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Abstract

In H_2 and H_∞ optimal control (semi-) stabilizing solutions of algebraic Riccati equations play an essential role. It is well-known that these solutions might have discontinuities as a function of the system parameters. The paper shows that these discontinuities are directly linked to non-left-invertibility and, in contrast to what one might think, unrelated to zeros on the imaginary axis.

Keywords: algebraic Riccati equations, quadratic matrix inequalities.

1 Introduction

In most H_2 and H_∞ control problems solutions of the algebraic Riccati equation play a crucial role. Note that in general for continuous time systems we have to use quadratic matrix inequalities instead of Riccati equations. However, these have a 1-1 relation to Riccati equations of a lower dimension (see [5]). In particular we are interested in the stabilizing solution of these Riccati equations and quadratic matrix inequalities. However, if the system has zeros on the imaginary axis (continuous time) or on the unit circle (discrete time), we have to study semi-stabilizing solutions. These are solutions of the Riccati equation/quadratic matrix inequality associated to eigenvalues in the closed left-half plane (continuous time) or in the closed unit circle (discrete time). The standard way to obtain semi-stabilizing solutions is a cheap control argument where we perturb the system parameters to obtain a system without problems induced by for instance the zeros on the boundary of the stability domain. A natural question is then whether the semi-stabilizing solutions depend continuously on the system parameters. There are simple examples where the solution does not depend continuously on the system parameters (see e.g. [3]). On the other hand, [8] identifies a class of perturbations which guarantee a continuous behaviour. We would like to study this question in more detail. We will clearly identify what kind of perturbations can yield discontinuous behaviour and in the process show that for a very large class of systems discontinuities never occur. We will consider both continuous and discrete time systems.
Notation in this paper is mostly standard. However we would like to note that by $M^\dagger$ we denote the Moore-Penrose inverse of M.

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2 Discrete time systems

2.1 Problem formulation

Consider the following discrete time Riccati equation:

$$P = A^{\mathrm{T}}PA + C^{\mathrm{T}}C - \begin{pmatrix} B^{\mathrm{T}}PA + D_1^{\mathrm{T}}C \\ E^{\mathrm{T}}PA + D_2^{\mathrm{T}}C \end{pmatrix}^{\mathrm{T}} G(P)^{\dagger} \begin{pmatrix} B^{\mathrm{T}}PA + D_1^{\mathrm{T}}C \\ E^{\mathrm{T}}PA + D_2^{\mathrm{T}}C \end{pmatrix}, \tag{2.1}$$

where

$$G(P) := \begin{pmatrix} D_1^{\mathrm{T}}D_1 & D_1^{\mathrm{T}}D_2 \\ D_2^{\mathrm{T}}D_1 & D_2^{\mathrm{T}}D_2 - \gamma^2 I \end{pmatrix} + \begin{pmatrix} B^{\mathrm{T}} \\ E^{\mathrm{T}} \end{pmatrix} P \begin{pmatrix} B & E \end{pmatrix}, \tag{2.2}$$

subject to

$$D_2^{\mathrm{T}}D_2 + E^{\mathrm{T}}PE - (D_2^{\mathrm{T}}D_1 + E^{\mathrm{T}}PB)(D_1^{\mathrm{T}}D_1 + B^{\mathrm{T}}PB)^{\dagger}(D_1^{\mathrm{T}}D_2 + B^{\mathrm{T}}PE) < \gamma^2 I. \tag{2.3}$$

We are interested in real symmetric semi-stabilizing solutions of this algebraic Riccati equation. These are solutions of the algebraic Riccati equation where the zeros of the matrix pencil

$$\begin{pmatrix} zI - A & -B & -E \\ B^{\mathrm{T}}PA + D_1^{\mathrm{T}}C & D_1^{\mathrm{T}}D_1 + B^{\mathrm{T}}PB & D_1^{\mathrm{T}}D_2 + B^{\mathrm{T}}PE \\ E^{\mathrm{T}}PA + D_2^{\mathrm{T}}C & D_2^{\mathrm{T}}D_1 + E^{\mathrm{T}}PB & D_2^{\mathrm{T}}D_2 + E^{\mathrm{T}}PE - \gamma^2 I \end{pmatrix} \tag{2.4}$$

are inside or on the unit circle. If the zeros are strictly inside the unit circle we will call P a stabilizing solution of the Riccati equation. This Riccati equation is associated to the following system:

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + Bu(k) + Ew(k), \\ z(k) = Cx(k) + D_1u(k) + D_2w(k). \end{cases} \tag{2.5}$$

Basically there exists a stabilizing feedback $u = F_1x + F_2w$ such that the closed loop H_∞ norm is less than γ if and only if there exists a positive semi-definite semi-stabilizing solution of the above Riccati equation and some additional conditions (see [6]).
For $\gamma = \infty$ the general Riccati equation (2.1) reduces to the H_2 Riccati equation:

$$P = A^{\mathrm{T}}PA + C^{\mathrm{T}}C - (A^{\mathrm{T}}PB + C^{\mathrm{T}}D_1)(B^{\mathrm{T}}PB + D_1^{\mathrm{T}}D_1)^{\dagger}(B^{\mathrm{T}}PA + D_1^{\mathrm{T}}C). \tag{2.6}$$

Moreover, the extra condition (2.3) becomes void. Finally, the stability requirement is imposed on the following matrix pencil:

$$\begin{pmatrix} zI - A & -B \\ B^{\mathrm{T}}PA + D_1^{\mathrm{T}}C & B^{\mathrm{T}}PB + D_1^{\mathrm{T}}D_1 \end{pmatrix}. \tag{2.7}$$

The Riccati equation is associated to the system Σ which is parameterized by (A, B, E, C, D_1, D_2). We define the set $\mathcal{D}$ to be the class of systems Σ for which (A, B, C, D_1) is left-invertible and for which there exists matrices F_1, F_2 such that $A + BF_1$ is stable and

$$\|(C + DF_1)(zI - A - BF_1)^{-1}(E + BF_2) + (D_2 + D_1F_2)\|_\infty < \gamma \tag{2.8}$$

For the H_2 problem, where $\gamma = \infty$ the set $\mathcal{D}$ consists of systems Σ for which (A, B, C, D_1) is left-invertible and (A, B) is stabilizable. In that case it is known (see [7]) that there exists a unique real symmetric semi-stabilizing solution P of the Riccati equation. Moreover, this solution is positive semi-definite.
For the H_∞ control problem (i.e. finite γ) the semi-stabilizing solution need not be unique. However, for elements of the set $\mathcal{D}$ there exists a semi-stabilizing, positive semi-definite solution of the Riccati equation. In this section, we will study the smallest, positive semi-definite rank-minimizing solution P of the quadratic matrix inequality which always exists and is obviously unique.
In the next subsection we study the behaviour of P when we vary the system parameters over the set $\mathcal{D}$. We will show that P depends continuously on the system parameters both for finite γ and for $\gamma = \infty$. Since we allow for zeros on the unit circle, this continuity is far from obvious. We consider systems outside the set $\mathcal{D}$ in the subsection thereafter.
For elements of the set $\mathcal{D}$, the system (A, B, C, D_1) is left-invertible. This implies that the generalized inverses in (2.1), (2.3) and (2.6) become standard inverses. Moreover for semi-stabilizing and stabilizing solutions of the Riccati equation we can simply study the eigenvalues of the following matrix

$$A - \begin{pmatrix} B & E \end{pmatrix} G(P)^{-1} \begin{pmatrix} B^{\mathrm{T}} P A + D_1^{\mathrm{T}} C \\ E^{\mathrm{T}} P A + D_2^{\mathrm{T}} C \end{pmatrix}. \tag{2.9}$$

2.2 Continuity

We first show that the stabilizing solution of the Riccati equation depends continuously on the system parameters if we do not have zeros on the unit circle.

Lemma 2.1 *Let $\mathcal{D}_0$ be the open subset in $\mathcal{D}$ of systems Σ for which (A, B, C, D_1) has no zeros on the unit circle.*
For each element of $\mathcal{D}_0$, the Riccati equation (2.1) has a unique solution P for which the matrix (2.9) is asymptotically stable. The function f from $\mathcal{D}_0$ to $\mathbb{R}^{n\times n}$ which assigns to each system in $\mathcal{D}_0$, the associated stabilizing solution of the Riccati equation is continuous.

Proof : The existence and uniqueness of the stabilizing solution can for instance be found in [7].
The continuity follows in a straightforward manner. The solution P is associated to the stable subspace of a symplectic pencil (see e.g. [7]). Since this symplectic pencil has no eigenvalues on the unit circle, the stable and antistable eigenvalues are strictly separated and the existence of a continuous basis for the stable subspace and hence the continuous dependence of the stabilizing solution of the Ricati equation can be found in e.g. [4]. ∎

Our main objective is to show that the extension of this function f to the whole set $\mathcal{D}$ is also continuous. We will need some technical lemmas. First of all a lemma related to the classical cheap control argument.

Lemma 2.2 *Let Σ be an arbitrary element of $\mathcal{D}$ such that D_1 is invertible and (A, B, C, D_1) has no zeros outside the unit circle. For $\varepsilon \neq 0$ small enough the following Riccati equation has a stabilizing solution P_ε:*

$$P_\varepsilon = A^T P_\varepsilon A + C^T C + \varepsilon^2 I - \begin{pmatrix} B^T P_\varepsilon A + D_1^T C \\ E^T P_\varepsilon A + D_2^T C \end{pmatrix}^T G(P_\varepsilon)^{-1} \begin{pmatrix} B^T P_\varepsilon A + D_1^T C \\ E^T P_\varepsilon A + D_2^T C \end{pmatrix}. \tag{2.10}$$

where G is defined by (2.2). Moreover $P_\varepsilon \to 0$ as $\varepsilon \to 0$.

Proof : The Riccati equation (2.10) is the Riccati equation associated with the system $\sigma(\Sigma, \varepsilon)$ parameterized by $(A, B, E, C_\varepsilon, D_{1,\varepsilon}, D_{2,\varepsilon})$ where $C_\varepsilon, D_{1,\varepsilon}, D_{2,\varepsilon}$ are such that

$$\begin{pmatrix} C_\varepsilon^T C_\varepsilon & C_\varepsilon^T D_{1,\varepsilon} & C_\varepsilon^T D_{2,\varepsilon} \\ D_{1,\varepsilon}^T C_\varepsilon & D_{1,\varepsilon}^T D_{1,\varepsilon} & D_{1,\varepsilon}^T D_{2,\varepsilon} \\ D_{2,\varepsilon}^T C_\varepsilon & D_{2,\varepsilon}^T D_{1,\varepsilon} & D_{2,\varepsilon}^T D_{2,\varepsilon} \end{pmatrix} = \begin{pmatrix} C^T C + \varepsilon^2 I & C^T D_1 & C^T D_2 \\ D_1^T C & D_1^T D_1 & D_1^T D_2 \\ D_2^T C & D_2^T D_1 & D_2^T D_2 \end{pmatrix}. \tag{2.11}$$

Since $\mathcal{D}$ is an open set, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in [0, \varepsilon^*]$ we have that $\sigma(\Sigma, \varepsilon)$ is in $\mathcal{D}$.
Because $\sigma(\Sigma, \varepsilon)$ is in $\mathcal{D}$, we know that the Riccati equation has a semi-stabilizing solution. It is actually a stabilizing solution since for $\varepsilon \neq 0$ the system $\sigma(\Sigma, \varepsilon)$ is such that $(A, B, C_\varepsilon, D_{1,\varepsilon})$ has no invariant zeros in the unit circle. We have:

$$\delta := \sup_{\varepsilon \in [0, \varepsilon^*]} \inf_{F_1, F_2} \left\{ \|(C_\varepsilon + D_{1,\varepsilon} F_1)(zI - A - BF_1)^{-1}(E + BF_2) + (D_2 + D_1 F_2)\|_\infty \;\middle|\; A + BF_1 \text{ stable} \right\} < \gamma$$

From [5] we have for finite γ and $\varepsilon \in (0, \varepsilon^*]$:

$$L_\varepsilon \leqslant P_\varepsilon \leqslant \tfrac{\gamma^2}{\gamma^2 - \delta^2} L_\varepsilon$$

where L_ε is the stabilizing solution of the following Riccati equation:

$$L_\varepsilon = A^T L_\varepsilon A + C^T C + \varepsilon^2 I - (A^T L_\varepsilon B + C^T D)(B^T L_\varepsilon B + D^T D)^{-1}(B^T L_\varepsilon A + D^T C)$$

This basically implies that we can restrict attention to the H_2 Riccati equation (if we start with $\gamma = \infty$ then obviously $P_\varepsilon = L_\varepsilon$).
Using the relation of L_ε with a standard linear quadratic control problem we note that L_ε is decreasing in ε and bounded from below. Therefore, there exists a limit $\bar{L}$. Clearly $\bar{L}$ is a semi-stabilizing solution of the following Riccati equation:

$$\bar{L} = A^T \bar{L} A + C^T C + \varepsilon^2 I - (A^T \bar{L} B + C^T D)(B^T \bar{L} B + D^T D)^{-1}(B^T \bar{L} A + D^T C)$$

On the other hand 0 is also a semi-stabilizing solution of this Riccati equation. By [7], we know the semi-stabilizing solution is unique and hence $\bar{L} = 0$, which implies $L_\varepsilon \to 0$ as $\varepsilon \to 0$ and hence $P_\varepsilon \to 0$. ■

Lemma 2.3 *Let $\mathcal{D}_1$ be a compact subset of $\mathcal{D}$ consisting of systems Σ for which the direct feedthrough matrix D_1 is invertible and (A, B, C, D_1) has no zeros outside the unit circle. There exists $\varepsilon^* > 0$ such that for all $\varepsilon \in [0, \varepsilon^*]$ and for all $\Sigma \in \mathcal{D}_1$, we have $\sigma(\Sigma, \varepsilon)$ in $\mathcal{D}$. For each element in $\mathcal{D}_1$ and $\varepsilon \in [0, \varepsilon^*]$ there exists a smallest positive semi-definite stabilizing solution of the algebraic Riccati equation:*

$$P = A^T PA + C^T C + \varepsilon^2 I - \begin{pmatrix} B^T PA + D_1^T C \\ E^T PA + D_2^T C \end{pmatrix}^T G(P)^{-1} \begin{pmatrix} B^T PA + D_1^T C \\ E^T PA + D_2^T C \end{pmatrix}, \tag{2.12}$$

where G is defined by (2.2), such that all the zeros of the matrix pencil (2.9) are in the closed unit disc. P defines a map from $\mathcal{D}_1 \times [0, \varepsilon^*]$ to $\mathbb{R}^{n \times n}$ which is well-defined and continuous.

Proof : The existence of ε^* is a consequence of the fact that $\mathcal{D}_1$ is compact and that $\mathcal{D}$ is open. Choose $(\Sigma_0, \varepsilon_0)$ in $\mathcal{D}_1 \times [0, \varepsilon^*]$. If $\varepsilon_0 > 0$ continuity at $(\Sigma_0, \varepsilon_0)$ is a consequence of lemma 2.1.

Assume that $\varepsilon_0 = 0$. Then it is easy to check that the smallest positive semi-definite solution of the algebraic Riccati equation associated to $\sigma(\Sigma, 0)$ equals 0. By definition, we always have $P \geqslant 0$. Hence we find that P is lower semi-continuous at $(\Sigma_0, 0)$, i.e. for each $\delta > 0$ there exists an open neigbourhood $\mathcal{S}$ of $(\Sigma_0, \varepsilon_0)$ in $\mathcal{D}_1 \times [0, \varepsilon^*]$ such that:

$$P(\Sigma, \varepsilon) \geqslant P(\Sigma_0, 0) - \delta I \tag{2.13}$$

for all (Σ, ε) in $\mathcal{S}$. After all (2.13) is trivially satisfied for $\mathcal{S} = \mathcal{D}_1 \times [0, \varepsilon^*]$ since

$$P(\Sigma_0, 0) = 0. \tag{2.14}$$

Remains to show upper semi-continuity. Choose $\delta > 0$. We have to construct an open neighbourhood $\mathcal{S}$ of $(\Sigma_0, 0)$ in $\mathcal{D}_1 \times [0, \varepsilon^*]$ such that for all (Σ, ε) in $\mathcal{S}$:

$$P(\Sigma, \varepsilon) \leqslant P(\Sigma_0, 0) + \delta I \tag{2.15}$$

From lemma 2.2 we find that, for the fixed system Σ_0, we have $P(\Sigma_0, \varepsilon) \to 0$ as $\varepsilon \to 0$. Let $\varepsilon_0 > 0$ be such that $\bar{P} := P(\Sigma_0, \varepsilon_0) < \delta/2I$. Define F_1, F_2 by:

$$\begin{aligned} F_1 &:= -(B_0^{\mathrm{T}} \bar{P} B_0 + D_{1,0}^{\mathrm{T}} D_{1,0})^{-1} (B_0^{\mathrm{T}} \bar{P} A_0 + D_{1,0}^{\mathrm{T}} C_0) \\ F_2 &:= -(B_0^{\mathrm{T}} \bar{P} B_0 + D_{1,0}^{\mathrm{T}} D_{1,0})^{-1} (B_0^{\mathrm{T}} \bar{P} E_0 + D_{1,0}^{\mathrm{T}} D_{2,0}) \end{aligned}$$

where the system $\sigma(\Sigma_0, \varepsilon_0)$ is parameterized by $(A_0, B_0, E_0, C_0, D_{1,0}, D_{2,0})$. We find that $A_0 + B_0 F_1$ is stable and

$$\|(C_0 + D_{1,0} F_1)(zI - A_0 - B_0 F_1)^{-1}(E_0 + B_0 F_2) + (D_{2,0} + D_{1,0} F_2)\|_\infty < \gamma$$

Let $\sigma(\Sigma, \varepsilon)$ be parameterized by $(A, B, E, C, D_{1,\varepsilon}, D_{2,\varepsilon})$. We will use the following notation for the system parameters we obtain after applying the feedback $u = F_1 x + F_2 w$ to $\sigma(\Sigma, \varepsilon)$:

$$A_F := A + BF_1, \quad E_F := E + BF_2, \quad C_{F,\varepsilon} := C_\varepsilon + D_{1,\varepsilon} F_1, \quad D_{F,\varepsilon} := D_{2,\varepsilon} + D_{1,\varepsilon} F_2.$$

We will use the interpretation of P as an equilibrium of a difference game (see e.g. [2]). We note that that the ℓ_2 norm is defined by:

$$\|f\|_2^2 := \sum_{k=0}^{\infty} \|f(k)\|^2$$

We say that $f \in \ell_2$ if the above sum is finite. We have

$$\begin{aligned}
\xi^{\mathrm{T}} P(\Sigma,\varepsilon)\xi &= \sup_{w\in\ell_2} \inf_{u\in\ell_2} \left\{ \|z\|_2^2 - \gamma^2 \|w\|_2^2 \;\middle|\; x(k) \to 0 \text{ as } k \to \infty \right\} \\
&\leqslant \sup_{w\in\ell_2} \left\{ \|z\|_2^2 - \gamma^2 \|w\|_2^2 \;\middle|\; u = F_1 x + F_2 w \right\} \\
&= \xi^{\mathrm{T}} \tilde{P}(\Sigma,\varepsilon)\xi
\end{aligned}$$

subject to (2.5) with $x(0) = \xi$ where $\tilde{P}(\Sigma,\varepsilon)$ is the unique stabilizing solution $\tilde{P}$ of the following algebraic Riccati equation:

$$\tilde{P} = A_F^{\mathrm{T}} \tilde{P} A_F + C_{F,\varepsilon}^{\mathrm{T}} C_{F,\varepsilon} + A_F^{\mathrm{T}} \tilde{P} E_F \left[\gamma^2 I - D_{F,\varepsilon}^{\mathrm{T}} D_{F,\varepsilon} - E_F^{\mathrm{T}} \tilde{P} E_F \right]^{-1} E_F^{\mathrm{T}} \tilde{P} A_F,$$

We know:

$$\tilde{P}(\Sigma_0,\varepsilon_0) = P(\Sigma_0,\varepsilon_0) < \delta/2I$$

Moreover, $\tilde{P}(\Sigma,\varepsilon_0)$ depends continuously on the system parameters for fixed ε_0 on the basis of lemma 2.1. Therefore there exists an open neighbourhood $\mathcal{D}_2$ of Σ_0 in $\mathcal{D}_1$ such that

$$P(\Sigma,\varepsilon_0) \leqslant \tilde{P}(\Sigma,\varepsilon_0) < \delta I.$$

The latter implies that $P(\Sigma,\varepsilon) \leqslant P(\Sigma,\varepsilon_0) < \delta I$ for all (Σ,ε) in $\mathcal{D}_2 \times [0,\varepsilon_0]$. Combined with (2.14), this implies (2.15). ■

Theorem 2.4 *Consider the set $\mathcal{D}$ of systems Σ for which (A,B,C,D_1) is left-invertible and, if γ is finite, for which there exists F_1, F_2 such that $A+BF$ is stable and (2.8) is satisfied. The smallest, positive semi-definite semi-stabilizing solution of the algebraic Riccati equation is a continuous function from $\mathcal{D}$ to $\mathbb{R}^{n\times n}$.*

Proof : We denote by $P(\Sigma)$ the semi-stabilizing solution associated to the system Σ. Consider an arbitrary system Σ_0 in $\mathcal{D}$ and $\delta > 0$. We have to construct an open neighbourhood $\mathcal{S}$ in $\mathcal{D}$ of Σ_0 such that:

$$\|P(\Sigma) - P(\Sigma_0)\| < \delta$$

for all Σ in $\mathcal{S}$.
Choose F_1, F_2 such that $A_0 + B_0 F_1$ is asymptotically stable and

$$\|(C_0 + D_{1,0}F_1)(zI - A_0 - B_0F_1)^{-1}(E_0 + B_0F_2) + (D_{2,0} + D_{1,0}F_2)\|_\infty < \gamma.$$

We will use the following notation for the system parameters we obtain after applying the feedback $u = F_1 x + F_2 w$ to Σ:

$$A_F := A + BF_1, \quad E_F := E + BF_2, \quad C_F := C + D_1F_1, \quad D_F := D_2 + D_1F_2.$$

As in the proof of lemma 2.3 we have:

$$P(\Sigma) \leqslant \tilde{P}(\Sigma)$$

where $\tilde{P}(\Sigma)$ is the unique stabilizing solution $\tilde{P}$ of the following algebraic Riccati equation:

$$\begin{aligned}\tilde{P} &= A_F^{\mathrm{T}} \tilde{P} A_F + C_F^{\mathrm{T}} C_F \\ &\qquad + (A_F^{\mathrm{T}} \tilde{P} E_F + C_F^{\mathrm{T}} D_F) \left(\gamma^2 I - D_F^{\mathrm{T}} D_F - E_F^{\mathrm{T}} \tilde{P} E_F\right)^{-1} (E_F^{\mathrm{T}} \tilde{P} A_F + D_F^{\mathrm{T}} C_F),\end{aligned}$$

such that

$$A_F + E_F \left(\gamma^2 I - D_F^{\mathrm{T}} D_F - E_F^{\mathrm{T}} \tilde{P} E_F\right)^{-1} (E_F^{\mathrm{T}} \tilde{P} A_F + D_F^{\mathrm{T}} C_F),$$

is asymptotically stable. Because of lemma 2.1 we know that $\tilde{P}$ depends continuously on the system parameters. Hence there exists a compact set $\mathcal{S}_1$ containing Σ_0 in its interior and $M > 0$ such that for all systems Σ in $\mathcal{S}_1$ we have that $A + BF$ is asymptotically stable and

$$P(\Sigma) \leqslant \tilde{P}(\Sigma) < M$$

for all Σ in $\mathcal{S}_1$.
We define:

$$\begin{aligned}
V &:= B^{\mathrm{T}} P B + D_1^{\mathrm{T}} D_1, \\
R &:= \gamma^2 I - D_2^{\mathrm{T}} D_2 - E^{\mathrm{T}} P E + (E^{\mathrm{T}} P B + D_2^{\mathrm{T}} D_1) V^{-1} (B^{\mathrm{T}} P E + D_1^{\mathrm{T}} D_2). \\
Z &:= E^{\mathrm{T}} P A + D_2^{\mathrm{T}} C - (E^{\mathrm{T}} P B + D_2^{\mathrm{T}} D_1) V^{-1} (B^{\mathrm{T}} P A + D_1^{\mathrm{T}} C), \\
\tilde{A} &:= A + E R^{-1} Z, \\
\tilde{E} &:= E R^{-1/2}, \\
\tilde{C} &:= V^{-1/2} (B^{\mathrm{T}} P A + D_1^{\mathrm{T}} C) + V^{-1/2} (B^{\mathrm{T}} P E + D_1^{\mathrm{T}} D_2) R^{-1} Z, \\
\tilde{D}_1 &:= V^{1/2} \\
\tilde{D}_2 &:= V^{-1/2} (B^{\mathrm{T}} P E + D_1^{\mathrm{T}} D_2) R^{-1/2},
\end{aligned}$$

Since P is bounded by M when the system Σ is restricted to the set $\mathcal{S}_1$ we find that $\tilde{\Sigma}$ parameterized by $(\tilde{A}, B, \tilde{E}, \tilde{C}, \tilde{D}_1, \tilde{D}_2)$ is in some compact set $\mathcal{T}$. Moreover, $\tilde{D}_1$ is invertible and $(\tilde{A}, B, \tilde{C}, \tilde{D}_1)$ has no zeros outside the unit circle.
We define on $\mathcal{T} \times [0, 1]$, the function Q which maps $(\tilde{\Sigma}, \varepsilon)$ to the semi-stabilizing solution of the following Riccati equation:

$$Q = \tilde{A}^{\mathrm{T}} Q \tilde{A} + \tilde{C}^{\mathrm{T}} \tilde{C} + \varepsilon^2 I - \begin{pmatrix} B^{\mathrm{T}} Q \tilde{A} + \tilde{D}_1^{\mathrm{T}} \tilde{C} \\ \tilde{E}^{\mathrm{T}} Q \tilde{A} + \tilde{D}_2^{\mathrm{T}} \tilde{C} \end{pmatrix}^{\mathrm{T}} \tilde{G}(Q)^{-1} \begin{pmatrix} B^{\mathrm{T}} Q \tilde{A} + \tilde{D}_1^{\mathrm{T}} \tilde{C} \\ \tilde{E}^{\mathrm{T}} Q \tilde{A} + \tilde{D}_2^{\mathrm{T}} \tilde{C} \end{pmatrix},$$

where

$$\tilde{G}(Q) := \begin{pmatrix} \tilde{D}_1^{\mathrm{T}} \tilde{D}_1 & \tilde{D}_1^{\mathrm{T}} \tilde{D}_2 \\ \tilde{D}_2^{\mathrm{T}} \tilde{D}_1 & \tilde{D}_2^{\mathrm{T}} \tilde{D}_2 - \gamma^2 I \end{pmatrix} + \begin{pmatrix} B^{\mathrm{T}} \\ \tilde{E}^{\mathrm{T}} \end{pmatrix} Q \begin{pmatrix} B & \tilde{E} \end{pmatrix}.$$

We know from lemma 2.3 that this is a continuous function and since we have a continuous function on a compact set it is also uniformly continuous. Moreover,

$$Q(\tilde{\Sigma}, 0) = 0.$$

Hence there exists $\varepsilon_1 > 0$ such that

$$\|Q(\tilde{\Sigma}, \varepsilon_1)\| < \delta/3I$$

for all $\tilde{\Sigma}$ in T. It is straightforward to check that

$$X(\Sigma) = P(\Sigma) + Q(\tilde{\Sigma}\varepsilon_1)$$

is the semi-stabilizing solution of the algebraic Riccati equation:

$$X = A^{\mathrm{T}}XA + C^{\mathrm{T}}C + \varepsilon_1^2 - \begin{pmatrix} B^{\mathrm{T}}XA + D_1^{\mathrm{T}}C \\ E^{\mathrm{T}}XA + D_2^{\mathrm{T}}C \end{pmatrix}^{\mathrm{T}} G(X)^{-1} \begin{pmatrix} B^{\mathrm{T}}XA + D_1^{\mathrm{T}}C \\ E^{\mathrm{T}}XA + D_2^{\mathrm{T}}C \end{pmatrix},$$

where G is defined by (2.2). X is the semi-stabilizing solution of an algebraic Riccati equation associated to a system which is left-invertible and has no zeros on the unit circle. Therefore lemma 2.1 guarantees it is continuous and we can find an open neighbourhood $\mathcal{S}$ of Σ_0 such that:

$$\|X(\Sigma) - X(\Sigma_0)\| < \delta/3$$

for all $\Sigma \in \mathcal{S}$. This implies:

$$\|P(\Sigma) - P(\Sigma_0)\| \leqslant \|X(\Sigma) - X(\Sigma_0)\| + \|Q(\tilde{\Sigma}, \varepsilon_1)\| + \|Q(\tilde{\Sigma}_0, \varepsilon_1)\| < \delta \qquad \blacksquare$$

2.3 Non-left-invertible systems

If a discrete time system is not left-invertible then we can almost always find a perturbation which yields a discontinuous jump in the semi-stabilizing solution of the algebraic Riccati equation. This is quite natural. After all if the system is not left-invertible then one has an input which does not have any affect on the to-be-controlled output z. After a small perturbation this input will have a (small) affect on the output z. It is a very small affect but since this input is not weighted in the performance criterion we can have high-gain feedback. The high gain can offset the fact that there is only a small affect on z and therefore a discontinuous jump. A simple example of this is given by the following system:

$$\Sigma : \begin{cases} x(k+1) = \quad + \quad w(k) \\ z(k) \quad = x(k) + \varepsilon u(k) \end{cases}$$

For $\varepsilon = 0$ the control input cannot affect z at all and we will have a non-zero cost. On the other hand for $\varepsilon \neq 0$ we can choose $u = -\varepsilon^{-1}x$ which guarantees $z = 0$. For this example we have that the solution of the algebraic Riccati equation is non-zero for any γ for $\varepsilon = 0$ and jumps to zero if we perturb ε away from 0.
The number of inputs that affect z is measured by the normal rank. Hence we might think that if a perturbation is such that the normal rank of $C(zI - A)^{-1}B + D_1$ does not change then this perturbation changes the solution of the algebraic Riccati equation in a continuous manner. For a special case this property is indeed true:

Theorem 2.5 *Consider the set $\mathcal{D}_{mp}$ of systems Σ for which (A, B) is stabilizable, which have no zeros outside the unit circle and for which , if γ is finite, there exists F_1, F_2 such that*

$A + BF_1$ is stable and (2.8) is satisfied. Consider a sequence of systems Σ_ε parameterized by $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_{1,\varepsilon}, D_{2,\varepsilon})$ which converges to Σ. Moreover assume that the normal rank of $C_\varepsilon(zI - A_\varepsilon)^{-1}B_\varepsilon + D_{1,\varepsilon}$ is equal to the normal rank of $C(zI - A)^{-1}B + D_1$ for all ε.
Then the smallest, positive semi-definite semi-stabilizing solution P_ε of the algebraic Riccati equation associated with Σ_ε converges to the smallest, positive semi-definite semi-stabilizing solution P of the algebraic Riccati equation associated with Σ.

Proof : Because the system (A, B, C, D_1) does not have zeros outside the closed unit circle we have:

$$\xi^{\mathrm{T}} P(\Sigma, \varepsilon)\xi = \sup_{w\in\ell_2} \inf_{u\in\ell_2} \|z\|_2^2 - \gamma^2\|w\|_2^2. \tag{2.16}$$

This is basically a consequence of results in [6]. Hence in contrast with earlier in the paper we do not need to require that the state x converges to 0. It suffices to require that $z \in L_2$. Let the open loop transfer matrix from u to z and from w to z be denoted by $G_{1,eps}$ and $G_{2,\varepsilon}$ respectively. We have:

$$\hat{z} = G_{1,\varepsilon}\hat{u} + G_{2,\varepsilon}\hat{w}.$$

where $\hat{u}, \hat{w}, \hat{z}$ denote the Laplace transform of u, w, z. There exists a stable transfer matrix U such that:

$$G_{1,0}U = \begin{pmatrix} H_{11,0} & 0 \end{pmatrix}$$

with $H_{11,0}$ left invertible. Define $H_{1,\varepsilon}$ by

$$G_{1,\varepsilon}U = H_{1,\varepsilon} = \begin{pmatrix} H_{11,\varepsilon} & H_{12,\varepsilon} \end{pmatrix}$$

Since the normal rank of $G_{1,\varepsilon}$ in independent of ε and since $G_{1,\varepsilon}$ converges to $G_{1,0}$ we find that for small ε and for all z outside the unit circle:

$$\operatorname{rank} H_{1,\varepsilon}(z) = \operatorname{rank} H_{11,\varepsilon}$$

The latter implies that there exists a matrix V_ε such that

$$H_{12,\varepsilon} = H_{11,\varepsilon}V_\varepsilon$$

We then find that without loss of generality we can assume $u = \begin{pmatrix} u_1 \\ 0 \end{pmatrix}$ in the optimization problem (2.16). Since the system mapping u_1 to z is left-invertible, we find as a consequence of theorem 2.4 that P is a continuous function of ε. ■

The above theorem does not hold without the requirement that the system (A, B, C, D_1) has no unstable zeros. As an example consider the following system

$$\Sigma : \begin{cases} x(k+1) = 2x(k) + (1 \ \ \varepsilon)u(k) + w(k) \\ \\ z(k) \quad = \quad (1 \ \ 0)u(k) \end{cases}$$

Clearly for this system the normal rank of the subsystem from u to z is equal to 1 for all ε. On the other hand the semi-stabilizing solution of the algebraic Riccati equation behaves discontinously. The reason is clearly that a major objective of this system is the requirement to stabilize the system. For $\varepsilon \neq 0$ there is suddenly an extra input available to stabilize the system. This input is not weighted in the cost criterion and hence the cost jumps to 0. Therefore an additional condition is needed which ensures that we do not change the number of inputs that can stabilize unstable zeros. We can connect non-minimum-phase zeros to the following subspace:

Definition 2.6 *Consider a linear system Σ characterized by the quadruple (A, B, C, D). Then, the strongly controllable subspace $\mathcal{R}^*(\Sigma)$ is defined as the maximal subspace of $\mathbb{R}^n$ for which there exists a matrix F such that*

- *$\mathcal{R}^*(\Sigma)$ is $(A + BF)$-invariant*
- *$\mathcal{R}^*(\Sigma)$ is contained in $\operatorname{Ker}(C + DF)$.*
- *For each $\bar{\lambda} \in \mathbb{R}$ there eixsts F_1 such that $\mathcal{R}^*(\Sigma)$ is $A + BF_1$ invariant, contained in $\operatorname{Ker}(C + DF_1)$ and the eigenvalues of $A + BF_1$ restricted to $\mathcal{R}^*(\Sigma)$ satisfy $\operatorname{Re} \lambda < \bar{\lambda}$.*

Note that this subspace is closely related to left-invertibility. In particular, a system is left-invertible if and only if $\mathcal{R}^* = \{0\}$ and $(B^{\mathrm{T}} \quad D^{\mathrm{T}})$ is surjective. Basically the example given before is such that part of the state space associated with a non-minimum phase zero suddenly becomes part of $\mathcal{R}^*$ by a small perturbation. We have to exclude this from happening. In particular, we can obtain the following theorem:

Theorem 2.7 *Consider the set $\mathcal{D}$ of systems Σ for which (A, B) is stabilizable and, if γ is finite, there exists F_1, F_2 such that $A + BF_1$ is stable and (2.8) is satisfied. Consider a sequence of perturbed systems Σ_ε with parameters $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_{1,\varepsilon}, D_{2,\varepsilon})$ which converges to Σ. Moreover assume that the normal rank of $C_\varepsilon(zI - A_\varepsilon)^{-1}B_\varepsilon + D_{1,\varepsilon}$ is equal to the normal rank of $C(zI - A)^{-1}B + D_1$ for all ε and*

$$\dim \mathcal{R}^*(A, B, C, D_1) = \dim \mathcal{R}^*(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_{1,\varepsilon})$$

for all ε.
Then the smallest, positive semi-definite semi-stabilizing solution P_ε of the algebraic Riccati equation associated with Σ_ε converges to the smallest, positive semi-definite semi-stabilizing solution P of the algebraic Riccati equation associated with Σ.

Proof : First of all we note that since $\mathcal{R}^*$ does not change dimension and since the normrank is fixed, we find that $\mathcal{R}^*$ depends continuously on ε. Hence there exists a basis transformation depending continuously on ε such that

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, A_\varepsilon = \begin{pmatrix} A_{11,\varepsilon} & A_{12,\varepsilon} \\ A_{21,\varepsilon} & A_{22,\varepsilon} \end{pmatrix}, B_\varepsilon = \begin{pmatrix} B_{11,\varepsilon} & B_{12,\varepsilon} \\ 0 & B_{22,\varepsilon} \end{pmatrix}, E_\varepsilon = \begin{pmatrix} E_{1,\varepsilon} \\ E_{2,\varepsilon} \end{pmatrix},$$
$$C = \begin{pmatrix} C_{1,\varepsilon} & C_{2,\varepsilon} \end{pmatrix}, D = \begin{pmatrix} 0 & D_{12,\varepsilon} \end{pmatrix}$$

such that for each ε the component x_1 corresponds to $\mathcal{R}^*$. Moreover it is then easy to check that:

$$\begin{pmatrix} B_{22,\varepsilon} \\ D_{12,\varepsilon} \end{pmatrix} = \operatorname{normrank} C_\varepsilon(zI - A_\varepsilon)^{-1}B_\varepsilon + D_{1,\varepsilon}$$

for all ε. Since the normal rank is independent of ε, we can without loss of generality assume that we chose a basis of the input space such that $\begin{pmatrix} B_{22,\varepsilon}^{\mathrm{T}} & D_{12,\varepsilon}^{\mathrm{T}} \end{pmatrix}$ is surjective.
It is well-known that the semi-stabilizing solution P_ε of the Riccati equation is zero on $\mathcal{R}^*$. In other words, we have

$$P_\varepsilon = \begin{pmatrix} 0 & 0 \\ 0 & P_{22,\varepsilon} \end{pmatrix}$$

for all ε. $P_{22,\varepsilon}$ is a solution of the algebraic Riccati equation associated with a reduced system with parameters $(A_{22,\varepsilon}, B_{22,\varepsilon}, E_{2,\varepsilon}, C_{2,\varepsilon}, D_{12,\varepsilon}, D_{2,\varepsilon})$. The system $(A_{22}, \varepsilon, B_{22,\varepsilon}, C_{2,\varepsilon}, D_{12,\varepsilon})$ is by construction left-invertible and hence continuity of P_{22} is a consequence of theorem 2.4.

■

3 Continuous time systems

3.1 Problem formulation

Consider the following quadratic matrix inequality

$$F_\gamma(P) := \begin{pmatrix} PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + \gamma^{-2}(PE + C^{\mathrm{T}}D_2)(E^{\mathrm{T}}P + D_2^{\mathrm{T}}C) & PB + C^{\mathrm{T}}D_1 \\ B^{\mathrm{T}}P + D_1^{\mathrm{T}}C & D_1^{\mathrm{T}}D_1 \end{pmatrix} \geqslant 0. \tag{3.1}$$

We are interested in rank-minimizing solutions which imposes the following rank condition on solutions of the quadratic matrix inequality:

$$\operatorname{rank}_{\mathbb{C}} F_\gamma(P) = \operatorname{rank}_{\mathbb{R}(s)} G_{ci} \tag{3.2}$$

As in the discrete time we want to have semi-stabilizing solutions. In this setting, semi-stabilizing solutions are rank-minimizing solutions which satisfy the following additional rank condition:

$$\operatorname{rank} \begin{pmatrix} L_\gamma(P,s) \\ F_\gamma(P) \end{pmatrix} = n + \operatorname{rank}_{\mathbb{R}(s)} G_{ci} \qquad \forall s \in \mathbb{C}^+. \tag{3.3}$$

where

$$L_\gamma(P,s) := \begin{pmatrix} sI - A - \gamma^{-2}EE^{\mathrm{T}}P & -B \end{pmatrix}.$$

If this last rank condition is also satisfied on the imaginary axis then we will call P a stabilizing solution of the quadratic matrix inequality. Like in the continuous time we can associate this quadratic matrix inequality to an H_∞ control problem for the following system:

$$\Sigma : \begin{cases} \dot{x} = Ax + \ \ Bu + \ \ Ew, \\ z = Cx + D_1u + D_2w. \end{cases} \tag{3.4}$$

Basically there exists a stabilizing feedback $u = F_1 x + F_2 w$ such that the closed loop H_∞ norm is less than γ if and only if there exists a positive semi-definite semi-stabilizing, rank-minimizing solution of the above quadratic matrix inequality and some additional conditions (see [6]).
For $\gamma = \infty$ the general quadratic matrix inequality (3.1) reduces to the H_2 linear matrix inequality:

$$F(P) := \begin{pmatrix} PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C & PB + C^{\mathrm{T}}D_1 \\ B^{\mathrm{T}}P + D_1^{\mathrm{T}}C & D_1^{\mathrm{T}}D_1 \end{pmatrix} \geqslant 0. \tag{3.5}$$

The quadratic matrix inequality is associated to the system Σ which is parameterized by the matrices (A, B, E, C, D_1, D_2). We define the set $\mathcal{D}$ to be the class of systems Σ for which (A, B, C, D_1) is left-invertible and for which there exists matrices F_1, F_2 such that $A + BF_1$ is stable and

$$\|(C + DF_1)(sI - A - BF_1)^{-1}(E + BF_2) + (D_2 + D_1F_2)\|_\infty < \gamma \tag{3.6}$$

For the H_∞ control problem (i.e. finite γ) the semi-stabilizing solution always exists for elements of the set $\mathcal{D}$ (see [6]) but it need not be unique. We will study the smallest, positive semi-definite semi-stabilizing solution P of the quadratic matrix inequality which is obviously unique.
For the H_2 problem, where $\gamma = \infty$, the set $\mathcal{D}$ consists of systems Σ for which (A, B, C, D_1) is left-invertible and (A, B) is stabilizable. In that case, it is known (see [1]) that for elements of the set $\mathcal{D}$ there exists a unique real symmetric semi-stabilizing solution P of the linear matrix inequality (3.5). Moreover, this solution is positive semi-definite.
Note that if D_1 is injective then we can characterize rank-minimizing solutions of the quadratic matrix inequality as those matrices P that satisfy the following standard Riccati equation:

$$\begin{aligned} 0 = PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C + \gamma^{-2}(PE + C^{\mathrm{T}}D_2)(E^{\mathrm{T}}P + D_2^{\mathrm{T}}C) \\ - (PB + C^{\mathrm{T}}D_1)(D_1^{\mathrm{T}}D_1)^{-1}(B^{\mathrm{T}}P + D_1^{\mathrm{T}}C) \end{aligned}$$

In this case a solution is semi-stabilizing or stabilizing if the following matrix

$$A + \gamma^{-2}E(E^{\mathrm{T}}P + D_2^{\mathrm{T}}C) - B(D_1^{\mathrm{T}}D_1)^{-1}(B^{\mathrm{T}}P + D_1^{\mathrm{T}}C)$$

has all eigenvalues in the closed or open right half plane respectively.
In the next subsection, we will show that P depends continuously on the system parameters for systems in the set $\mathcal{D}$ both for finite γ and for $\gamma = \infty$. Since we allow for zeros on the imaginary axis, this continuity is far from obvious. In the subsection thereafter we study continuity questions for systems outside the set $\mathcal{D}$.

3.2 Continuity

We first show that the stabilizing, rank-minimizing solution of the quadratic matrix inequality depends continuously on the system parameters if we do not have zeros on the (extended) imaginary axis.

Lemma 3.1 *Let $\mathcal{D}_0$ be the open subset in $\mathcal{D}$ of systems Σ for which (A, B, C, D_1) has no zeros on the imaginary axis and D_1 is injective.*
For each element of $\mathcal{D}_0$, the quadratic matrix inequality (3.1) *has a unique rank-minimizing, stabilizing solution P. The function f from $\mathcal{D}_0$ to $\mathbb{R}^{n\times n}$ which assigns to each system in $\mathcal{D}_0$, the associated rank-minimizing stabilizing solution of the quadratic matrix inequality is continuous.*

Proof : The existence and uniqueness of the stabilizing solution can for instance be found in [5].
The continuity follows in a straightforward manner. The solution P is associated to the stable subspace of a Hamiltonian matrix (note that when D_1 is injective, the stabilizing solution of the quadratic matrix inequality is simply the stabilizing solution of a Riccati equation, see e.g. [5]). Since this Hamiltonian matrix has no eigenvalues on the imaginary axis, the stable and antistable eigenvalues are strictly separated and the existence of a continuous basis for the stable subspace and hence the continuous dependence of the stabilizing solution of the quadratic matrix inequality can be found in e.g. [4]. ∎

Our main objective is to show that the extension of this function f to the whole set $\mathcal{D}$ is also continuous. The derivation will be mutatis mutandis equivalent to the discrete time. First we need some technical lemmas. The following lemma is related to the classical cheap control argument.

Lemma 3.2 *Let Σ be an arbitrary element of $\mathcal{D}$ such that D_1 is invertible and (A, B, C, D_1) has no zeros in the right half plane. For $\varepsilon \neq 0$ small enough the following quadratic matrix inequality has a stabilizing solution P_ε:*

$$\begin{pmatrix} PA + A^TP + C^TC + \varepsilon^2 I + \gamma^{-2}(PE + C^TD_2)(E^TP + D_2^TC) & PB + C^TD_1 \\ B^TP + D_1^TC & D_1^TD_1 + \varepsilon^2 I \end{pmatrix} \geqslant 0. \tag{3.7}$$

Moreover $P_\varepsilon \to 0$ as $\varepsilon \to 0$.

Proof : The quadratic matrix inequality (3.7) is basically the quadratic matrix inequality associated with the system $\sigma(\Sigma, \varepsilon)$ parameterized by $(A, B, E, C_\varepsilon, D_{1,\varepsilon}, D_{2,\varepsilon})$ where $C_\varepsilon, D_{1,\varepsilon}, D_{2,\varepsilon}$ are such that

$$\begin{pmatrix} C_\varepsilon^T C_\varepsilon & C_\varepsilon^T D_{1,\varepsilon} & C_\varepsilon^T D_{2,\varepsilon} \\ D_{1,\varepsilon}^T C_\varepsilon & D_{1,\varepsilon}^T D_{1,\varepsilon} & D_{1,\varepsilon}^T D_{2,\varepsilon} \\ D_{2,\varepsilon}^T C_\varepsilon & D_{2,\varepsilon}^T D_{1,\varepsilon} & D_{2,\varepsilon}^T D_{2,\varepsilon} \end{pmatrix} = \begin{pmatrix} C^TC + \varepsilon^2 I & C^TD_1 & C^TD_2 \\ D_1^TC & D_1^TD_1 + \varepsilon^2 I & D_1^TD_2 \\ D_2^TC & D_2^TD_1 & D_2^TD_2 \end{pmatrix}. \tag{3.8}$$

Since $\mathcal{D}$ is an open set, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in [0, \varepsilon^*]$ we have that $\sigma(\Sigma, \varepsilon)$ is in $\mathcal{D}$.

Because $\sigma(\Sigma,\varepsilon)$ is in $\mathcal{D}$, we know that the quadratic matrix inequality has a semi-stabilizing solution. It is actually a stabilizing solution since for $\varepsilon \neq 0$ the system $\sigma(\Sigma,\varepsilon)$ is such that $(A, B, C_\varepsilon, D_{1,\varepsilon})$ has no invariant zeros on the imaginary axis. We have:

$$\delta := \sup_{\varepsilon\in[0,\varepsilon^*]} \inf_{F_1,F_2} \Big\{ \|(C_\varepsilon + D_{1,\varepsilon}F_1)(sI - A - BF_1)^{-1}(E + BF_2) + (D_2 + D_1F_2)\|_\infty \\ \mid A + BF_1 \text{ stable } \Big\} < \gamma$$

From [5] we have for finite γ and $\varepsilon \in (0, \varepsilon^*]$:

$$L_\varepsilon \leqslant P_\varepsilon \leqslant \tfrac{\gamma^2}{\gamma^2-\delta^2}\, L_\varepsilon$$

where L_ε is the stabilizing solution of the following linear matrix inequality:

$$\begin{pmatrix} \bar{L}_\varepsilon A + A^{\mathrm{T}}\bar{L}_\varepsilon + C^{\mathrm{T}}C + \varepsilon^2 I & \bar{L}_\varepsilon B + C^{\mathrm{T}}D_1 \\ B^{\mathrm{T}}\bar{L}_\varepsilon + D_1^{\mathrm{T}}C & D_1^{\mathrm{T}}D_1 + \varepsilon^2 I \end{pmatrix} \geqslant 0.$$

This basically implies that we can restrict attention to the H_2 linear matrix inequality (if we start with $\gamma = \infty$ then obviously $P_\varepsilon = L_\varepsilon$).
Using the relation of L_ε with a standard linear quadratic control problem we note that L_ε is decreasing in ε and bounded from below. Therefore, there exists a limit $\bar{L}$. Clearly $\bar{L}$ is a semi-stabilizing solution of the following linear matrix inequality

$$\begin{pmatrix} \bar{L}A + A^{\mathrm{T}}\bar{L} + C^{\mathrm{T}}C & \bar{L}B + C^{\mathrm{T}}D_1 \\ B^{\mathrm{T}}\bar{L} + D_1^{\mathrm{T}}C & D_1^{\mathrm{T}}D_1 \end{pmatrix} \geqslant 0.$$

On the other hand 0 is also a semi-stabilizing solution of this linear matrix inequality. By [1], we know that the real symmetric semi-stabilizing solution is unique and hence $\bar{L} = 0$, which implies $L_\varepsilon \to 0$ as $\varepsilon \to 0$ and hence $P_\varepsilon \to 0$. ■

Lemma 3.3 *Let $\mathcal{D}_1$ be a compact subset of $\mathcal{D}$ consisting of systems Σ for which (A, B, C, D_1) has no zeros in the open right half plane. There exists $\varepsilon^* > 0$ such that for all $\varepsilon \in [0, \varepsilon^*]$ and for all $\Sigma \in \mathcal{D}_1$, we have $\sigma(\Sigma,\varepsilon)$ in $\mathcal{D}$. For each element in $\mathcal{D}_1$ and $\varepsilon \in [0, \varepsilon^*]$ there exists a smallest positive semi-definite stabilizing solution of the quadratic matrix inequality.:*

$$\begin{pmatrix} PA + A^TP + C^TC + \varepsilon^2 I + \gamma^{-2}(PE + C^TD_2)(E^TP + D_2^TC) & PB + C^TD_1 \\ B^TP + D_1^TC & D_1^TD_1 + \varepsilon^2 I \end{pmatrix} \geqslant 0. \tag{3.9}$$

P defines a map from $\mathcal{D}_1 \times [0, \varepsilon^*]$ to $\mathbb{R}^{n\times n}$ which is well-defined and continuous.

Proof : The existence of ε^* is a consequence of the fact that $\mathcal{D}_1$ is compact and that $\mathcal{D}$ is open. Choose $(\Sigma_0, \varepsilon_0)$ in $\mathcal{D}_1 \times [0, \varepsilon^*]$. If $\varepsilon_0 > 0$ continuity at $(\Sigma_0, \varepsilon_0)$ is a consequence of lemma 3.1.

Assume that $\varepsilon_0 = 0$. Then it is easy to check that the smallest positive semi-definite solution of the quadratic matrix inequality associated to $\sigma(\Sigma, 0)$ equals 0. By definition, we always have $P \geqslant 0$. Hence we find that P is lower semi-continuous at $(\Sigma_0, 0)$, i.e. for each $\delta > 0$ there exists an open neigbourhood $\mathcal{S}$ of $(\Sigma_0, \varepsilon_0)$ in $\mathcal{D}_1 \times [0, \varepsilon^*]$ such that:

$$P(\Sigma, \varepsilon) \geqslant P(\Sigma_0, 0) - \delta I \tag{3.10}$$

for all (Σ, ε) in $\mathcal{S}$. After all (3.10) is trivially satisfied for $\mathcal{S} = \mathcal{D}_1 \times [0, \varepsilon^*]$ since

$$P(\Sigma_0, 0) = 0. \tag{3.11}$$

Remains to show upper semi-continuity. Choose $\delta > 0$. We have to construct an open neighbourhood $\mathcal{S}$ of $(\Sigma_0, 0)$ in $\mathcal{D}_1 \times [0, \varepsilon^*]$ such that for all (Σ, ε) in $\mathcal{S}$:

$$P(\Sigma, \varepsilon) \leqslant P(\Sigma_0, 0) + \delta I \tag{3.12}$$

From lemma 2.2 we find that, for the fixed system Σ_0, we have $P(\Sigma_0, \varepsilon) \to 0$ as $\varepsilon \to 0$. Let $\varepsilon_0 > 0$ be such that $\bar{P} := P(\Sigma_0, \varepsilon_0) < \delta/2I$. Define F_1, F_2 by:

$$\begin{aligned} F_1 &:= -(D_{1,0}^{\mathrm{T}} D_{1,0})^{-1}(B_0^{\mathrm{T}} \bar{P} + D_{1,0}^{\mathrm{T}} C_0) \\ F_2 &:= -(D_{1,0}^{\mathrm{T}} D_{1,0})^{-1} D_{1,0}^{\mathrm{T}} D_{2,0} \end{aligned}$$

where the system $\sigma(\Sigma_0, \varepsilon_0)$ is parameterized by $(A_0, B_0, E_0, C_0, D_{1,0}, D_{2,0})$. We find that $A_0 + B_0 F_1$ is stable and

$$\|(C_0 + D_{1,0} F_1)(sI - A_0 - B_0 F_1)^{-1}(E_0 + B_0 F_2) + (D_{2,0} + D_{1,0} F_2)\|_\infty < \gamma$$

Let $\sigma(\Sigma, \varepsilon)$ be parameterized by $(A, B, E, C, D_{1,\varepsilon}, D_{2,\varepsilon})$. We will use the following notation for the system parameters we obtain after applying the feedback $u = F_1 x + F_2 w$ to $\sigma(\Sigma, \varepsilon)$:

$$A_F := A + BF_1, \quad E_F := E + BF_2, \quad C_{F,\varepsilon} := C_\varepsilon + D_{1,\varepsilon} F_1, \quad D_{F,\varepsilon} := D_{2,\varepsilon} + D_{1,\varepsilon} F_2.$$

We will use the interpretation of P as an equilibrium of a differential game (see e.g. [2]). We have

$$\begin{aligned} \xi^{\mathrm{T}} P(\Sigma, \varepsilon) \xi &= \sup_{w \in L_2} \inf_{u \in L_2} \left\{ \|z\|_2^2 - \gamma^2 \|w\|_2^2 \;\middle|\; x(t) \to 0 \text{ as } t \to \infty \right\} \\ &\leqslant \sup_{w \in L_2} \left\{ \|z\|_2^2 - \gamma^2 \|w\|_2^2 \;\middle|\; u = F_1 x + F_2 w \right\} \\ &= \xi^{\mathrm{T}} \tilde{P}(\Sigma, \varepsilon) \xi \end{aligned}$$

where $\tilde{P}(\Sigma, \varepsilon)$ is the unique stabilizing solution $\tilde{P}$ of the following Riccati equation:

$$PA_F + A_F^{\mathrm{T}} P + C_F^{\mathrm{T}} C_F + (PE_F + C^{\mathrm{T}} D_2)(\gamma^2 I - D_2^{\mathrm{T}} D_2)^{-1}(E^{\mathrm{T}} P + D_2^{\mathrm{T}} C) = 0.$$

We know:

$$\tilde{P}(\Sigma_0, \varepsilon_0) = P(\Sigma_0, \varepsilon_0) < \delta/2I$$

Moreover, $\tilde{P}(\Sigma, \varepsilon_0)$ depends continuously on the system parameters for fixed ε_0 on the basis of lemma 3.1. Therefore there exists an open neighbourhood $\mathcal{D}_2$ of Σ_0 in $\mathcal{D}_1$ such that

$$P(\Sigma, \varepsilon_0) \leqslant \tilde{P}(\Sigma, \varepsilon_0) < \delta I.$$

The latter implies that $P(\Sigma, \varepsilon) \leqslant P(\Sigma, \varepsilon_0) < \delta I$ for all (Σ, ε) in $\mathcal{D}_2 \times [0, \varepsilon_0]$. Combined with (3.11), this implies (3.12). ■

Theorem 3.4 *Consider the set $\mathcal{D}$ of systems Σ for which (A, B, C, D_1) is left-invertible and, if γ is finite, for which there exists F_1, F_2 such that $A + BF$ is stable and (3.6) is satisfied. The smallest, positive semi-definite semi-stabilizing solution of the algebraic Riccati equation is a continuous function from $\mathcal{D}$ to $\mathbb{R}^{n\times n}$.*

Proof : Denote by $P(\Sigma)$ the semi-stabilizing solution of the quadratic matrix inequality associated to the system Σ. Consider an arbitrary system Σ_0 in $\mathcal{D}$ and $\delta > 0$. We have to construct an open neighbourhood $\mathcal{S}$ in $\mathcal{D}$ of Σ_0 such that:

$$\|P(\Sigma) - P(\Sigma_0)\| < \delta$$

for all Σ in $\mathcal{S}$. Choose F_1, F_2 such that $A_0 + B_0F_1$ is asymptotically stable and

$$\|(C_0 + D_{1,0}F_1)(sI - A_0 - B_0F_1)^{-1}(E_0 + B_0F_2) + (D_{2,0} + D_{1,0}F_2)\|_\infty < \gamma.$$

We will use the following notation for the system parameters we obtain after applying the feedback $u = F_1x + F_2w$ to Σ:

$$A_F := A + BF_1, \quad E_F := E + BF_2, \quad C_F := C + D_1F_1, \quad D_F := D_2 + D_1F_2.$$

As in the proof of lemma 2.3 we have:

$$P(\Sigma) \leqslant \tilde{P}(\Sigma)$$

where $\tilde{P}(\Sigma)$ is the unique stabilizing solution $\tilde{P}$ of the following algebraic Riccati equation:

$$\tilde{P} = A_F^{\mathrm{T}}\tilde{P} + \tilde{P}A_F + C_F^{\mathrm{T}}C_F + (\tilde{P}E_F + C_F^{\mathrm{T}}D_F)\left(\gamma^2 I - D_F^{\mathrm{T}}D_F\right)^{-1}(E_F^{\mathrm{T}}\tilde{P} + D_F^{\mathrm{T}}C_F),$$

such that

$$A_F + E_F\left(\gamma^2 I - D_F^{\mathrm{T}}D_F\right)^{-1}(E_F^{\mathrm{T}}\tilde{P} + D_F^{\mathrm{T}}C_F),$$

is asymptotically stable. Because of lemma 2.1 we know that $\tilde{P}$ depends continuously on the system parameters. Hence there exists a compact set $\mathcal{S}_1$ containing Σ_0 in its interior and $M > 0$ such that for all systems Σ in $\mathcal{S}_1$ we have that $A + BF$ is asymptotically stable and

$$P(\Sigma) \leqslant \tilde{P}(\Sigma) < M$$

for all Σ in $\mathcal{S}_1$.
We define $\tilde{C}$, $\tilde{D}_1$ and $\tilde{D}_2$ such that

$$\begin{pmatrix} \tilde{C}^{\mathrm{T}}\tilde{C} & \tilde{C}^{\mathrm{T}}\tilde{D}_1 & \tilde{C}^{\mathrm{T}}\tilde{D}_2 \\ \tilde{D}_1^{\mathrm{T}}\tilde{C} & \tilde{D}_1^{\mathrm{T}}\tilde{D}_1 & \tilde{D}_1^{\mathrm{T}}\tilde{D}_2 \\ \tilde{D}_2^{\mathrm{T}}\tilde{C} & \tilde{D}_2^{\mathrm{T}}\tilde{D}_1 & \tilde{D}_2^{\mathrm{T}}\tilde{D}_2 \end{pmatrix} = \begin{pmatrix} PA + A^{\mathrm{T}}P + V & PB + C^{\mathrm{T}}D_1 & 0 \\ B^{\mathrm{T}}P + D_1^{\mathrm{T}}C & D_1^{\mathrm{T}}D_1 & D_1^{\mathrm{T}}D_2 \\ 0 & D_2^{\mathrm{T}}D_1 & D_2^{\mathrm{T}}D_2 - \gamma^2 I \end{pmatrix}.$$

where $V := C^{\mathrm{T}}C + \gamma^{-2}(PE + C^{\mathrm{T}}D_2)(E^{\mathrm{T}}P + D_2^{\mathrm{T}}C)$. Note that $\tilde{C}, \tilde{D}_1, \tilde{D}_2$ are not uniquely determined but at least locally we can guarantee that they depend continuously on the system parameters of Σ. We define:

$$\tilde{A} := A + \gamma^{-2}E(E^{\mathrm{T}}P + D_2^{\mathrm{T}}C)$$

Since P is bounded by M when the system Σ is restricted to the set $\mathcal{S}_1$ we find that $\tilde{\Sigma}$ parameterized by $(\tilde{A}, B, \tilde{E}, \tilde{C}, \tilde{D}_1, \tilde{D}_2)$ is in some compact set $\mathcal{T}$. Moreover, $(\tilde{A}, B, \tilde{C}, \tilde{D}_1)$ is left-invertible and has no zeros in the open right half plane.
We define on $\mathcal{T} \times [0,1]$, the function Q which maps $(\tilde{\Sigma}, \varepsilon)$ to the semi-stabilizing solution of the following quadratic matrix inequality:

$$\begin{pmatrix} Q\tilde{A} + \tilde{A}^{\mathrm{T}} Q + \tilde{C}^{\mathrm{T}} \tilde{C} + \varepsilon^2 I + \gamma^{-2}(Q\tilde{E} + \tilde{C}^{\mathrm{T}} \tilde{D}_2)(\tilde{E}^{\mathrm{T}} Q + \tilde{D}_2^{\mathrm{T}} \tilde{C}) & QB + \tilde{C}^{\mathrm{T}} \tilde{D}_1 \\ B^{\mathrm{T}} Q + \tilde{D}_1^{\mathrm{T}} \tilde{C} & \tilde{D}_1^{\mathrm{T}} \tilde{D}_1 + \varepsilon^2 I \end{pmatrix} \geqslant 0.$$

We know from lemma 3.3 that this is a continuous function and since we have a continuous function on a compact set it is also uniformly continuous. Moreover,

$$Q(\tilde{\Sigma}, 0) = 0.$$

Hence there exists $\varepsilon_1 > 0$ such that

$$\|Q(\tilde{\Sigma}, \varepsilon_1)\| < \delta/3I$$

for all $\tilde{\Sigma}$ in T. It is straightforward to check that

$$X(\Sigma) = P(\Sigma) + Q(\tilde{\Sigma}\varepsilon_1)$$

is the semi-stabilizing solution of the quadratic matrix inequality:

$$\begin{pmatrix} PA + A^{\mathrm{T}} P + C^{\mathrm{T}} C + \varepsilon^2 I + \gamma^{-2}(PE + C^{\mathrm{T}} D_2)(E^{\mathrm{T}} P + D_2^{\mathrm{T}} C) & PB + C^{\mathrm{T}} D_1 \\ B^{\mathrm{T}} P + D_1^{\mathrm{T}} C & D_1^{\mathrm{T}} D_1 + \varepsilon^2 I \end{pmatrix} \geqslant 0.$$

X is the semi-stabilizing solution of an quadratic matrix inequality associated to a system which is left-invertible and has no zeros on the unit circle. Therefore lemma 3.1 guarantees it is continuous and we can find an open neighbourhood $\mathcal{S}$ of Σ_0 such that:

$$\|X(\Sigma) - X(\Sigma_0)\| < \delta/3$$

for all $\Sigma \in \mathcal{S}$. This implies:

$$\|P(\Sigma) - P(\Sigma_0)\| \leqslant \|X(\Sigma) - X(\Sigma_0)\| + \|Q(\tilde{\Sigma}, \varepsilon_1)\| + \|Q(\tilde{\Sigma}_0, \varepsilon_1)\| < \delta$$

■

3.3 Non-left-invertible systems

If a continuous time system is not left-invertible then we can almost always find a perturbation which yields a discontinuous jump in the semi-stabilizing solution of the quadratic matrix inequality. This is quite natural. Basically the same arguments as in the discrete time case apply. Discontinuities only occur if we obtain an additional input that can either affect to the be controlled output z or can stabilize the non-minimum-phase zeros. The examples given in subsection 2.3 can easily be adapted to the continuous time and the two theorems are repeated below in a continuous time setting.

Theorem 3.5 *Consider the set $\mathcal{D}_{mp}$ of systems Σ for which (A, B) is stabilizable, (A, B, C, D_1) has no zeros in the open right half plane and, if γ is finite, there exists F_1, F_2 such that $A + BF$ is stable and (3.6) is satisfied. Consider a sequence of systems Σ_ε parameterized by $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_{1,\varepsilon}, D_{2,\varepsilon})$ which converges to Σ. Moreover assume that the normal rank of $C_\varepsilon(sI - A_\varepsilon)^{-1}B_\varepsilon + D_{1,\varepsilon}$ is equal to the normal rank of $C(sI - A)^{-1}B + D_1$ for all ε.*
Then the smallest, positive semi-definite semi-stabilizing solution P_ε of the quadratic matrix inequality associated with Σ_ε converges to the smallest, positive semi-definite semi-stabilizing solution P of the quadratic matrix inequality associated with Σ.

Proof : This is identical to the proof of theorem 2.5 ■

We need the definition of $\mathcal{R}^*$ given in definition 2.6. We can then obtain the following theorem:

Theorem 3.6 *Consider the set $\mathcal{D}$ of systems Σ for which (A, B) is stabilizable and if γ is finite, for which there exists F_1, F_2 such that $A + BF_1$ is stable and (3.6) is satisfied. Consider a sequence of perturbed systems Σ_ε with parameters $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_{1,\varepsilon}, D_{2,\varepsilon})$ which converges to Σ. Moreover assume that the normal rank of $C_\varepsilon(sI - A_\varepsilon)^{-1}B_\varepsilon + D_{1,\varepsilon}$ is equal to the normal rank of $C(sI - A)^{-1}B + D_1$ for all ε and*

$$\dim \mathcal{R}^*(A, B, C, D_1) = \dim \mathcal{R}^*(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_{1,\varepsilon})$$

for all ε.
Then the smallest, positive semi-definite semi-stabilizing solution P_ε of the algebraic Riccati equation associated with Σ_ε converges to the smallest, positive semi-definite semi-stabilizing solution P of the algebraic Riccati equation associated with Σ.

Proof : This is identical to the proof of theorem 2.7 ■

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