (2.11), and we have

$$(2.14) \quad g_p^{(m)}(s) = \pi \int_{-\infty}^{\infty} \frac{(s^2/4+u^2)^{m-p-1}}{\cosh(2\pi u) - \cos(\pi s)} du.$$

Obviously, $g_p^{(m)}(s)$ depends only on the difference $m-p$; this was to be expected from the basic recursion formula $F_p'(t) = F_{p-1}(t)$. Finally, for brevity we introduce

$$(2.15) \quad \nu = m - p - \frac{1}{2}, \quad g_\nu(s) = g_p^{(m)}(s);$$

then the representation (2.14) becomes

$$(2.16) \quad g_\nu(s) = \pi \int_{-\infty}^{\infty} \frac{(s^2/4+u^2)^{\nu-1/2}}{\cosh(2\pi u) - \cos(\pi s)} du.$$

3. Integral representation for $g_\nu(s)$. The representation (2.16) for $g_\nu(s)$ is further reduced by another application of Parseval's formula. It is convenient to distinguish three cases.

Case i. $\nu < \frac{1}{2}$. From [4, Formulas 1.9(6), 1.3(7)] we quote the Fourier cosine transforms

$$(3.1) \quad \int_0^{\infty} \frac{\cos(xu)}{\cosh(2\pi u) - \cos(\pi s)} du = \frac{1}{2 \sin(\pi s)} \frac{\sinh[(1-s)x/2]}{\sinh(x/2)},$$

$$(3.2) \quad \int_0^{\infty} \left(\frac{s^2}{4} + u^2\right)^{\nu-1/2} \cos(xu) du = \frac{\pi^{1/2}}{\Gamma(\frac{1}{2}-\nu)} \left(\frac{x}{s}\right)^{-\nu} K_\nu\left(\frac{sx}{2}\right)$$

where we used that $K_{-\nu}(z) = K_\nu(z)$ by [7, Formula 3.71(8)]. Next, by means of Parseval's formula applied to (2.16) we are led to

$$(3.3) \quad g_\nu(s) = \frac{2\pi^{1/2}}{\Gamma(\frac{1}{2}-\nu)} \frac{s^\nu}{\sin(\pi s)} \int_0^{\infty} \frac{\sinh[(1-s)x/2]}{\sinh(x/2)} x^{-\nu} K_\nu\left(\frac{sx}{2}\right) dx.$$

It is easily seen that the integral (3.3) is convergent if $\nu < \frac{1}{2}$.

Case ii. $\nu > \frac{1}{2}$, $\nu - \frac{1}{2} \notin \mathbb{N}$. Let k be the smallest integer greater than or equal to ν ; then we set $\nu = k - q$, where $0 \leq q < 1$ and $q \neq \frac{1}{2}$. To apply Parseval's formula in (2.16), we need the Fourier cosine transform

$$(3.4) \quad \int_0^{\infty} \frac{(s^2/4+u^2)^k}{\cosh(2\pi u) - \cos(\pi s)} \cos(xu) du = \frac{1}{2 \sin(\pi s)} \left(\frac{s^2}{4} - \frac{d^2}{dx^2}\right)^k \left\{ \frac{\sinh[(1-s)x/2]}{\sinh(x/2)} \right\}$$

obtainable from (3.1), and the transform (3.2) with ν replaced by $-q$. As a result it is found that the representation (2.16) passes into

$$(3.5) \quad g_\nu(s) = \frac{2\pi^{1/2}}{\Gamma(q+\frac{1}{2})} \frac{s^{-q}}{\sin(\pi s)} \int_0^{\infty} \left(\frac{s^2}{4} - \frac{d^2}{dx^2}\right)^k \left\{ \frac{\sinh[(1-s)x/2]}{\sinh(x/2)} \right\} x^q K_{-q}\left(\frac{sx}{2}\right) dx.$$

To further reduce (3.5), we would like to integrate by parts so that the differential operator acts on $x^q K_{-q}(sx/2)$. Here a difficulty comes up, since the resulting integral is divergent at the lower limit $x=0$ and the intermediate endpoint contributions at $x=0$ become infinite. To overcome this, we introduce the “finite part” (in the sense of Hadamard) of the resulting integral and end point contributions, defined as follows.

For $\delta \geq 0$, let $f(\delta)$ have an asymptotic expansion as $\delta \downarrow 0$, that consists of terms $\delta^r (\log \delta)^j$ with real r and integer j . Suppose the expansion contains a finite number of singular terms (i.e., terms with $r < 0$ or with $r = 0, j \geq 1$), and let $f_\infty(\delta)$ denote the sum of the singular terms. Then we define the finite part of $f(\delta)$ as $\delta \downarrow 0$ by

$$(3.6) \quad \text{fin} f(\delta) = \lim_{\delta \downarrow 0} [f(\delta) - f_\infty(\delta)].$$

Likewise, if $\int_0^\infty h(x) dx$ is divergent or convergent at $x=0$, we define the finite part of the integral as

$$(3.7) \quad \int_0^\infty h(x) dx = \text{fin} \int_\delta^\infty h(x) dx.$$

When integrating by parts in (3.5), the finite part of a typical endpoint contribution looks like

$$(3.8) \quad \text{fin}_{\delta \downarrow 0} \left(\frac{d}{dx} \right)^j \left\{ \frac{\sinh [(1-s)x/2]}{\sinh (x/2)} \right\} \left(\frac{d}{dx} \right)^l \left\{ x^q K_{-q} \left(\frac{sx}{2} \right) \right\} \Big|_{x=\delta}$$

where j and l are nonnegative integers with $j+l$ odd. We expand this in a power series in powers of $x = \delta$. Then the expansion is found to contain terms $\delta^{2n-j-l}, \delta^{2n+2q-j-l}$ and, if $q=0$, also $\delta^{2n-j-l} \log \delta$, whereby $n=0, 1, 2, \dots$. Because $q \neq \frac{1}{2}$ and $j+l$ is odd, none of the exponents $2n-j-l$ or $2n+2q-j-l$ is zero and the finite part (3.8) vanishes. In this way we find, through integration by parts in (3.5), that

$$(3.9) \quad g_\nu(s) = \frac{2\pi^{1/2}}{\Gamma(q+\frac{1}{2})} \frac{s^{-q}}{\sin(\pi s)} \int_0^\infty \frac{\sinh [(1-s)x/2]}{\sinh (x/2)} \left(\frac{s^2}{4} - \frac{d^2}{dx^2} \right)^k \left\{ x^q K_{-q} \left(\frac{sx}{2} \right) \right\} dx.$$

Setting $t = sx/2$, by repeated use of the recurrence formula [7, § 3.71]

$$(3.10) \quad \left(1 - \frac{d^2}{dt^2} \right) \frac{K_\nu(t)}{t^\nu} = \frac{2\nu+1}{t} \frac{d}{dt} \left(\frac{K_\nu(t)}{t^\nu} \right) = -(2\nu+1) \frac{K_{\nu+1}(t)}{t^{\nu+1}}$$

we find

$$(3.11) \quad \left(\frac{s^2}{4} - \frac{d^2}{dx^2} \right)^k \left\{ x^q K_{-q} \left(\frac{sx}{2} \right) \right\} = \frac{\Gamma(q+\frac{1}{2})}{\Gamma(q-k+\frac{1}{2})} s^k x^{q-k} K_{k-q} \left(\frac{sx}{2} \right).$$

Inserting (3.11) into (3.9) and restoring the notation $\nu = k - q$, we are led to the integral representation

$$(3.12) \quad g_\nu(s) = \frac{2\pi^{1/2}}{\Gamma(\frac{1}{2}-\nu)} \frac{s^\nu}{\sin(\pi s)} \int_0^\infty \frac{\sinh [(1-s)x/2]}{\sinh (x/2)} x^{-\nu} K_\nu \left(\frac{sx}{2} \right) dx.$$

This result is identical to the corresponding representation (3.3) for Case i, except that now the finite part of the divergent integral is to be taken (as indicated by the notation \int). From the original representation (2.16) it is clear that $g_\nu(s)$ is an analytic function of ν in the whole complex ν -plane. Therefore the finite part integral (3.12) is also the analytic continuation of the integral (3.3) which is analytic for $\text{Re } \nu < \frac{1}{2}$.

Case iii. $\nu = n + \frac{1}{2}$, $n = 0, 1, 2, \dots$. In this case the integral (2.16) can be evaluated by means of (3.4), viz.

$$\begin{aligned}
 (3.13) \quad g_{n+1/2}(s) &= \pi \int_{-\infty}^{\infty} \frac{(s^2/4 + u^2)^n}{\cosh(2\pi u) - \cos(\pi s)} du \\
 &= \frac{\pi}{\sin(\pi s)} \left(\frac{s^2}{4} - \frac{d^2}{dx^2} \right)^n \left\{ \frac{\sinh[(1-s)x/2]}{\sinh(x/2)} \right\} \Big|_{x=0}.
 \end{aligned}$$

Thus for $n = 0$, $\nu = \frac{1}{2}$, we have

$$(3.14) \quad g_{1/2}(s) = \frac{\pi}{\sin(\pi s)} (1-s).$$

To evaluate the derivative in (3.13), we substitute

$$(3.15) \quad \frac{\sinh[(1-s)x/2]}{\sinh(x/2)} = \frac{e^{(1-s/2)x} - e^{sx/2}}{e^x - 1} = -2 \sum_{k=0}^{\infty} B_{2k+1} \left(\frac{s}{2} \right) \frac{x^{2k}}{(2k+1)!}$$

where $B_{2k+1}(s/2)$ is the Bernoulli polynomial [3, § 1.13]. Then we find

$$(3.16) \quad g_{n+1/2}(s) = -\frac{2\pi}{\sin(\pi s)} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{s^2}{4} \right)^{n-k} \frac{B_{2k+1}(s/2)}{2k+1}.$$

4. Expansion of $g_\nu(s)$ if $2\nu \notin \mathbb{Z}$. To determine the series expansion of $g_\nu(s)$ around $s = 0$, we start from the integral representation (3.12) which includes the representation (3.3) as a special case. For convenience it is assumed that 2ν is not integral. By substitution of

$$(4.1) \quad \frac{\sinh[(1-s)x/2]}{\sinh(x/2)} = e^{-sx/2} - 2 \sinh\left(\frac{sx}{2}\right) \frac{e^{-x}}{1 - e^{-x}}$$

the representation (3.12) is rewritten as

$$\begin{aligned}
 (4.2) \quad g_\nu(s) &= \frac{2\pi^{1/2}}{\Gamma(\frac{1}{2} - \nu)} \frac{s^\nu}{\sin(\pi s)} \int_0^\infty e^{-sx/2} x^{-\nu} K_\nu\left(\frac{sx}{2}\right) dx \\
 &\quad - \frac{4\pi^{1/2}}{\Gamma(\frac{1}{2} - \nu)} \frac{s^\nu}{\sin(\pi s)} \int_0^\infty \sinh\left(\frac{sx}{2}\right) K_\nu\left(\frac{sx}{2}\right) \frac{x^{-\nu} e^{-x}}{1 - e^{-x}} dx.
 \end{aligned}$$

From [4, Formula 6.8(28)] we have

$$(4.3) \quad \int_0^\infty e^{-sx/2} x^{-\nu} K_\nu\left(\frac{sx}{2}\right) dx = \frac{\pi^{1/2} \Gamma(1 - 2\nu)}{\Gamma(\frac{3}{2} - \nu)} s^{\nu-1},$$

valid for $\text{Re } \nu < \frac{1}{2}$. By analytic continuation the result (4.3) also holds for $\text{Re } \nu \geq \frac{1}{2}$, $2\nu \notin \mathbb{N}$, provided that the finite part of the integral is taken as in (4.2). To evaluate the second integral in (4.2), we expand the product $\sinh(sx/2) K_\nu(sx/2)$ in a power series. Starting from the definition

$$(4.4) \quad K_\nu(z) = \frac{\pi}{2 \sin(\nu\pi)} [I_{-\nu}(z) - I_\nu(z)] \quad (\nu \notin \mathbb{Z})$$

we employ Watson's expansion [7, Formula 5.41(1)] for the products $J_\mu(z) J_{\pm\nu}(z)$ with $\mu = \frac{1}{2}$, $z = isx/2$. As a result, we obtain

$$\begin{aligned}
 (4.5) \quad \sinh\left(\frac{sx}{2}\right) K_\nu\left(\frac{sx}{2}\right) &= \frac{\pi^{1/2}}{2 \sin(\nu\pi)} \left[\sum_{k=0}^{\infty} \frac{\Gamma(2k - \nu + \frac{3}{2})}{(2k+1)! \Gamma(2k - 2\nu + 2)} (sx)^{2k - \nu + 1} \right. \\
 &\quad \left. - \sum_{k=0}^{\infty} \frac{\Gamma(2k + \nu + \frac{3}{2})}{(2k+1)! \Gamma(2k + 2\nu + 2)} (sx)^{2k + \nu + 1} \right].
 \end{aligned}$$

The latter expansion is inserted into the second integral in (4.2) and we apply a term-by-term integration using the auxiliary integral

$$(4.6) \quad \int_0^\infty x^\alpha \frac{e^{-x}}{1 - e^{-x}} dx = \Gamma(\alpha + 1)\zeta(\alpha + 1) \quad (\text{Re } \alpha > 0)$$

where $\zeta(\alpha + 1)$ denotes Riemann's zeta function [3, § 1.12]. By analytic continuation the result (4.6) also holds for $\text{Re } \alpha \leq 0, \alpha \notin \mathbb{Z}$, provided that the finite part of the integral is taken.

Finally, by compiling the previous results we are led to the desired expansion

$$(4.7) \quad g_\nu(s) = \frac{2\pi}{\Gamma(\frac{1}{2} - \nu)} \frac{s}{\sin(\nu\pi) \sin(\pi s)} \left[\sum_{k=0}^\infty \frac{\Gamma(2k + \nu - \frac{1}{2})}{\Gamma(2k + 2\nu)} \zeta(2k) s^{2k+2\nu-2} - \sum_{k=0}^\infty \frac{\Gamma(2k - \nu + \frac{3}{2})}{(2k+1)!} \zeta(2k - 2\nu + 2) s^{2k} \right],$$

valid if $2\nu \notin \mathbb{Z}$. It is readily seen that the expansion (4.7) is convergent for $0 < |s| < 1$.

5. Expansion of $g_\nu(s)$ if $2\nu \in \mathbb{Z}$. Since $g_\nu(s)$ is a continuous function of the parameter ν , the series expansion of $g_\nu(s)$ when $2\nu = N \in \mathbb{Z}$ can be found by taking limits in (4.7) as $\nu \rightarrow N/2$. We distinguish four cases.

Case i. $\nu = n, n = 1, 2, 3, \dots$. Rewrite the expansion (4.7) as

$$(5.1) \quad g_\nu(s) = \frac{2\pi}{\Gamma(\frac{1}{2} - \nu)} \frac{s}{\sin(\pi s) \sin(\nu\pi)} \cdot \left[- \sum_{k=0}^{n-2} \frac{\Gamma(2k - \nu + \frac{3}{2})}{(2k+1)!} \zeta(2k - 2\nu + 2) s^{2k} + \sum_{k=n-1}^\infty \left\{ \frac{\Gamma(2k - 2n + \nu + \frac{3}{2})}{\Gamma(2k - 2n + 2\nu + 2)} \zeta(2k - 2n + 2) s^{2k-2n+2\nu} - \frac{\Gamma(2k - \nu + \frac{3}{2})}{(2k+1)!} \zeta(2k - 2\nu + 2) s^{2k} \right\} \right]$$

where it is noted that the terms of the finite sum and of the infinite series vanish when $\nu = n$. By properly taking limits as $\nu \rightarrow n$, the expansion (5.1) passes into

$$(5.2) \quad g_n(s) = \frac{4}{\pi} \Gamma\left(n + \frac{1}{2}\right) \frac{s}{\sin(\pi s)} \cdot \left[\sum_{k=0}^{n-2} \frac{\Gamma(2k - n + \frac{3}{2})}{(2k+1)!} \zeta'(2k - 2n + 2) s^{2k} + \sum_{k=n-1}^\infty \frac{\Gamma(2k - n + \frac{3}{2})}{(2k+1)!} \zeta(2k - 2n + 2) s^{2k} \cdot \left\{ \log s + \psi\left(2k - n + \frac{3}{2}\right) - \psi(2k + 2) + \frac{\zeta'(2k - 2n + 2)}{\zeta(2k - 2n + 2)} \right\} \right],$$

valid for $n = 1, 2, 3, \dots$; here, $\psi(z)$ denotes the logarithmic derivative of the Γ -function, i.e., $\psi(z) = \Gamma'(z)/\Gamma(z)$. By means of [3, Formula 1.12(23)] we have

$$(5.3) \quad \zeta'(2k - 2n + 2) = (-1)^{n-k-1} \frac{(2n - 2k - 2)!}{2(2\pi)^{2n-2k-2}} \zeta(2n - 2k - 1) \quad (k = 0, 1, \dots, n - 2),$$

which is used in the finite sum in (5.2).

Case ii. $\nu = -n, n = 0, 1, 2, \dots$. Rewrite the expansion (4.7) as

$$\begin{aligned}
 g_\nu(s) = & \frac{2\pi}{\Gamma(\frac{1}{2}-\nu)} \frac{s}{\sin(\pi s)} \frac{1}{\sin(\nu\pi)} \\
 & \cdot \left[\sum_{k=0}^n \frac{\Gamma(2k+\nu-\frac{1}{2})}{\Gamma(2k+2\nu)} \zeta(2k) s^{2k+2\nu-2} \right. \\
 & + \sum_{k=0}^\infty \left\{ \frac{\Gamma(2k+2n+\nu+\frac{3}{2})}{\Gamma(2k+2n+2\nu+2)} \zeta(2k+2n+2) s^{2k+2n+2\nu} \right. \\
 & \left. \left. - \frac{\Gamma(2k-\nu+\frac{3}{2})}{(2k+1)!} \zeta(2k-2\nu+2) s^{2k} \right\} \right]
 \end{aligned}
 \tag{5.4}$$

and take limits as $\nu \rightarrow -n$. Then, as in Case i, we are led to the expansion

$$\begin{aligned}
 g_{-n}(s) = & \frac{4(-1)^n}{\Gamma(n+\frac{1}{2})} \frac{s}{\sin(\pi s)} \left[\sum_{k=0}^n \Gamma\left(2k-n-\frac{1}{2}\right) (2n-2k)! \zeta(2k) s^{2k-2n-2} \right. \\
 & + \sum_{k=0}^\infty \frac{\Gamma(2k+n+\frac{3}{2})}{(2k+1)!} \zeta(2k+2n+2) s^{2k} \\
 & \left. \cdot \left\{ \log s + \psi\left(2k+n+\frac{3}{2}\right) - \psi(2k+2) + \frac{\zeta'(2k+2n+2)}{\zeta(2k+2n+2)} \right\} \right],
 \end{aligned}
 \tag{5.5}$$

valid for $n = 0, 1, 2, \dots$. For $n = 0$, the expansion (5.5) agrees with [6, Formula (33)]. Note that the expansion of $g_\nu(s)$ contains logarithmic terms in the case where ν is integral.

Case iii. $\nu = n + \frac{1}{2}, n = 0, 1, 2, \dots$. When taking limits in (4.7) as $\nu \rightarrow n + \frac{1}{2}$, proper care should be taken because some of the Γ - and ζ -functions become singular. First consider the case $\nu = \frac{1}{2}$; then we find

$$\begin{aligned}
 g_{1/2}(s) = & \frac{2\pi s}{\sin(\pi s)} \lim_{\nu \rightarrow 1/2} \frac{1}{\Gamma(\frac{1}{2}-\nu)} \left[\frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(2\nu)} \zeta(0) s^{2\nu-2} - \Gamma\left(\frac{3}{2}-\nu\right) \zeta(2-2\nu) \right] \\
 = & \frac{\pi s}{\sin(\pi s)} \left(\frac{1}{s} - 1 \right),
 \end{aligned}
 \tag{5.6}$$

in accordance with (3.14). Generally, for $\nu = n + \frac{1}{2}, n \geq 1$, the expansion (4.7) passes into

$$\begin{aligned}
 g_{n+1/2}(s) = & (-1)^n \frac{2\pi s}{\sin(\pi s)} \\
 & \cdot \lim_{\nu \rightarrow n+1/2} \frac{1}{\Gamma(\frac{1}{2}-\nu)} \left[- \sum_{k=0}^{[(n-1)/2]} \frac{\Gamma(2k-\nu+\frac{3}{2})}{(2k+1)!} \zeta(2k-2\nu+2) s^{2k} \right. \\
 & \left. - \frac{\Gamma(2n-\nu+\frac{3}{2})}{(2n+1)!} \zeta(2n-2\nu+2) s^{2n} \right] \\
 = & \frac{2\pi s}{\sin(\pi s)} \left[(-1)^n \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} \zeta(2k-2n+1) s^{2k} - \frac{(n!)^2}{2(2n+1)!} s^{2n} \right],
 \end{aligned}
 \tag{5.7}$$

valid for $n = 1, 2, 3, \dots$. In the final line of (5.7) we may set, by [3, Formula 1.12(22)],

$$\zeta(2k-2n+1) = -\frac{B_{2n-2k}}{2(n-k)} \quad (k = 0, 1, \dots, n-1)
 \tag{5.8}$$

where B_{2n-2k} is the Bernoulli number. It can be shown that the expansion (5.7) agrees with (3.16).

Case iv. $\nu = -n - \frac{1}{2}, n = 0, 1, 2, \dots$. In this case the expansion (4.7) remains valid, provided that the ratio $\Gamma(2k + \nu - \frac{1}{2})/\Gamma(2k + 2\nu)$ is handled with proper care. Thus by means of

$$(5.9) \quad \lim_{\nu \rightarrow -n-1/2} \frac{\Gamma(2k + \nu - \frac{1}{2})}{\Gamma(2k + 2\nu)} = \begin{cases} 2(2k - 2n - 1)_n, & k \leq n, \\ (2k - 2n - 1)_n, & k \geq n + 1 \end{cases}$$

we obtain the expansion

$$(5.10) \quad g_{-n-1/2}(s) = \frac{(-1)^n}{n!} \frac{2\pi s}{\sin(\pi s)} \left[-2 \sum_{k=0}^n (2k - 2n - 1)_n \zeta(2k) s^{2k-2n-3} + \sum_{k=1}^{\infty} (-1)^k (k)_n \zeta(k + 2n + 1) s^{k-2} \right],$$

valid for $n = 0, 1, 2, \dots$. In the special case $n = 0, \nu = -\frac{1}{2}$, the expansion (5.10) reduces to

$$(5.11) \quad g_{-1/2}(s) = \frac{2\pi s}{\sin(\pi s)} \left[s^{-3} + \sum_{k=1}^{\infty} (-1)^k \zeta(k + 1) s^{k-2} \right] = -\frac{2\pi s^{-1}}{\sin(\pi s)} [\psi(s) + \gamma]$$

by use of [3, Formula 1.17(5)]. The same result can also be found by a direct evaluation of the integral (3.3) with $\nu = -\frac{1}{2}$.

Finally, it is pointed out that the infinite series expansions of $g_\nu(s)$, as presented in (5.2), (5.5), and (5.10), are convergent for $0 < |s| < 1$. In Case iii the infinite series reduces to a finite sum; see (5.6) and (5.7).

6. Asymptotic expansion of $G_{\mu,p}^{(m)}(\eta)$. The asymptotic expansion of $G_{\mu,p}^{(m)}(\eta)$ as $\eta \rightarrow \infty$ is determined through a term-by-term conversion, based on theorems of Abelian asymptotics [2, Kap. 7], of the series expansion of $s^{-\mu} g_\nu(s)$ around $s = 0$. The conversion is most easily carried out by use of the ‘‘dictionary’’ in Table 1. The left column of the table shows a specific term of the expansion around $s = 0$; the right column shows the corresponding term of the asymptotic expansion as $\eta \rightarrow \infty$.

In the expansions of $g_\nu(s)$ as determined in §§ 4 and 5, replace $\pi s/\sin(\pi s)$ by

$$(6.1) \quad \frac{\pi s}{\sin(\pi s)} = 2 \sum_{k=0}^{\infty} (1 - 2^{1-2k}) \zeta(2k) s^{2k}, \quad |s| < 1$$

TABLE 1
Inverse Laplace transforms.

$f(s)$	$(1/2\pi i) \int_{c-i\infty}^{c+i\infty} f(s) e^{\eta s} ds$
s^λ	$[1/\Gamma(-\lambda)] \eta^{-\lambda-1}, \quad \lambda \neq 0, 1, 2, \dots$ $0, \quad \lambda = 0, 1, 2, \dots$
$s^\lambda \log s$	$-[1/\Gamma(-\lambda)] \eta^{-\lambda-1} [\log \eta - \psi(-\lambda)], \quad \lambda \neq 0, 1, 2, \dots$ $(-1)^{\lambda+1} \lambda! \eta^{-\lambda-1}, \quad \lambda = 0, 1, 2, \dots$

and multiply the series involved. Then for $2\nu \notin \mathbb{Z}$, the expansion (4.7) of $g_\nu(s)$ takes the form

$$(6.2) \quad g_\nu(s) = \sum_{k=0}^{\infty} A_k s^{2k+2\nu-2} + \sum_{k=0}^{\infty} B_k s^{2k}, \quad 0 < |s| < 1$$

with coefficients

$$(6.3) \quad A_k = \frac{4}{\Gamma(\frac{1}{2}-\nu) \sin(\nu\pi)} \sum_{l=0}^k \frac{\Gamma(2l+\nu-\frac{1}{2})}{\Gamma(2l+2\nu)} \zeta(2l)(1-2^{1-2k+2l})\zeta(2k-2l),$$

$$B_k = \frac{-4}{\Gamma(\frac{1}{2}-\nu) \sin(\nu\pi)} \sum_{l=0}^k \frac{\Gamma(2l-\nu+\frac{3}{2})}{(2l+1)!} \zeta(2l-2\nu+2)(1-2^{1-2k+2l})\zeta(2k-2l).$$

Similar expansions hold in the case where $2\nu \in \mathbb{Z}$. From (5.2) and (5.5) it follows that the expansion of $g_\nu(s)$ contains logarithmic terms if ν is integral.

Starting from (6.2) multiplied by $s^{-\mu}$, we find by use of Table 1 the desired asymptotic expansion

$$(6.4) \quad G_{\mu,p}^{(m)}(\eta) \sim \sum_{k=0}^{\infty} \frac{A_k}{\Gamma(\mu-2\nu-2k+2)} \eta^{\mu-2\nu-2k+1} + \sum_{k=0}^{\infty} \frac{B_k}{\Gamma(\mu-2k)} \eta^{\mu-2k-1} \quad (\eta \rightarrow \infty),$$

valid if $2\nu \notin \mathbb{Z}$. It is pointed out that the first (second) asymptotic series in (6.4) terminates to a finite sum if $\mu-2\nu$ (μ) is an integer. Similar asymptotic expansions hold in the case where $2\nu \in \mathbb{Z}$. If ν is an integer, it is found from (5.2) and (5.5) that the asymptotic expansion of $G_{\mu,p}^{(m)}(\eta)$ contains logarithmic terms $[1/\Gamma(\mu-2k)] \times \eta^{\mu-2k-1} \log \eta$, with $k \geq \max(\nu-1, 0)$.

As an example, we determine the asymptotic expansion of the integral [5]

$$(6.5) \quad G_{2,-1/2}^{(2)}(\eta) = \int_{-\infty}^{\eta} (\eta-t)[F_{-1/2}''(t)]^2 dt,$$

for which $\nu=2$. In the expansion (5.2) with $n=2$, replace $\pi s/\sin(\pi s)$ by (6.1), and multiply the series involved. Then the expansion of $g_2(s)$ takes the form

$$(6.6) \quad g_2(s) = \sum_{k=1}^{\infty} c_k s^{2k} \log s + \sum_{k=0}^{\infty} d_k s^{2k}, \quad |s| < 1$$

with coefficients

$$(6.7) \quad c_k = 6\pi^{-3/2} \sum_{l=1}^k \frac{\Gamma(2l-\frac{1}{2})}{(2l+1)!} \zeta(2l-2)(1-2^{1-2k+2l})\zeta(2k-2l),$$

$$d_k = 6\pi^{-3/2} \sum_{l=0}^k \frac{\Gamma(2l-\frac{1}{2})}{(2l+1)!} \zeta(2l-2) \left[\psi\left(2l-\frac{1}{2}\right) - \psi(2l+2) + \frac{\zeta'(2l-2)}{\zeta(2l-2)} \right] \cdot (1-2^{1-2k+2l})\zeta(2k-2l).$$

Next, by use of Table 1 in a term-by-term conversion of the expansion of $s^{-2}g_2(s)$, we are led to the asymptotic expansion

$$(6.8) \quad G_{2,-1/2}^{(2)}(\eta) \sim d_0 \eta - \sum_{k=1}^{\infty} (2k-2)! c_k \eta^{-2k+1} \quad (\eta \rightarrow \infty).$$

By evaluating the leading terms in (6.8), we find

$$(6.9) \quad G_{2,-1/2}^{(2)}(\eta) = \frac{3\zeta(3)}{2\pi^3} \eta + \frac{1}{8\pi} \eta^{-1} + \frac{5\pi}{192} \eta^{-3} + \frac{43\pi^3}{1920} \eta^{-5} + \frac{323\pi^5}{7168} \eta^{-7} + O(\eta^{-9}) \quad (\eta \rightarrow \infty).$$

The asymptotic expansion (6.8) can also be derived in a more elementary manner. To that end we start from the two-sided Laplace transform

$$(6.10) \quad \int_{-\infty}^{\infty} e^{-st} F''_{-1/2}(t) dt = \frac{\pi s^{3/2}}{\sin(\pi s)},$$

obtainable from (2.12) and (2.13). Using (6.1) and Table 1, we expand (6.10) in a power series around $s = 0$, whereupon a term-by-term conversion yields the asymptotic expansion

$$(6.11) \quad F''_{-1/2}(t) \sim -\frac{2}{\pi} \sum_{k=0}^{\infty} (1 - 2^{1-2k}) \Gamma\left(2k + \frac{3}{2}\right) \zeta(2k) t^{-2k-3/2} \quad (t \rightarrow \infty).$$

By squaring (6.11) we find

$$(6.12) \quad [F''_{-1/2}(t)]^2 \sim \sum_{k=0}^{\infty} b_k t^{-2k-3} \quad (t \rightarrow \infty)$$

with coefficients

$$(6.13) \quad b_k = \frac{4}{\pi^2} \sum_{l=0}^k (1 - 2^{1-2l}) \Gamma\left(2l + \frac{3}{2}\right) \zeta(2l) \cdot (1 - 2^{1-2k+2l}) \Gamma\left(2k - 2l + \frac{3}{2}\right) \zeta(2k - 2l).$$

Next it is observed from (6.5) that $G_{2,-1/2}^{(2)}(\eta)$ is the repeated integral of order 2, of $[F''_{-1/2}(t)]^2$. As it has been shown in the Appendix of [6], the asymptotic expansion of $G_{2,-1/2}^{(2)}(\eta)$ can now be derived by a twice repeated termwise integration of the expansion (6.12). Thus we find

$$(6.14) \quad G_{2,-1/2}^{(2)}(\eta) \sim C_1 \eta + C_0 + \sum_{k=0}^{\infty} \frac{b_k}{(2k+1)(2k+2)} \eta^{-2k-1} \quad (\eta \rightarrow \infty)$$

where the constants C_0 and C_1 are yet to be determined. By dividing (6.14) by η and taking limits as $\eta \rightarrow \infty$, it readily follows that

$$(6.15) \quad C_1 = \int_{-\infty}^{\infty} [F''_{-1/2}(t)]^2 dt = g_2(0) = \frac{3\zeta(3)}{2\pi^3}$$

where $g_2(0)$ has been evaluated by means of (2.16) and [4, Formula 6.6(4)]. In a similar manner it is found that

$$(6.16) \quad C_0 = - \int_{-\infty}^{\infty} t [F''_{-1/2}(t)]^2 dt = g'_2(0) = 0.$$

The asymptotic expansion (6.14) does agree with (6.8) provided that

$$(6.17) \quad -(2k)! c_{k+1} = \frac{b_k}{(2k+1)(2k+2)}, \quad k = 0, 1, 2, \dots$$

The latter identity can be proved by a generating function technique.

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