

## Covering a circle with random arcs

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by

R. Doornbos and J.H. van Lint

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1. Introduction. The problem discussed in this note and some of the ideas used in proofs were suggested by a problem posed by F. Göbel in Nieuw Archief voor Wiskunde (Vol. 18, 1970). Consider random points on the unit circle drawn independently from a uniform distribution. Let the circumference of the circle be 1 and suppose each point is "first endpoint" (in a fixed orientation) of an arc of length  $\alpha$  ( $\alpha$  fixed). What is the expected value of the number of arcs necessary to cover the circle?

If we call the number of arcs  $\underline{n}$  then

$$(1.1) \quad E(\underline{n}) = 1 + \sum_{k=1}^{[\alpha^{-1}]} (-1)^{k-1} (1 - k\alpha)^{k-1} (k\alpha)^{-k-1}.$$

Several results in the literature are sufficient to demonstrate (1.1) e.g. S.S. Wilks, Mathematical Statistics, 8.43 or F.W. Steutel, Random division of an interval, Stat. Neerl. 21 (1967), 231 - 244 (cf. last formula on p. 236). In the latter it is proved that

$$(1.2) \quad E(\underline{n}) = \frac{1}{\alpha} \left\{ \log \frac{1}{\alpha} + \log \log \frac{1}{\alpha} + \gamma + o(1) \right\} \quad (\alpha \rightarrow 0).$$

In § 2 we give a simple proof of (1.1). In § 3 we give a second proof by considering a more difficult problem namely the expected value of the number of arcs necessary to cover a certain part of the circle. Finally in § 4 we consider the problem of covering the circle with arcs of random length.

2. Proof of (1.1). Suppose  $(\ell + 1)^{-1} \leq \alpha < \ell^{-1}$ ,  $\ell$  an integer. Instead of speaking of the circle we consider the real numbers mod 1 and random variables  $\underline{x}_1, \underline{x}_2, \dots$  chosen from a uniform distribution on  $[0, 1]$ . We consider the event  $\underline{n} > n$ . For  $i = 1, 2, \dots, n$  we define the event  $E_i$  as  $\underline{x}_k \notin [\underline{x}_i, \underline{x}_i + \alpha]$  for  $k \neq i$ . Then by the principle of inclusion and exclusion

$$P(\underline{n} > n) = \sum_{i=1}^n P(E_i) - \sum_{i < j} P(E_i \cap E_j) + \dots$$

where  $P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}) = 0$  if  $k > \ell$ .

For  $k \leq \ell$  we have

$$P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}) = (1 - k\alpha)^{n-1}.$$

Hence

$$P(\underline{n} > n) = \sum_{k=1}^{\ell} (-1)^{k-1} \binom{n}{k} (1 - k\alpha)^{n-1} \quad \text{for } n = 1, 2, \dots.$$

Trivially

$$P(\underline{n} > 0) = 1. \quad \text{We now find}$$

$$\begin{aligned} E(\underline{n}) &= \sum_{n=1}^{\infty} n \{P(\underline{n} > n-1) - P(\underline{n} > n)\} = \\ &= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{\ell} (-1)^{k-1} \binom{n}{k} (1 - k\alpha)^{n-1} \\ &= 1 + \sum_{k=1}^{\ell} (-1)^{k-1} (1 - k\alpha)^{k-1} (k\alpha)^{-k-1}. \end{aligned}$$

For  $\ell$  we can write  $[\alpha^{-1}]$  since if  $\alpha = (\ell+1)^{-1}$  the term with  $k = \ell+1$  is 0.

3. Covering part of the circle. We now consider the problem of covering

$[0, x]$  where  $0 < x \leq 1$  and define if  $\underline{n}$  is the number of intervals necessary for the covering :

$$(3.1) \quad F(x) := E(\underline{n}).$$

We also consider the case of two intervals. Suppose  $\alpha \leq 1/2$  and let  $x$  and  $y$  both be less than  $\alpha$ . Furthermore let  $p$  be a number with

$$x + \alpha \leq p \leq 1 - y - \alpha.$$

Let  $\underline{m}$  be the number of random intervals necessary to cover  $[0, x]$  and  $[p, p+y]$ .

We define

$$(3.2) \quad G(x, y) := E(\underline{m}) .$$

The function  $G$  does not depend on  $p$  .

We now consider the functional equations for  $F$  and  $G$  .

For  $x \leq \alpha$  we have

$$(3.3) \quad F(x) = \int_0^x \{1 + F(t)\} dt + (1-\alpha-x) \{1 + F(x)\} + \\ + \int_{1-\alpha}^{1-\alpha+x} \{1 + F(1-\alpha+x-t)\} dt + (\alpha-x) ,$$

i.e.

$$(3.4) \quad (\alpha+x) F(x) = 1 + 2 \int_0^x F(t) dt .$$

Hence

$$(3.5) \quad F(x) = \alpha^{-2} (x+\alpha) \quad \text{for } 0 \leq x \leq \alpha .$$

If  $\alpha \geq 1/2$  we can substitute  $x = 1 - \alpha$ . This gives us

$$(3.6) \quad F(1) = 1 + F(1-\alpha) = 1 + \alpha^{-2} ,$$

i.e. (1.1) for  $1/2 \leq \alpha \leq 1$ .

Now assume  $\alpha \leq x \leq 2\alpha$ . Then we have

$$(3.7) \quad F(x) = \int_0^{x-\alpha} \{1 + G(t, x-\alpha-t)\} dt + 2 \int_0^{\alpha} \{1 + F(x-t)\} dt + \\ + (1-\alpha-x) \{1 + F(x)\} ,$$

i.e. by (3.5)

$$(3.8) \quad (\alpha+x) F(x) = 5 - \alpha^{-2} x^2 + 2 \int_{\alpha}^x F(t) dt + \int_0^{x-\alpha} G(t, x-\alpha-t) dt .$$

We use the same method to compute  $G(x,y)$  for small values of  $x$  and  $y$  .

As in (3.3) we find :

$$(3.9) \quad G(x,y) = (\alpha-x) \{1 + F(y)\} + (\alpha-y) \{1 + F(x)\} + 2 \int_0^y \{1 + G(x,t)\} dt + \\ + 2 \int_0^x \{1 + G(t,y)\} dt + (1-2\alpha-x-y) \{1 + G(x,y)\} ,$$

i.e.

$$(3.10) \quad (2\alpha + x + y) G(x,y) = 3 - 2\alpha^{-2} xy + 2 \int_0^y G(x,t) dt + 2 \int_0^x G(t,y) dt .$$

The functional equation (3.10) can be solved by standard methods (differentiation etc.) yielding

$$(3.11) \quad G(x,y) = -\frac{1}{4} \alpha^{-3} xy + \frac{3}{4} \alpha^{-2} (x+y) + \frac{3}{2} \alpha^{-1} \text{ for } 0 \leq x \leq \alpha , \\ 0 \leq y \leq \alpha .$$

Substitution of (3.11) in (3.8) gives us

$$(3.12) \quad (\alpha+x) F(x) = \frac{103}{24} - \frac{1}{8} \alpha^{-1} x - \frac{1}{8} \alpha^{-2} x^2 - \frac{1}{24} \alpha^{-3} x^3 + 2 \int_{\alpha}^x F(t) dt ,$$

from which we find  $F$  in the interval  $[\alpha, 2\alpha]$  :

$$(3.13) \quad F(x) = \frac{9}{8} \alpha^{-1} + \alpha^{-2} x - \frac{1}{8} \alpha^{-3} x^2 \quad (\alpha \leq x \leq 2\alpha) .$$

The treatment of  $2\alpha \leq x \leq 3\alpha$  etc. is essentially the same .

Suppose  $\frac{1}{3} \leq \alpha \leq 1/2$  . By (3.13) we have

$$\begin{aligned}
 (3.14) \quad F(1) &= 1 + F(1-\alpha) = \\
 &= 1 + \frac{5}{4} \alpha^{-2} - \frac{1}{8} \alpha^{-3}, \\
 \text{i.e. (1.1) for } \frac{1}{3} &\leq \alpha \leq 1/2.
 \end{aligned}$$

Concerning (3.11) we make the following remark. Suppose random arcs of length  $\alpha < 1/2$  are chosen on the circle of circumference 1 and two opposite points on the same diameter are to be covered. Clearly the expected value of the number of intervals necessary for the covering is

$$2 \sum_{k=0}^{\infty} (1-\alpha)^k - \sum_{k=0}^{\infty} (1-2\alpha)^k = \frac{3}{2} \alpha^{-1} = G(0,0).$$

If we replace the two points by small intervals with length  $x$  and  $y$  then  $G(x,y)$  is the expected value. The other extreme treated by  $G$

is the case where two opposite arcs of length  $\frac{1}{4}$  are to be covered by random arcs which also have length  $\frac{1}{4}$ . Then we have

$$x = y = \alpha = \frac{1}{4}, \quad p = \frac{1}{2} \quad \text{and the expected value is 11}.$$

#### 4. Covering the circle by arcs of random length

We consider again random points on the circle, but now arcs are formed between pairs of points :

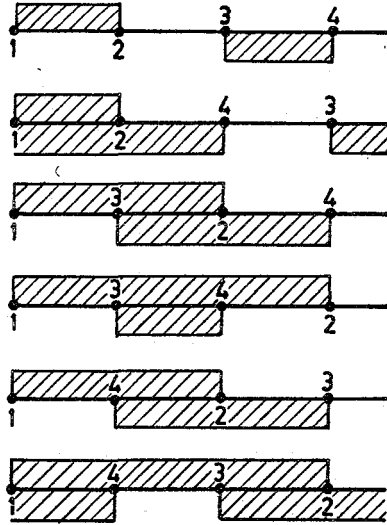
$$[\underline{x}_{2i-1}, \underline{x}_{2i}] \quad (i = 1, 2, \dots)$$

We are interested in the expected number of arcs necessary to cover the circle.

In order to make the principle clear we consider first the case of 2 arcs <sup>1)</sup>. Instead of the circle we take again the interval  $[0,1]$  and we identify  $\underline{x}_1$  with 0 (and consequently also with 1) .

1) We will still use the name arc when the arcs are straightened in the interval  $[0,1]$  .

Then all permutations of the remaining 3 points have the same probability. Thus there are 6 possible cases, shown in the figure, where  $x_i$  is denoted by  $i$ .



In 1 out of the 6 cases the "circle" is covered, viz. in the last case. In 4 cases one endpoint is still "open", that means that it is not lying inside an other interval, in 1 case two endpoints are open. In the general case we denote by  $P_{k,n}$  the probability that  $k$  specified endpoints are open when  $n$  arcs have been drawn. Thus for instance

$$P_{1,2} = \frac{3}{6} \quad \text{and} \quad P_{2,2} = \frac{1}{6}. \quad \text{Then we have :}$$

$$(4.1) \quad P(\underline{n} > n) = \binom{n}{1} P_{1,n} - \binom{n}{2} P_{2,n} + \dots + (-1)^{n-1} \binom{n}{n} P_{n,n}.$$

$$\text{For } n = 2 \text{ this becomes } P(\underline{n} > 2) = 2 \cdot \frac{3}{6} - 1 \cdot \frac{1}{6} = \frac{5}{6}.$$

Next we determine  $P_{k,n}$ . In the first place the  $k$  endpoints must be open when the other  $(n-k)$  arcs are left out. This can be arrived at in  $(k-1)!$  different ways.

We may namely consider the arcs ending in  $\underline{x}_2, \underline{x}_4, \dots, \underline{x}_{2k}$ .

Their configuration is determined by the sequence of the  $(k-1)$  points  $\underline{x}_4, \dots, \underline{x}_{2k}$ .



The probability that the endpoints  $\underline{x}_2, \dots, \underline{x}_{2k}$  remain open does not depend on their order and we assume therefore that

$$0 < \underline{x}_2 < \underline{x}_4 < \dots < \underline{x}_{2k} < 1.$$

Each of the remaining  $(n-k)$  arcs must lie inside one of the intervals

$$[\underline{x}_2, \underline{x}_4], [\underline{x}_4, \underline{x}_6], \dots, [\underline{x}_{2k}, \underline{x}_2].$$

We denote the number of arcs within the  $i$ -th of these intervals by  $n_i$ .

Thus 
$$n_i \geq 0, \sum_1^k n_i = n-k.$$

The number of ways in which  $n_i$  arcs can be fitted in such an interval is

$$(2n_i + 1)! / 2^{n_i}.$$

This can be seen in the following way. The number of permutations of the begin and endpoints of the  $n_i$  arcs and the beginpoint  $\underline{x}_{2i+1}$

which lies already in the interval  $[\underline{x}_{2i}, \underline{x}_{2i+2}]$  is  $(2n_i + 1)!$ .

But only those permutations are permitted that leave the order of begin- and endpoint of each arc unaltered. This gives the factor  $2^{-n_i}$ . The total number of favourable possibilities is therefore :

$$(4.2) \quad \frac{(k-1)!(n-k)!}{2^{n-k}} \sum_{\sum n_i = n-k} \frac{(2n_1 + 1)! \dots (2n_k + 1)!}{n_1! \dots n_k!}$$

In order to get  $P_{k,n}$  this has to be divided by the total number of possibilities which is  $(2n-1)!$  ( $\underline{x}_1$  being fixed at 0). Substitution of this result into (4.1) gives

$$(4.3) \quad P(\underline{n} \geq n) = \sum_{k=1}^n \binom{n}{k} \frac{(-1)^{k-1} (k-1)!(n-k)!}{2^{n-k} (2n-1)!} \sum_{\sum n_i = n-k} \frac{(2n_1 + 1)! \dots (2n_k + 1)!}{n_1! \dots n_k!} =$$

$$= \frac{n!}{2^n (2n-1)!} \sum_{k=1}^n \frac{2^k (-1)^{k-1}}{k} \sum_{\sum n_i = n-k} \prod_{i=1}^k \frac{(2n_i + 1)!}{n_i!} .$$

Now

$$\begin{aligned} (4.4) \quad E(\underline{n}) &= \sum_{n=0}^{\infty} P(n > n) = \\ &= 1 + \sum_{n=1}^{\infty} \frac{n!}{2^n (2n-1)!} \sum_{k=1}^n \frac{2^k (-1)^{k-1}}{k} \sum_{\sum n_i = n-k} \prod_{i=1}^k \frac{(2n_i + 1)!}{n_i!} . \end{aligned}$$

A numerical calculation up to  $n = 15$  showed that

$$(4.5) \quad E(\underline{n}) = 4.535\dots$$

Finally (4.4) can also be written as

$$(4.6) \quad E(\underline{n}) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^n (-1)^{k-1} t_{k,n} , \text{ say} .$$

Rearranging terms we can write alternatively :

$$(4.7) \quad E(\underline{n}) = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \sum_{n=k}^{\infty} t_{k,n} .$$

Now 
$$\sum_{n=1}^{\infty} t_{1,n} = \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = 4 .$$

This gives as a first approximation for  $E(\underline{n})$  the value 5, which is the value found for  $E(\underline{n})$  in (1.1) for  $\alpha = 1/2$ , the average arc length in our case .