An optimization problem related to the storage of solar energy

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ABSTRACT

In this paper an optimization problem is studied that was obtained when modelling the seasonal storage of solar energy in the ground. After guessing the optimal strategy a result from positive dynamic programming is used to establish its optimality.

1. Introduction

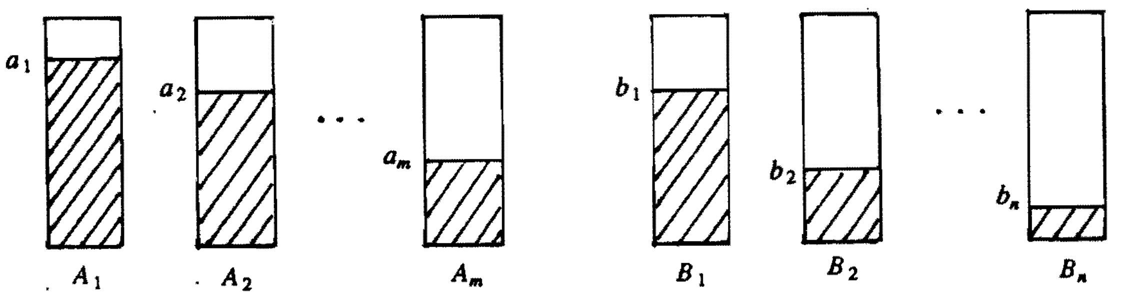
We will consider an optimization problem which resulted from analyzing the seasonal ground storage of solar energy.

A very rough description of the physical problem follows. Solar energy is collected by heating water in the summer period and has to be stored for the winter. One way of doing this, is to let the heated water flow through pipes in the ground. As a result the energy is absorbed by the earth. In the winter period cold water flows through the same pipes and the earth returns the energy to the water. Such systems are in operation today. A more detailed discussion of these energy storage systems can be found in Logtenberg and Van Delft [1987].

An optimization problem formulated to study this way of storing energy, and communicated to me by Logtenberg, will be given below.

There are $m + n$ identical tanks partially filled with water. The first m tanks represent the collectors, the last n tanks are used for storage.

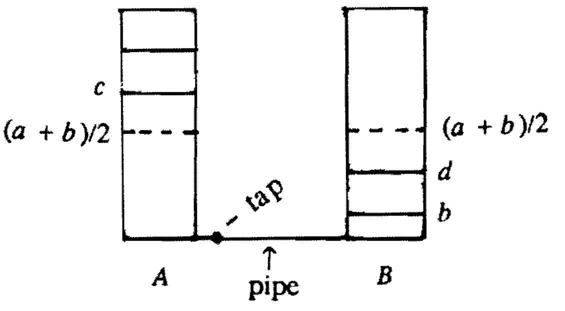
The collectors are called the A tanks and are numbered $A_1, A_2, \cdots, A_m$. The storage tanks are the B-tanks, $B_1, B_2, \cdots, B_n$.



The water levels in the A tanks are $a_1, a_2, \cdots, a_m$ and in the B tanks $b_1, b_2, \cdots, b_n$. The tanks are ordered such that $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_n$.

The water level in the tanks corresponds to the temperature of the water, or earth, in the energy storage problem. We assume the tanks to be cilinder shaped with area 1 so that level and contents are equal.

In the optimization problem as much water as possible has to be transfered from the A tanks to the B tanks. However there is a restriction in the way these transfers can take place. Remember, water levels correspond to temperatures. Transfers follow Archimedes law. So when combining an A tank with level a and a B tank with level $b < a$ the result can be any pair of levels, c in A and d in B, with $a > c \geq d > b$ and $c + d = a + b$ depending on how much water is transfered.



Because of the restrictions on c and d mentioned above we also have $c \geq (a + b)/2 \geq d$.

Now the optimization problem can be formulated as follows.

In which order should the A and B tanks be combined to maximize the total flow from A to B.

2. Model, notations, preliminaries

There are m A tanks, $A_1, \cdots, A_m$ with water levels $a_1 \geq a_2 \geq \cdots \geq a_m$ and n B tanks, $B_1, \cdots, B_n$ with levels $b_1 \geq b_2 \geq \cdots \geq b_n$.

Define

$$E_r := \{\underline{e} = (e_1, \cdots, e_r) \mid e_1 \geq e_2 \geq \cdots \geq e_r \geq 0\} \ .$$

Then we are interested in finding the function $V^* : E_m \times E_n \to \mathbb{R}^+$ with

$V^*(\underline{c},\underline{d})=$ the maximal amount of water that can be transfered from the A tanks to the B tanks starting from state $\underline{c} = (c_1, \cdots, c_m) \in E_m$, $\underline{d} = (d_1, \cdots, d_n) \in E_n$.

Similarly $V_g(\underline{c},\underline{d})$ denotes the amount transfered for strategy g.

As we will show, the strategy, which in each state $(\underline{c},\underline{d})$ combines the most empty A tank from which still some water can be transfered to the B tanks with the most full B tank to which in can let some water flow, is optimal.

Formally, in state $(\underline{c},\underline{d})$ strategy f prescribes to combine those A_k and B_l for which

(i) $c_i \leq d_n$ for all $i > k$

(ii) $c_k \leq d_j$ for all $j < l$

(iii) $c_k > d_l$.

The process is stopped if $c_1 \leq d_n$.

Note that strategy f always transfers the system from one state in $E_m \times E_n$ to another, i.e. the ordering of the A tanks and the ordering of the B tanks does not change.

In oder to prove that $V_f(\underline{c},\underline{d}) = V^*(\underline{c},\underline{d})$ for all $(\underline{c},\underline{d}) \in E_m \times E_n$ we use a result of Blackwell [1967] for positive dynamic programming. Therefore we have to define the set of actions that can be taken in state $(\underline{c},\underline{d})$.

In state $(\underline{c},\underline{d})$ the set of feasible actions is characterized by the set of triples (k,l,x) with $1 \leq k \leq m$, $1 \leq l \leq n$ and $x > 0$, where k and l must satisfy

$$\begin{aligned} &(i) \quad c_k > d_l \\ &(ii) \quad c_{k+1} < c_k \ \text{ and } \ d_{l-1} > d_l \ , \end{aligned} \tag{1}$$

extended with action "stop".

Action (k,l,x) implies that water is transfered from A_k to B_l until one of the following 4 events occurs

$$
\begin{array}{ll}
(i) & \text{the levels in } A_k \text{ and } B_l \text{ are equal} \\
(ii) & \text{the levels in } A_k \text{ and } A_{k+1} \text{ are equal} \\
(iii) & \text{the levels in } B_l \text{ and } B_{l-1} \text{ are equal} \\
(iv) & \text{the amount } x \text{ has flown from } A_k \text{ to } B_l \ .
\end{array}
\tag{2}
$$

Note that the restrictions (1, ii), (2, ii) and (2, iii) guarantee that the state remains in $E_m \times E_n$.

For a feasible action (k,l,x) in state $(\underline{c},\underline{d})$ the immediate reward r is defined as the amount of water transfered by that action, so

$$
r(\underline{c},\underline{d};k,l,x) = \begin{cases} (c_k - d_l)/2 & \text{in case the execution of the action} \\ & \text{terminates according to (2,i)} \\ c_{k+1} - c_k & \text{in case (2,ii)} \\ d_{l-1} - d_l & \text{in case (2,iii)} \\ x & \text{in case (2,iv)} \end{cases} \tag{3}
$$

and

$$r(\underline{c},\underline{d};\text{stop}) = 0 \ .$$

The state transformation (the transition is deterministic), denoted by T (state; action), is defined by

$$T(\underline{c},\underline{d};\text{stop}) = (\underline{c},\underline{d})$$

and

$$T(\underline{c},\underline{d};k,l,x) = (\underline{c} - \delta\underline{e}_k, \underline{d} + \delta\underline{e}_l) \ ,$$

where $\underline{e}_r$ denotes the row vector with $e_i = 0,\ i \neq k$ and $e_k = 1$ of the appropriate dimension and $\delta = r(\underline{c},\underline{d};k,l,x)$.

The result from Blackwell [1967] we use is the following

Lemma 1

If for all $(\underline{c},\underline{d}) \in E_m \times E_n$ and all feasible (k,l,x) in state $(\underline{c},\underline{d})$

$$r(\underline{c},\underline{d};k,l,x) + V_f(T(\underline{c},\underline{d};k,l,x)) \leq V_f(\underline{c},\underline{d}) \tag{4}$$

then

$$V_f(\underline{c},\underline{d}) = V^*(\underline{c},\underline{d}) \text{ for all } (\underline{c},\underline{d}) \in E_m \times E_n \ .$$

In order to be able to use this lemma, i.e. verify (4), to prove that f is optimal, we first have to determine V_f.

3. Finding V_f

To determine V_f explicitly we assume (without loss of generality) that after combining two tanks both tanks contain half of what each of them had initially. Note that for strategy f execution of an action always terminates according to (2,i). In the sequel it is studied what happens with the water that was in A_i and B_j initially. To be precise, which part of it ends in the tanks $B_1, B_2, \cdots, B_n$.

3.1. What happens with the initial contents of A_i?

Consider as initial state $(\underline{c}, \underline{d})$ and define $p(i; \underline{c}, \underline{d})$ by

$$p(i; \underline{c}, \underline{d}) := \# \{ j \mid d_j < c_i \} \ . \tag{5}$$

In the remainder of this section we write $p(i)$, suppressing the dependence on $(\underline{c}, \underline{d})$.

Then, when using f, the contents of A_i are subsequently mixed with $B_{n-p(i)+1}, B_{n-p(i)+2}, \cdots, B_n$. Next half of the A_i water which now is in $B_{n-p(i)+1}$ is transfered to A_{i-1} followed by mixing of A_{i-1} with $B_{n-p(i)+2}, \cdots, B_n$, etc.

Lemma 2.

Let $f(r,s)$ denote the fraction of the initial A_i water that is in $B_{n-p(i)+s}$ after combining subsequently $A_i, A_{i-1}, \cdots, A_{i-r+1}$ with $B_{n-p(i)+1}, \cdots, B_n$.

Then

$$f(r,s) = 2^{-(r+s-1)} \begin{pmatrix} r+s-2 \\ r-1 \end{pmatrix} . \tag{6}$$

Proof.

We prove (6) by induction to r and s.

First $r = 1$.

It is clear that after combining A_i with $B_{n-p(i)+1}, \cdots, B_n$ a fraction $f(1,s) = 2^{-s}$ will be found in $B_{n-p(i)+s}$. (First half of the A_i water goes to $B_{n-p(i)+1}$, next half of what is left, so 1/4, goes to $B_{n-p(i)+2}$, etc.)

Next $r \geq 2, s = 1$.

Just before combining A_{i-r+1} with $B_{n-p(i)+1}$ tank A_{i-r+1} still contains no A_i water, and $B_{n-p(i)+1}$ contains a fraction $f(r-1,1)$. So after combining A_{i-r+1} and $B_{n-p(i)+1}$ tank $B_{n-p(i)+1}$ contains the fraction

$$f(r,1) = \tfrac{1}{2} f(r-1,1) \quad . \tag{7}$$

Finally $r \geq 2, s \geq 2$.

Immediately before A_{i-r+1} and $B_{n-p(i)+s}$ are combined, tanks A_{i-r+1} and $B_{n-p(i)+s-1}$ contain the same fraction of what was in A_i initially, so $f(r,s-1)$.

After mixing, $B_{n-p(i)+s}$ contains half of what it had after mixing with A_{i-r+2} plus half of what was in A_{i-r+1} just before. So

$$f(r,s) = \tfrac{1}{2} f(r,s-1) + \tfrac{1}{2} f(r-1,s) \quad . \tag{8}$$

Using $f(1,s) = 2^{-s}$, formulas (7) and (8) and

$$\begin{bmatrix} p+q \\ p \end{bmatrix} = \begin{bmatrix} p+q-1 \\ p \end{bmatrix} + \begin{bmatrix} p+q-1 \\ p-1 \end{bmatrix}$$

one obtains (6). □

3.2. What happens with the initial contents of B_j?

Similary we can follow the B_j water. Define

$$q(j;\underline{c},\underline{d}) := \# \{ i \mid c_i > d_j \} \quad .$$

In this section we write $q(j)$.

So f first combines B_j with $A_{q(j)}$, after which B_j and $A_{q(j)}$ both contain half of the B_j water. Next $A_{q(j)}$ mixes with $B_{j+1}, \cdots, B_n$ after which a fraction 2^{-s} of the initial content of B_j is found in B_{j+s-1}. Subsequently $A_{q(j)-1}, \cdots, A_1$ mix with $B_j, \cdots, B_n$.

The situation is the same as in section 3.1 with $A_i, \cdots, A_1$ replaced by $A_{q(j)}, \cdots, A_1$ and $B_j, \cdots, B_n$ instead of $B_{n-p(i)+1}, \cdots, B_n$. So we have

Lemma 3.

After combining $B_j, \cdots, B_n$ with subsequently $A_{q(j)}, \cdots, A_1$ a fraction $f(q(j),s)$ of the initial contents of B_j ends up in B_{j+s-1}, $s = 1, \cdots, n-j+1$ with $f(q(j),s)$ defined as in (6).

3.3. Conclusion.

Combining Lemmas 2 and 3 yields

Theorem 1.

$$V_f(\underline{c},\underline{d}) = \sum_{i=1}^{m} c_i \sum_{s=1}^{p(i)} f(i,s)$$

$$- \sum_{j=1}^{n} d_j (1 - \sum_{s=1}^{n-j+1} f(q(j),s)) \ . \qquad (9)$$

Proof.

The first term is just the total amount of the initial contents of $A_1, \cdots, A_n$ that at the end of the process is found in $B_1, \cdots, B_n$, the second term is the amount that went from the B tanks to the A tanks. The net result is $V_f(\underline{c},\underline{d})$. ▯

4. Proof of condition (4) of lemma 1

Consider feasible action (k,l,x) in state $(\underline{c},\underline{d})$. Assume we are in the simplest case that the action terminates due to (3,iv) and not (simultaneously) by (3,i), (3,ii) or (3,iii).

Then $p(i;\underline{c},\underline{d}) = p(i;T(\underline{c},\underline{d};k,l,x))$ and $q(j;\underline{c},\underline{d}) = q(j;T(\underline{c},\underline{d};k,l,x))$, i.e., the order including $<$ and $\leq$ signs remains unchanged.

So $r(\underline{c},\underline{d};k,l,x) = x$ and we can write $p(i)$ and $q(j)$ both for state $(\underline{c},\underline{d})$ and state $T(\underline{c},\underline{d};k,l,x) = (\underline{c} - x\underline{e}_k, \underline{d} + x\underline{e}_k)$.

From this and (9)

$$
\begin{aligned}
r(\underline{c},\underline{d};k,l,x) &+ V_f(T(\underline{c},\underline{d};k,l,x)) \\
&= x + V_f(\underline{c} - x\underline{e}_k, \underline{d} + x\underline{e}_l) \qquad (10)\\
&= x + \sum_{i=1}^{m} c_i \sum_{s=1}^{p(i)} f(i,s) - x \sum_{s=1}^{p(k)} f(k,s) \\
&\quad - \sum_{j=1}^{n} d_j (1 - \sum_{s=1}^{n-j+1} f(q(j),s)) - \delta(1 - \sum_{s=1}^{n-l+1} f(q(j),s)) \\
&= V_f(\underline{c},\underline{d}) - x[\sum_{s=1}^{p(k)} f(k,s) - \sum_{s=1}^{n-l+1} f(q(l),s)] \ .
\end{aligned}
$$

Now $c_k > d_l$ (cf. (1,i)) implies $p(k) \geq n = l + 1$ and $q(l) \geq k$.

So we see from (10) that to prove (4) it suffices to show that for all $r \geq k$ and all t

$$\sum_{s=1}^{t} [f(k,s) - f(r,s)] \geq 0 \ .$$

Or even simpler

$$y(r,t) := \sum_{s=1}^{t} [f(r,s) - f(r+1,s)] \geq 0 \text{ for all } r,t \geq 1 \ . \qquad (11)$$

One may verify that

$$y(1,t) = \sum_{s=1}^{t} [f(1,s) - f(2,s)] = \sum_{s=0}^{t} [2^{-s} - s2^{-s-1}) = t2^{-t-1} > 0 \qquad (12)$$

$$y(r,1) = f(r,1) - f(r+1,1) = 2^{-r} - 2^{-r-1} > 0 \qquad (13)$$

and for $r,t \geq 2$, using (8) and (11),

$$y(r,t) = \tfrac{1}{2}\, y(r,t-1) + \tfrac{1}{2}\, y(r-1,t) \ . \qquad (14)$$

Clearly (12)-(14) yields $y(r,t) \geq 0$ for all $r,t \geq 1$.

Hence in this case, i.e. action termination due to (3,iv), condition (4) holds.

The other 3 cases, (3,i)-(3,iii), seem to be far more complicated, since $p(i)$ and $q(j)$ will be different for $(\underline{c},\underline{d})$ and $T(\underline{c},\underline{d};k,l,x)$. However, if starting in $T(\underline{c},\underline{d};k,l,x)$ we combine A and B tanks in exactly the same order as strategy f prescribes when starting in state $(\underline{c},\underline{d})$, the final result in liters per tank will be the same as when we start in $T(\underline{c},\underline{d};k,l,x)$ and follow strategy f. The only difference is that a number of times an A and B tank are combined although the levels in the two tanks are equal.

From this it can be shown that (10) also holds in the other three cases with $p(i) = p(i;\underline{c},\underline{d}), q(j) = q(j;\underline{c},\underline{d})$ and x replaced by $r(\underline{c},\underline{d};k,l,x)$.

The rest of the argument will be the same, so we can conclude that (4) holds for all feasible (k,l,x) in $(\underline{c},\underline{d})$ and for all $(\underline{c},\underline{d})$.

Hence

Theorem 2.

$$V_f(\underline{c},\underline{d}) = V^*(\underline{c},\underline{d}) \text{ for all } (\underline{c},\underline{d}) \in E_m \times E_n \quad .$$

5. Alternative optimal strategies

There is a whole class of optimal strategies which apart from the order in which tanks are combined are the same as f.

To characterize this set of optimal strategies let us introduce the concept of neighbours.

In state $(\underline{c},\underline{d})$ tanks A_k and B_l are called neighbours if $c_k > d_l, c_{k+1} \leq d_l$ and $d_{l-1} \geq c_k$.

Any strategy which in each state combines neighbours, i.e. in each state $(\underline{c},\underline{d})$ prescribes an action (k,l,x) with (k,l) neighbours in $(\underline{c},\underline{d})$ and x so large that the action always continues until one of the cases (2,i)-(2,iii) occurs, is optimal.

It is easily seen that this is true, since if A_k and B_l are neighbours in $(\underline{c},\underline{d})$ also strategy f will combine A_k and B_l, though may be much later, and at that moment A_k still contains c_k and B_l still d_l.

Practically this means that more than one pair can be combined at the same time. This is useful since the flow from one tank to another may take some time, recall that in the energy problem we are dealing with a diffusion process.

Conclusion

We have determined the optimal strategy for a problem that resulted from studying energy storage models. The strategy has a simple form and its value is determined explicitly.

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