About the set of obtainable reference trajectories for linear discrete time systems

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by

J.C. Engwerda

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ABSTRACT

This paper gives insight in the question which trajectories can be tracked exactly, respectively approximately in case the considered system has a linear structure.

I. Introduction

In control literature much attention is paid to the question of designing controllers which show a prescribed behaviour (see e.g. Wonham [5], Schumacher [4] or Kwakernaak [2]). However, before this design question is posed, the question arises whether it is possible to achieve this desired behaviour.

In this paper we shall distinguish between two types of desired behaviour, namely tracking a desired target path exactly during a certain time interval, and tracking a desired target path in the limit. The trajectories that can be tracked in the limit are called admissible, while the trajectories that can be tracked exactly will be called strongly admissible.

The paper is organized as follows.

In section II some definitions and results are stated which will be used in the rest of the paper. Section III answers then the question whether or not a given reference trajectory will be strongly admissible, while in section IV the admissible trajectories are characterized. The paper ends with a section containing some final remarks and conclusions.

II Definitions and tools

In this section we give a precise definition of the concepts introduced in the first section. Furthermore some results, which will be used in the rest of the paper, are stated.

Notation: y_i^* will denote in the sequel a reference value for variable y at time i. A trajectory, $y_k^*, y_{k+1}^*, \ldots, y_n^*$, will be abbreviated by $y^*[k,n]$, while the matrix product $A_{k+n}A_{k+n-1}\ldots A_k$ will be abbreviated by $A(k+n,k)$. An infinite trajectory will be written short as $y[k,.]$.

Definition 1: A reference trajectory $y^*[k,n]$ is called strongly admissible for the initial condition x if the output of the system, $y[k,n]$, equals $y^*[k,n]$. Note that we do not require that n is finite in this definition.

Definition 2: A reference trajectory $y^*[k,.]$ is called admissible (in the large) if there exists a control sequence $u[k-1,.]$ such that $\|y_i - y_i^*\| \to 0$ for $i \to \infty$.

The next item is to introduce the system we will be considering in this paper. We will assume that the underlying system is described by the following linear discrete time recurrence equation:

$$\Sigma: \begin{array}{lcl} x_{k+1} & = & A_k x_k + B_k u_k + G_k d_k; \qquad x_0 = \bar{x}_0 \\ y_k & = & C_k x_k \end{array}$$

Here x_k is the state of the system, u_k the applied control, d_k an exogenous noise variable, and y_k the output at time k.

Now that we have defined the concepts, introduced in the first section, and the system we want to investigate, we shall proceed by giving two lemma's which will be used in the forthcoming sections.

<u>Lemma 1:</u>

Let $\|A_k\| \leq M$ for all k, and let $\{e_k\}$ satisfy

$$e_{k+1} = A_k e_k + v_k \quad .$$

Then $e_k \to 0$ implies $v \to 0$. □

<u>Lemma 2:</u>

Let B be an $n \times m$ matrix.

Then the equation $B\ u = y$ is solvable if and only if rank $[B \mid y] =$ rank $[B]$.

Moreover the solution will be uniquely determined if and only if rank $[B] = m$.

The solution is then given by $u = (B^T B)^{-1} B^T y$.

In case matrix B is not full column rank there always exists a transformation S in the input space U such that BS equals $[B' \mid 0]$, where now matrix B' is full column rank. So it is clear that in this case there exist infinite many solutions to the problem. □

III. The strongly admissible reference trajectories

In this section we shall give a characterization of the strongly admissible reference trajectories for the linear discrete time system Σ. We shall start this section however with a discussion of some related problems.

A first remark in this context is that, if the input should describe some desired behaviour $u^*[k,n]$, the strongly admissible reference trajectory for the output, $y^*[k+1,n+1]$, is then of course fixed by this desired control sequence. We shall state this observation in a proposition.

<u>Proposition 1:</u>

If the to be applied control in Σ is described by $u^*[k,n]$, then there exists only one strongly admissible reference trajectory, which is described by:

$$x_{i+1} = A_i x_i + B_i u_i^* + G_i d_i \, ; \; x_k = \overline{x}_k$$

$$y_i = C_i x_i \, , \; i = k, \ldots, n$$

□

The second related problem which we want to glance at is the problem of output controllability. In this problem one is interested in the answer to the question whether it is possible to achieve any output in the future by an appropriate choice of the input variables starting from some initial state x_0. The solution is summarized in proposition 2.

<u>Proposition 2:</u>

For any initial state $x_0 \in I\!R^n$ and for any desired output $y^* \in I\!R^r$ there exists a time t and a control sequence $u[0,t-1]$ such that $y_t = y^*$ if and only if rank $[CB \mid \ldots\ldots \mid CA^{n-1}B] = r$. □

The proof of this proposition is straightforward.

After these introductionary problems, we arrive now at the main part of this section, namely the characterization of the strongly admissible reference trajectories.

Of course these trajectories are totally characterized by Σ. To stress this fact, we shall formulate it in a proposition. Since for practical reasons it may be more convenient to have an input/output - rather than a state space description for the strongly admissible reference trajectories, we shall give this equivalent form of the trajectory too.

Proposition 3:

A reference trajectory $y^*[k+1,n+1]$ is strongly admissible if and only if it is for some $u^*[k,n]$ generated as follows:

$$y_i^* = C_i x_i^*$$
$$x_{i+1}^* = A_i x_i^* + B_i u_i^* + G_i d_i^*;\ x_k^* = \overline{x}_k \quad .$$

Proof:

The sufficiency of the condition is trivial. That the condition is also necessary is seen by the following reasoning.

We know that

$$x_{i+1} = A_i x_i + B_i u_i + G_i d_i;\ x_k = \overline{x}_k$$
$$y_i = C_i x_i$$

So the following equations hold too for random y_i^* :

$$x_{i+1} = A_i x_i + B_i u_i + G_i d_i;\ x_k = \overline{x}_k$$
$$y_i - y_i^* = C_i x_i - y_i^*$$

Now consider time $k+1$. Since y_{k+1}^* is strongly admissible, we have that $y_{k+1} - y_{k+1}^*$ has to be zero. So $y_{k+1}^* = C_{k+1}\overline{x}_{k+1}$ for some $\overline{x}_{k+1}$ generated by the system.

Since $\overline{x}_{k+1} = A_k x_k + B_k u_k + G_k d_k$, we can conclude now that there exists a u_k^* such that $x_{k+1} = \overline{x}_{k+1}$. By induction it is then easily verified that the relation as stated above holds. □

Proposition 4:

Define the following matrices:

$$W_k = \begin{bmatrix} C_{k+1}A(k,k) \\ \vdots \\ \vdots \\ \vdots \\ C_{k+n}A(k+N-1,k) \end{bmatrix} \text{, and}$$

$$X_{k,k+n}(E_i) = [C_{k+N+1}A(k+N,k)E_k \mid \ldots\ldots \mid C_{k+N+1}E_{k+N}] -$$
$$- C_{k+N+1}A(k+N,k)\,(W_k^T W_k)^{-1}\,W_k^T \begin{bmatrix} C_{k+1}B_k & 0\ldots0 & 0 \\ \vdots & & \vdots \\ C_{k+N}A(k+N-1,k+1)B_k & \ldots C_{k+N}B_{k+N-1} & 0 \end{bmatrix}$$

Assume that there exists an integer N such that for any k matrix W_k is full row rank.

Then the following input/output relation holds for Σ :

$$y_{k+N+1} = C_{k+N+1}A(k+N,k)\,(W_k^T W_k)^{-1}\,W_k^T \begin{bmatrix} y_{k+1} \\ \vdots \\ y_{k+N} \end{bmatrix} + X_{k,k+N}(B_i) \begin{bmatrix} u_k \\ \vdots \\ u_{k+N} \end{bmatrix} + X_{k,k+N}\,(G_i) \begin{bmatrix} d_k \\ \vdots \\ d_{k+N} \end{bmatrix}$$

Proof:

By induction it is not difficult to prove that x_{i+1} equals

$$A(i,k)x_k + \sum_{j=k+1}^{i} [A(i,j)\{B_{j-1}u_{j-1} + G_{j-1}d_{j-1}\}] + B_i u_i + G_i d_i$$

So

$$y_{i+1} - C_{i+1}A(i,k)x_k - C_{i+1}[A(i,k+1)G_k \mid \cdots \mid G_i] \begin{bmatrix} d_k \\ \vdots \\ d_i \end{bmatrix} =$$

$$= C_{i+1}[A(i,k+1)B_k \mid \cdots \mid B_i] \begin{bmatrix} u_k \\ \vdots \\ u_i \end{bmatrix}$$

From this equation we deduce that:

$$\begin{bmatrix} y_{k+1} \\ \vdots \\ y_{k+N} \end{bmatrix} - \begin{bmatrix} C_{k+1}B_k & 0 \cdots 0 \\ \vdots & . \\ C_{k+N}A(k+N-1,k+1)B_k & \cdots C_{k+N}B_{k+N-1} \end{bmatrix} \begin{bmatrix} u_k \\ \vdots \\ u_{k+N-1} \end{bmatrix} -$$

$$- \begin{bmatrix} C_{k+1}G_k & 0 \cdots 0 & d_k \\ \vdots & . & \vdots \\ C_{k+N}A(k+N-1,k+1)G_k & C_{k+N}G_{k+N-1} & d_{k+N-1} \end{bmatrix} = \begin{bmatrix} C_{k+1}A(k,k) \\ \vdots \\ C_{k+N}A(k+N-1,k) \end{bmatrix} x_k \; .$$

Since by assumption matrix W_k is left invertible, we can write the state of the system, x_k, now as a function of $y[k+1,k+N]$, $u[k,k+N-1]$ and $d[k,k+N-1]$.

Substitution of this expression into the equation for y_{k+N+1} yields then, after reordening some terms and using the definition of $X_{k,k+N}(E_i)$, the input/output relation as stated in the proposition. □

In the foregoing we gave necessary and sufficient conditions how a reference trajectory has to be generated in order to be strongly admissible. In practical situations however these conditions are not very handsome. Therefore it would be nice if we had a criterium, from which we can conclude strong admissibility of a trajectory immediately.

This is the subject of the next theorem.

Theorem 1:

A reference trajectory $y^*[k+1,k+N+1]$ is strongly admissible if and only if

$$\text{rank} \begin{bmatrix} C_{k+1}B_k & 0 \cdots 0 & \mid & z_k \\ \vdots & . & \mid & \vdots \\ C_{k+N+1}A(k+N,k)B_k & \cdots C_{k+N+1}B_{k+N} & \mid & z_{k+N+1} \end{bmatrix} = \text{rank} \begin{bmatrix} C_{k+1}B_k & 0 \cdots 0 \\ \vdots & . \\ C_{k+N+1}A(k+N,k)B_k & \cdots C_{k+N+1}B_{k+N} \end{bmatrix}$$

where $z_i = y_{i+1} - C_{i+1}A(i,k)x_k - C_{i+1}\sum_{j=k+1}^{i} A(i,j)\,G_{j-1}d_{j-1} - C_{i+1}G_i d_i$.

Proof:

From proposition 4 we have that $z_i = C_{i+1}\{ \sum_{j=k+1}^{i} A(i,j)B_{j-1}u_{j-1} + B_i u_i \}$.

From this identity it follows that a reference trajectory $y^*[k+1,k+N+1]$ is strongly admissible if and only if the following set of equations possesses a solution:

$$z_k = C_{k+1}B_k u_k$$

$$\vdots$$

$$z_{k+N+1} = C_{k+N+1}\{ \sum_{j=k+1}^{k+N} A(k+N,j)B_{j-1}u_{j-1} + B_{k+N}u_{k+N} \}$$

This is the case if and only if the rank equality as stated above in the theorem holds (see lemma 2). □

Now the question can be posed under which conditions any reference trajectory will be strongly admissible. Simple reasoning immediately gives rise to the supposition that this will be the case if it is at any point in time possible to steer the output completely in one timestep. This is the subject of the following corollary.

Corollary 1:

Any reference trajectory is strongly admissible if and only if matrix $C_{i+1}B_i$ is full rank at any point in time.

Proof:

From the proof of theorem 1 it follows that any reference trajectory will be admissible if and only if matrix

$$H \triangleq \begin{bmatrix} C_{k+1}B_k & 0 \cdots 0 \\ \vdots & . \\ C_{k+N+1}A(k+N,k+1)B_k & \cdots C_{k+N+1}B_{k+N} \end{bmatrix}$$ is full row rank.

But this implies that at least matrix $C_{k+1}B_k$ has to be full row rank.

Since the rank of a matrix does not change if we subtract or add rows it is obvious that the rank of H is equal to the rank of matrix

$$H' \triangleq \begin{bmatrix} C_{k+1}B_k & & 0......0 \\ 0 & C_{k+2}B_{k+1} & \\ \vdots & \vdots & \\ 0 & C_{k+N+1}A(k+N,k+1)B_{k+1} & \cdots C_{K+N+1}B_{k+N} \end{bmatrix}$$

We can now proceed in the same way, and by induction it is seen that matrix H will be full row rank if and only if all matrices $C_{i+1}B_i \quad i = k, \cdots, k+N$ are full row rank.

That this condition is also sufficient is trivial. □

We conclude this section by noting that the condition that matrix $C_{k+1}B_k$ has to be full row rank is satisfied only if the number of inputs is equal to or exceeds the number of outputs. This is due to the fact that always the rank inequality rank $(CB) \leq$ min (rank C, rank B) holds. In economic literature this is known as the Tinbergen condition.

IV. The admissible (in the large) reference trajectories

In the previous section we derived a criterion to check whether a certain reference trajectory could be tracked exactly or not.

Moreover an exact characterisation was given how a reference trajectory has to be generated in order to be strongly admissible.

We shall now treat the problem of tracking a reference trajectory "in the end". To tackle this problem, we shall assume that the input is chosen as a mixture of static/dynamic, state/output feedback.

That is:

$$u_k = E_k w_k + F_k x_k + H_k z_k + D_k y_k + g_k$$

where

$$w_{k+1} = M_k w_k + N_k x_k$$

$$z_{k+1} = P_k z_k + Q_k y_k$$

Then, for random $u_k^*, u_{k-1}^*, w_{k+1}^*, w_k^*, z_{k+1}^*, z_k^*, x_{k+1}^*, x_k^*, y_k^*, y_{k-1}^*$, the following closed loop system results:

$$\begin{bmatrix} I & 0 & 0 & -B_k & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & -Q_k \\ 0 & 0 & 0 & I & -D_k \\ 0 & 0 & 0 & 0 & I \end{bmatrix} e_{k+1} = \begin{bmatrix} A_k & 0 & 0 & 0 \\ N_k & M_k & 0 & 0 \\ 0 & 0 & P_k & 0 \\ P_k & G_k & M_k & 0 \\ C_k & 0 & 0 & 0 \end{bmatrix} e_k + \begin{bmatrix} A_k x_k^* + B_k u_k^* + G_k d_k^* - x_{k+1}^* \\ M_k w_k^* + N_k x_k^* - w_{k+1}^* \\ P_k z_k^* + Q_k y_k^* - z_{k+1}^* \\ F_k x_k^* + E_k w_k^* + H_k z_k^* + D_k d_k^* + g_k \\ C_k x_k^* - y_k^* \end{bmatrix} .$$

where $e_{k+1}^T = [(x_{k+1} - x_{k+1}^*)^T, (w_{k+1} - w_{k+1}^*)^T, (z_{k+1} - z_{k+1}^*)^T, (u_k - u_k^*)^T, (y_k - y_k^*)^T]$.

This error equation can be rewritten as $e_{k+1} = \overline{A}_k e_k + v_k$. From this equation it is clear that the error e_k converges to zero if and only if v_k is such that it stabilizes this system. Though this criterium is rather vague, it characterizes exact what properties the reference trajectory should satisfy in order to be admissible. To give some more insight in the properties of an admissible trajectory the next theorem, which immediately results from lemma 2, is stated.

Theorem 2

In order to be admissible a reference trajectory has to be generated as follows:

$$x_{k+1}^* = A_k x_k^* + B_k u_k^* + G_k d_k + v_{k,1}$$

$$y_k^* = C_k x_k^* + v_{k,2}$$

with $v_{k,i} \to 0$ when k tends to infinity. □

Note that this condition is also sufficient if the input is chosen such that the closed loop system is stabilized.

Using proposition 4, theorem 2 can be reformulated in the following way:

a necessary condition for a reference trajectory to be admissible is that it is generated in the limit by the same input/output recurrence relation.

We will proceed now with giving an example of how theorem 2 might be applicated in practice.

In Engwerda [ ], the infinite time quadratic tracking problem was solved under some conditions. As a special case the following problem was treated:

$$\min_{u(\cdot)} \lim_{N\to\infty} \sum_{k=0}^{N-1} \{(y_k - y_k^*)^T Q (y_k - y_k^*)^T + (u_k - u_k^*)^T R (u_k - u_k^*)\} + (y_N - y_N^*)^T Q (y_N - y_N^*) ,$$

where Q and R are positive definite symmetric matrices and y_{k+1} is given by the system equation $y_{k+1} = Ay_k + Bu_k + Gd_k$.

The optimal solution to this problem turned out to be

$$u_i = -(R + B^T KB)^{-1} B^T (KAy_i + KGd_i - h_{i+1} - B(B^T B)^{-1} Ru_i^*) ,$$

where K is the positive definite solution of the Algebraic Riccati Equation $K = A^T \{K - KB(R + B^T KB)^{-1} B^T K\} A + Q$, and h_i is given by the recurrence equation

$$h_1 = \sum_{k=1}^{\infty} \{(A-BH)^T\}^{-1} \{Qy_k^* - (RH)^T u_k^* - (A-BH)^T KHd_k\}$$

$$h_{i+1} = \{(A-BH)^T\}^{-1} \{h_i - Qy_i^* + (RH)^T u_i^*\} + KGd_i$$

Here H denotes the matrix $(R+B^T KB)^{-1} B^T KA$.

In this paper it was proved that this control stabilizes the closed loop system. So, application of theorem 2 yields now that the error $[(y_k - y_k^*)^T, (u_k - u_k^*)]$ converges to zero if and only if the following vector converges to zero when k tends to infinity:

$$\begin{bmatrix} v_{k1} \\ v_{k2} \end{bmatrix} \triangleq \begin{bmatrix} Ay_k^* + Bu_k^* + Gd_k - y_{k+1}^* \\ (R + B^T KB)^{-1} B^T [h_{k+1} - KAy_k^* - KGd_k + B(B^T B)^{-1} Ru_k^*] - u_k^* \end{bmatrix}$$

Now substitution of v_{k1} into v_{k2} yields:

$$v_{k2} = -(R + B^T KB)^{-1} B^T (Ky_{k+1}^* - h_{k+1} - v_{k1}) .$$

So that we can conclude that v_{k2} converges to zero if and only if $B^T (Ky_{k+1}^* - h_{k+1})$ converges to zero.

Summarizing, we have the following result:

<u>Result:</u>

A reference trajectory is admissible in the infinite time quadratic tracking problem if and only if the following two conditions hold for the reference trajectory:

i) $y_{k+1}^* - Ay_k^* - Bu_k^* - Gd_k \to 0$ when $k \to \infty$.

ii) $B^T (Ky_{k+1}^* - h_{k+1}) \to 0$ when $k \to \infty$.

□

We shall conclude this section by giving a geometric property of the set of admissible reference trajectories.

Therefore, for the moment, assume that the system $\sum$ has no exogenous noise component, i.e. $G = 0$.

As was shown at the beginning of this section, a reference trajectory is admissible if and only if e_k is stabilized by $v[k,.]$ in the linear system $e_{i+1} = A_i e_i + v_i$. Using this equivalence , it is now not difficult to show that the set of admissible reference trajectories form a linear subspace.

Furthermore we observe, by considering the following example, that this subspace is not closed in the topology of pointwise convergence:

$$e_{k+1,n} = 2\, e_{k,n} + v_{k,n}, \text{ with } v_{k,n} = -(\, n/(n-1)\,)e_{k,n}$$
$$e_{0,n} = 1 \ .$$

For this example it is easily verified that for any n the pair $(e_{o,n}, v_n[0,.])$ is admissible, while the pair $(e_{0,\infty}, v_{\infty}[0,.])$ does not possess this property.

So the set of admissible reference trajectories is, in case $G = 0$, a linear subspace that is not closed. Now $y_{k+1} - y^*_{k+1}$ equals, for any $G_k d_k, y_{k+1} + G_k d_k - G_k d_k - y^*_{k+1}$. So, if $y^*[k,.]$ is an admissible reference trajectory for the system without exogenous noise, then $(y^* + Gd)[k,.]$ is an admissible trajectory for the system when an exogenous component, $G_k dk_k$, is present. On the other hand it is by the same reasoning seen, that if a trajectory $y^*[k,.]$ is admissible for the system with noise, then $(y^* - Gd)[k,.]$ is admissible for the system without noise. Therefore we can conclude now that the set of admissible trajectories consists of the linear subspace of admissible reference trajectories for the system without noise, shifted by the exogenous noise trajectory. We will formulate this result in a theorem.

Theorem 3

The set of admissible reference trajectories is a linear variety of the set of all functions from $\mathbb{N} \to \mathbb{R}^n$, which is not closed in general in the topology of pointwise convergence.

V. Conclusions

In this paper a rank condition is given for checking the strong admissibility of a reference path. It was shown that a reference trajectory must evolve similar to the system in order to be strong admissible. This evaluation condition proved to be extendable to admissibility (in the large) of a reference trajectory. In this case, however, we had to assume that the input obeyed some feedback law.

Moreover the condition proved to be only necessary for this trajectory property. When the closed loop system is stabilized by the feedback law, the condition proved to be also sufficient.

A last result obtained is that the set of admissible reference trajectories is a linear subspace, which is in general not closed in the topology of pointwise convergence.

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