

Asymptotic behaviour of the utility vector in a dynamic programming model

Citation for published version (APA):

Zijm, W. H. M. (1980). *Asymptotic behaviour of the utility vector in a dynamic programming model*. (Memorandum COSOR; Vol. 8004). Technische Hogeschool Eindhoven.

Document status and date:

Published: 01/01/1980

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

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Memorandum COSOR 80-04

Asymptotic behaviour of the utility vector
in a dynamic programming model

by

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Eindhoven, April 1980

The Netherlands

ASYMPTOTIC BEHAVIOUR OF THE UTILITY VECTOR IN A

DYNAMIC PROGRAMMING MODEL

by

W.H.M. Zijm

0. Abstract

In mathematical economics (e.g. Leontief substitution systems) and in Markov decision theory we often deal with dynamic programming recursions of the following form

$$x(n + 1) = \max_{P \in M} Px(n) ; \quad n = 0, 1, 2, \dots$$

where $x(0)$ is assumed to be a strictly positive vector. M is a set of matrices, generated by all possible interchanges of corresponding rows, taken from a fixed finite set of nonnegative square matrices (not necessarily stochastic). We investigate the asymptotic behaviour of the vector $x(n)$ in terms of generalized eigenvectors of a particular matrix $\hat{P} \in M$, with respect to its spectral radius $\sigma(\hat{P})$. This paper extends earlier results of Sladký [11] and Zijm [13].

1. Introduction

Consider a set M of matrices, which is generated by all possible interchanges of corresponding rows, taken from a fixed finite set of nonnegative $N \times N$ - matrices. In this paper we investigate the asymptotic behaviour of the utility vector $x(n)$ (a column N -vector), defined by the following dynamic programming recursion

$$(1) \quad x(n+1) = \max_{P \in M} Px(n) ; \quad n = 0, 1, 2, \dots$$

where $x(0)$ is a fixed, strictly positive vector. From the structure of M it is obvious that we may take the maximum component-wise in (1).

Nonnegative matrices, and especially recursion (1), play an important role in mathematical economics, e.g. the analysis of (closed) Leontief substitution systems. Also dynamic programming recursions in Markov decision theory and in open Leontief substitution models, which often have the following well-known form

$$v(n+1) = \max_{Q \in M} \{r(Q) + Q v(n)\} ; \quad n = 0, 1, 2, \dots$$

may be written as equation (1), by applying the following translation

$$P = \begin{pmatrix} Q & r(Q) \\ 0 & 1 \end{pmatrix} \quad x(n) = \begin{pmatrix} v(n) \\ 1 \end{pmatrix} ; \quad n = 0, 1, 2, \dots$$

as noted by Sladký [11]. Here $r(Q)$ denotes the immediate reward vector, or consumption vector, associated with the matrix Q , which will be (sub)stochastic in general. (Note that we do not assume anything about the matrices $P \in M$, in particular, they don't have to be (sub)stochastic or irreducible). For more details, compare e.g. Burmeister and Dobell [2], and Bellman [1].

The recursion (1) was investigated by Sladký ([10], [11]) under very restrictive assumptions, whereas in [13] we studied the first order asymptotic behaviour of $x(n)$ in a more general context. In [10] all the matrices

were assumed to be irreducible^{*}, in [11] the matrices were allowed to be reducible, however every matrix contained exactly one basic class (see for definitions below). These assumptions were dropped in [13], in which the first order behaviour of $x(n)$ was characterized in terms of the maximal spectral radius and the maximal index associated with this spectral radius, using a number of decomposition results for the set of matrices M (Zijm [12]). Finally, characterizations of these decomposition results in terms of generalized eigenvectors were given in [14] (compare also Rothblum [7],[8]).

In this paper we describe the complete asymptotic behaviour of $x(n)$ (not only its first order term). In the case that M contains exactly one matrix, P say, one may easily prove that we have for $x(n)$, $n = 1, 2, \dots$

$$(2) \quad x(n) = \binom{n}{\nu - 1} \sigma^n x_1 + \binom{n}{\nu - 2} \sigma^n x_2 + \dots + \binom{n}{0} \sigma^n x_\nu + o\left(\binom{n}{t} \rho^n\right)$$

where P is supposed to be aperiodic^{*}. Here σ denotes the spectral radius of P , ν the index associated with σ , while ρ is the maximum of the absolute values of the eigenvalues with absolute value smaller than σ (hence $\rho < \sigma$) and t is a fixed number. The vectors x_i , $i = 1, \dots, \nu$ are called generalized eigenvectors of P , associated with σ . Formula (2) may be proved easily by using the so-called Jordan or normal form of P (compare e.g. Pease [6]).

In the case that M contains more than one matrix, we are still able to derive a result similar to (2); σ then denotes the maximal spectral radius, taken over the whole set of matrices, whereas ρ is some number, defined by the set of matrices (again $\rho < \sigma$). Since we do not have something like a simultaneous Jordan form for a whole set of matrices, the approach has to be different. The approach in this paper will be based on a number of decomposition results for the set of matrices M ([12], [14]), which will be mentioned in the next section, after giving some definitions and notational conventions. After that we prove that $x(n)$ is bounded by a polynomial, similar to the right-hand side of (2); once having this result the polynomial expansion of $x(n)$ can be given immediately (section 3). To be more specific

*

For concepts like (ir)reducibility, aperiodicity, etc., compare Gantmacher [4].

it is possible to decompose the state space $S = \{1, \dots, N\}$ in sets $C^{(1)}, \dots, C^{(r)}$, such that, if $x^{(i)}$ denotes the restriction of a vector x to $C^{(i)}$ ($i = 1, \dots, r$), there exist strictly positive real numbers σ_i, ρ_i , and, also positive, integers v_i , together with finite sets of vectors $\{x_j^{(i)} ; j = 1, \dots, v_i\}$, for $i = 1, \dots, r$, such that

$$(3) \quad x^{(i)}(n) = \binom{n}{v_i - 1} \sigma_i^n x_1^{(i)} + \dots + \binom{n}{0} \sigma_i^n x_{v_i}^{(i)} + \sigma(\rho_i^n) ; i = 1, \dots, r.$$

with $\sigma_1 > \sigma_2 > \dots > \sigma_r$ and $\rho_i < \sigma_i$; $i = 1, \dots, r$. We will end with some concluding remarks. One technical detail will be proved in an appendix.

2. Preliminaries

We will work in the Euclidian space \mathbb{R}^N . Matrices, resp. (column)vectors, are denoted by upper, resp. lower, case letters. We say that matrix A is non-negative (positive)- denoted by $A \geq 0$ ($A \gg 0$) - if all its coordinates are nonnegative (positive). We say that A is semi-positive - denoted by $A > 0$ - if $A \geq 0$ and $A \neq 0$. Similar definitions apply to vectors. Finally, $[A]_i$ denotes the i -th row of the matrix A , $[A]_{ij}$ its ij -th element.

M is defined as a set of nonnegative $N \times N$ - matrices with the following property: if V is an arbitrary subset of $\{1, \dots, N\}$, then $P_1, P_2 \in M$ implies that P , defined by $[P]_i = [P_1]_i$, for $i \in V$, $[P]_i = [P_2]_i$, for $i \notin V$, is also an element of M .

We will refer to the indices $1, \dots, N$ as states; $S = \{1, \dots, N\}$ will be called the state space.

Next we summarize some results about nonnegative matrices. Let $\sigma(P)$ be the spectral radius of P . According to the well-known Perron-Frobenius theorem $\sigma(P)$ equals the largest positive eigenvalue of P and we can choose the corresponding eigenvector $\mu(P) > 0$. Recall that if P is irreducible then even $\mu(P) \gg 0$ and $\sigma(P)$ is a simple eigenvalue. If P is reducible then, eventually after permuting the states, we may write

$$P = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1s} \\ & P_{22} & \cdots & P_{2s} \\ & & \ddots & \\ & & & P_{ss} \end{pmatrix}$$

where each P_{ii} itself is irreducible with spectral radius $\sigma_i(P)$. We say that P_{ii} has access to $P_{\ell\ell}$ (or $P_{\ell\ell}$ is accessible from P_{ii}) if for some sequence of integers $k_0 = i < k_1 < \dots < k_\delta = \ell$ we have: $P_{k_{j-1}, k_j} > 0, j = 1, \dots, \delta$.

The sequence $\{P_{k_j, k_j}; j = 1, \dots, \delta\}$ is called a chain.

We say that P_{ii} is basic, resp. non-basic if $\sigma_i(P) = \sigma(P)$, resp. $\sigma_i(P) < \sigma(P)$. It is well-known (compare e.g. Gantmacher [4]) that $\mu(P) \gg 0$ if and only if each non-basic class of P has access to some basic class and no basic class is accessible to any other irreducible class of P .

The length of a chain is the number of basic classes it contains. The index $\nu(P)$ of P is defined as the length of its longest chain of irreducible classes *.

Having these concepts, we next formulate the decomposition result for sets of nonnegative matrices, which will be basic in this paper. It holds

Lemma 1 : There exists a matrix $\hat{P} \in M$, with spectral radius $\hat{\sigma}$ and index $\hat{\nu}$, and a partition $\{D_0, D_1, \dots, D_{\hat{\nu}}\}$ of the state space S , such that, after eventually permuting the states, we may write, for all $P \in M$:

$$(4) \quad P = \begin{pmatrix} P_{\hat{\nu}, \hat{\nu}} & P_{\hat{\nu}, \hat{\nu}-1} & \cdots & P_{\hat{\nu}, 1} & P_{\hat{\nu}, 0} \\ & P_{\hat{\nu}-1, \hat{\nu}-1} & \cdots & P_{\hat{\nu}-1, 1} & P_{\hat{\nu}-1, 0} \\ & & \ddots & & \\ & & & P_{1, 1} & P_{1, 0} \\ & & & & P_{0, 0} \end{pmatrix}$$

where $P_{i,j}$ is defined on $D_i \times D_j$ and $P_{i,j} = 0$ for $i < j, i, j = 0, \dots, \hat{\nu}$, and

* We remark that the given definition of $\nu(P)$ is a rather unusual one. For non-negative matrices however, it can be shown that this definition is completely equivalent with the traditional one (compare Rothblum [7])

for all $P \in M$. Furthermore there exist vectors $\hat{\mu}_i \gg 0$, such that

$$(5) \quad \max_{P \in M} P_{i,i} \hat{\mu}_i = \hat{P}_{i,i} \hat{\mu}_i = \hat{\sigma} \hat{\mu}_i \text{ for } i = 1, \dots, \hat{\nu}$$

while

$$(6) \quad \max_{P \in M} \sigma(P_{0,0}) = \sigma(\hat{P}_{0,0}) < \hat{\sigma}.$$

Finally, every basic class of $\hat{P}_{i,i}$ has access to some basic class of $\hat{P}_{i-1,i-1}$ for $i = 2, \dots, \hat{\nu}$.

Proof: The proof of this lemma may be found in Zijm [12]. Note that (5) implies that $\max_{P \in M} \sigma(P_{i,i}) = \sigma(\hat{P}_{i,i}) = \hat{\sigma}$ for $i = 1, \dots, \hat{\nu}$, since $\hat{\mu}_i \gg 0$.

□

Remark: It will be clear that we may formulate a similar decomposition result for the set of matrices $P_{0,0}$, defined on $D_0 \times D_0$, and with respect to $\sigma_0 = \max_{P \in M} \sigma(P_{0,0})$. Continuing in this way we obtain the complete block-triangular decomposition as it was formulated in [12].

We have to mention one elementary result for nonnegative matrices.

Lemma 2: Let P be an $N \times N$ -nonnegative matrix with spectral radius $\sigma > 0$ and strictly positive right eigenvector μ . Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} \sigma^{-m} P^m$$

exists and is again a nonnegative matrix. Denoting this matrix by P^* we have

$$P \cdot P^* = P^* \cdot P = \sigma \cdot P^*$$

If P is aperiodic then even

$$P^* = \lim_{n \rightarrow \infty} \sigma^{-n} P^n$$

Proof: Let $[\mu]_i$ denote the i -th element of μ . Define \bar{P} by

$$[\bar{P}]_{ij} = \sigma^{-1} [\mu]_i^{-1} [P]_{ij} [\mu]_j$$

then P is a stochastic matrix, for which the results are well-known (Kemeny and Snell [5]).

□

Having these preliminary results, we next turn to the dynamic programming recursion (1). In the following section we will determine the asymptotic behaviour of $x_i(n)$, the restriction of $x(n)$ to D_i ($i = 1, \dots, \hat{v}$) by an inductive argument.

3. Asymptotic behaviour of $x(n)$

In this section we suppose all matrices to be aperiodic. Denote by $x_i(n)$ the restriction of $x(n)$ to D_i ($i = 0, \dots, \hat{v}$), as defined in lemma 1. First we recall a convergence result for the sequence $\{x_i(n) ; n = 0, 1, 2, \dots\}$, which was proved in Zijm [13]. After that we establish, step by step, the polynomial boundedness and the polynomial expansion of $\{x_i(n) ; n = 0, 1, \dots\}$, for $i = 2, \dots, \hat{v}$. For simplicity we assume in the following $\hat{\sigma} = 1$ and hence $\max_{P \in M} \sigma(P_{0,0}) < 1$ (e.g. by multiplying all matrices with $\hat{\sigma}^{-1}$) First we have

Lemma 3: Let $\{x(n) ; n = 0, 1, 2, \dots\}$ be defined by recursion (1), and let $x_i(n)$ be the restriction of $x(n)$ to D_i , as defined in lemma 1, for $i = 0, \dots, \hat{v}$; $n = 0, 1, 2, \dots$. Then

$$\lim_{n \rightarrow \infty} x_i(n)$$

exists. Denote this limit by x_i , then $x_i \gg 0$.

Proof: Compare Zijm [13], theorem 5.

□

Lemma 4: Let $\{x(n) ; n = 0, 1, 2, \dots\}$ and $\{x_i(n) ; n = 0, 1, 2, \dots\} ; i = 0, 1, \dots, \hat{v}$ be defined as in lemma 3. Then there exists a δ , $0 < \delta < 1$ and a constant A , such that

$$\|x_0(n)\| \leq A\delta^n \quad (\|\dots\| \text{ denotes the usual sup-norm})$$

Proof: In Zijm [13], lemma 4, it was proved that

$$\|x_0(n)\| \leq B \binom{n+t-1}{t-1} \sigma_0^n$$

where B is some constant, $\sigma_0 = \max_{P \in M} \sigma(P_{0,0}) < 1$ and t the maximal index, associated with σ_0 . The result follows immediately for some δ with $\sigma_0 < \delta < 1$.

□

We are now ready to formulate the main theorem of this section. It holds :

Theorem 5: For $i = 1, \dots, \hat{\nu}$ there exist sequences of vectors $\{x_{i,j}; j = 1, \dots, i\}$ such that

$$(7) \quad x_i(n) = \binom{n}{i-1} x_{i,i} + \dots + \binom{n}{1} x_{i,2} + \binom{n}{0} x_{i,1} + \mathcal{O}(\rho^n)$$

with $\rho < 1$ and $x_{i,i} \gg 0$ ($i = 1, \dots, \hat{\nu}$).

Proof: The proof will be given in 3 steps, the first one dealing with $x_1(n)$.
 $\underline{1^{\circ}}$ From (1) we have for $x_1(n)$:

$$x_1(n) = \max_{P \in M} \{P_{1,1} x_1(n-1) + P_{1,0} x_0(n-1)\}$$

with $P_{1,0} x_0(n)$ converging to zero geometrically, for all $P \in M$ (lemma 4).
 Furthermore $\lim_{n \rightarrow \infty} x_1(n)$ exists (lemma 3) and will be denoted by x_1 ($x_1 \gg 0$).

Finally $x_1(n)$ approaches x_1 geometrically fast; in fact this is a direct consequence of a result of Federgruen and Schweitzer [3], as will be pointed out in the appendix (lemma A).

In the following two steps we suppose that (7) holds for $i = 1, \dots, k < \hat{\nu}$ and for some $\rho < 1$. In step 2 we prove that the difference between $x_{k+1}(n)$ and a very specific polynomial is bounded.

2^o Consider the following set of functional equations

$$\begin{aligned}
 (8.1) \quad & \max_{P \in M} \left\{ P_{k+1,k+1} y_{k+1,k+1} \right\} = y_{k+1,k+1} \\
 (8.2) \quad & \max_{P \in A_1} \left\{ P_{k+1,k+1} y_{k+1,k} + P_{k+1,k} x_{k,k} \right\} = y_{k+1,k} + y_{k+1,k+1} \\
 \vdots & \\
 (8.(k+1)) \quad & \max_{P \in A_k} \left\{ P_{k+1,k+1} y_{k+1,1} + P_{k+1,k} x_{k,1} + \dots + P_{k+1,1} x_{1,1} \right\} = y_{k+1,1} + y_{k+1,2}
 \end{aligned}$$

where $\{x_{i,j} ; i, j = 1, \dots, k ; i \geq j\}$ are supposed to be known (by the induction hypothesis) and $y_{k+1,j} ; j = 1, \dots, k+1$ are the unknown vectors. Furthermore A_i denotes the set of matrices which are maximizing the left-hand side of (8.i), for $i = 1, \dots, k$.

In [14] we proved that the equations (8) together possess a solution which determines $\{y_{k+1,j} ; j = 2, \dots, k+1\}$ completely, with $y_{k+1,k+1} \gg 0$. Furthermore it is clear that with $y_{k+1,1}$ also $y_{k+1,1} + \alpha y_{k+1,k+1}$ satisfies equation (8.(k+1)) hence we may choose $y_{k+1,1}$ so large that

$$(9) \quad x_{k+1}(0) \leq y_{k+1,1} .$$

By the induction hypothesis we have for some $\rho < 1$ and some constant $C_1 > 0$, for all n , and for $i = 1, \dots, k$:

$$(10) \quad x_i(n) \leq \binom{n}{i-1} x_{i,i} + \dots + \binom{n}{1} x_{i,2} + \binom{n}{0} x_{i,1} + C_1 \rho^n e$$

(where e denotes the vector with all components equal to one), and

$$(11) \quad x_0(n) \leq C_1 \rho^n e \quad (\text{lemma 4}) .$$

Finally, define $\gamma = \|\max_{P \in M} P e\|$.

Then, combining (8), (9), (10), (11) and

$$(12) \quad x_{k+1}(n) = \max_{P \in M} \left\{ P_{k+1,k+1} x_{k+1}(n-1) + \dots + P_{k+1,1} x_1(n-1) + P_{k+1,0} x_0(n-1) \right\}$$

we find immediately by induction:

$$x_{k+1}(n) \leq \binom{n}{k} y_{k+1,k+1} + \dots + \binom{n}{1} y_{k+1,2} + \binom{n}{0} y_{k+1,1} + (k+1) \gamma C_1 \frac{1 - \rho^n}{1 - \rho}$$

which implies that

$$(13) \quad x_{k+1}(n) - \left\{ \binom{n}{k} y_{k+1,k+1} + \dots + \binom{n}{1} y_{k+1,2} \right\}$$

is bounded from above.

In the same way we can establish a lower bound for the expression (13), since we may choose $y'_{k+1,1}$ (in the solution of (8.(k+1))) and $C_2 > 0$ in such a way that (9), (10) and (11) hold with \leq replaced by \geq , C_1 by C_2 and $y_{k+1,1}$ by $y'_{k+1,1}$. This completes the proof of step 2.

For the rest of the proof we define:

$$x_{k+1,j} = y_{k+1,j} \quad ; \quad j = 2, \dots, k+1$$

$$w_i(n) = x_i(n) - \left\{ \binom{n}{i-1} x_{i,i} + \dots + \binom{n}{1} x_{i,2} \right\}; i = 2, \dots, k+1; n = 0, 1, 2, \dots$$

$$w_1(n) = x_1(n) \quad ; \quad n = 0, 1, 2, \dots$$

3° In this step we finally prove (7) for $i = k+1$, which in our notations simply means that $w_{k+1}(n)$ must converge to some $x_{k+1,1}$ geometrically fast. Note that the induction hypothesis already implies the geometric convergence of $w_i(n)$ to $x_{i,1}$, for $i = 1, \dots, k$, whereas in step 2 the boundedness of $w_{k+1}(n)$ has been established. Since

$$\lim_{n \rightarrow \infty} \binom{n}{\ell} / \binom{n}{\ell-1} = \infty \quad ; \quad \ell = 1, \dots, k$$

it is easy to see that in the right-hand side of (12) we will first maximize terms of order $\binom{n-1}{k}$, then terms of order $\binom{n-1}{k-1}$, etc., for n large enough. Reasoning in this way it can be derived immediately that for some n_0 and $n \geq n_0$, (12) reduces to

$$x_{k+1,2} + w_{k+1}(n) = \max_{P \in A_k} \left\{ P_{k+1,k+1} w_{k+1}(n-1) + \dots + P_{k+1,1} w_1(n-1) + P_{k+1,0} x_0(n-1) \right\}$$

(by successive substitution of (8.1), (8.2) until (8.k)).

This equation can be written in the following form:

$$(14) \quad w_{k+1}(n) = \max_{P \in A_k} \left\{ P_{k+1,k+1} w_{k+1}(n-1) + r_{k+1}(P; n-1) \right\}$$

with, for all $P \in A_k$:

$$(15) \quad r_{k+1}(P; n-1) = P_{k+1,k} w_k(n-1) + \dots \\ + P_{k+1,1} w_1(n-1) + P_{k+1,0} x_0(n-1) - x_{k+1,2} .$$

Note that $r_{k+1}(P; n)$ converges geometrically fast to $r_{k+1}(P)$, defined by

$$(16) \quad r_{k+1}(P) = P_{k+1,k} x_{k,1} + \dots + P_{k+1,1} x_{1,1} - x_{k+1,2} .$$

Furthermore, let $\bar{P} \in A_k$ maximize the left-hand side of (8.(k+1)) for some $y_{k+1,1}$. Multiplying both sides of (8.(k+1)) with $\bar{P}_{k+1,k+1}^*$ gives, together with (16)

$$\bar{P}_{k+1,k+1}^* r_{k+1}(\bar{P}) = 0$$

and in general, for $P \in A_k$:

$$P_{k+1,k+1}^* r_{k+1}(P) \leq 0.$$

Having these results the geometric convergence of $w_{k+1}(n)$ to some $x_{k+1,1}$ is again a consequence of the result of Federgruen and Schweitzer [3], mentioned above. In our context the geometric convergence of $w_{k+1}(n)$ in (14) follows immediately from lemma A in the Appendix. This completes the proof of step 3.

Combining 1^o, 2^o and 3^o theorem 5 now follows immediately by induction. \square

Theorem 5 may be written in a more convenient way. Define

$$C_1 = D_1 \cup D_2 \cup \dots \cup D_{\hat{0}}$$

and vectors $x_j^{(1)}$ on C_1 by

$$x_j^{(1)} = \begin{bmatrix} x_{\hat{\nu},j} \\ x_{\hat{\nu}-1,j} \\ \vdots \\ x_{2,j} \\ x_{1,j} \end{bmatrix} ; \quad j = 1, \dots, \hat{\nu}$$

with $x_{i,j} \equiv 0$ for $i < j$, and $x_{i,j}$ defined by (7) for $i \geq j$; $i, j = 1, \dots, \hat{\nu}$. Then if we denote the restriction of $x(n)$ to C_1 by $x^{(1)}(n)$, we may write instead of (7):

$$(17) \quad x^{(1)}(n) = \binom{n}{\hat{\nu}-1} x_{\hat{\nu}}^{(1)} + \dots + \binom{n}{1} x_2^{(1)} + \binom{n}{0} x_1^{(1)} + \mathcal{O}(\rho^n)$$

with $\{x_j^{(1)} ; j = 1, \dots, \hat{\nu}\}$ satisfying the following equations:

$$(18) \quad \begin{cases} \max_{P \in M} P^{(1)} x_{\hat{\nu}}^{(1)} = x_{\hat{\nu}}^{(1)} \\ \max_{P \in M} P^{(1)} x_j^{(1)} = x_j^{(1)} + x_{j+1}^{(1)} ; \quad j = 1, \dots, \hat{\nu}-1. \end{cases}$$

(these equations can be derived immediately from (8), for $k+1 = 1, \dots, \hat{\nu}$).

In general we have (for $\hat{\sigma} \neq 1$):

$$(19) \quad x^{(1)}(n) = \binom{n}{\hat{\nu}-1}_{\hat{\sigma}} x_{\hat{\nu}}^{(1)} + \dots + \binom{n}{1}_{\hat{\sigma}} x_2^{(1)} + \binom{n}{0}_{\hat{\sigma}} x_1^{(1)} + \mathcal{O}(\rho^n)$$

with $\rho < \hat{\sigma}$.

Recall that for all P , $[P]_{ij} = 0$ for $i \in D_0 = S \setminus C_1$ and $j \in C_1$ (compare lemma 1), which means in particular that $x^{(0)}(n)$ (the restriction of $x(n)$ to D_0 , for $n = 0, 1, 2, \dots$) is completely determined by the matrices $P_{0,0}$. We may apply the same decomposition procedure to the set of matrices, defined on $D_0 \times D_0$, etc., etc., (the ultimate result can be found in Zijm [12]). It will be clear however that we may find a second set $C_2 \subset D_0$, together with a real

number σ_2 (the maximal spectral radius of the matrices, defined on $D_0 \times D_0$) and an integer ν_2 (the associated maximal index) such that for $x^{(2)}(n)$, the restriction of $x(n)$ to C_2 , we have a result similar to (19). Continuing in this way we will find the following theorem.

Theorem 6: There exists a partition $\{C_1, \dots, C_r\}$ of the state space S , such that, after eventually permuting the states, we may write, for all $P \in M$:

$$P = \begin{pmatrix} P^{(1)} & Q^{(1)} & \text{-----} & R^{(1)} \\ & P^{(2)} & \text{-----} & R^{(2)} \\ & & \text{-----} & \\ & & & P^{(r)} \end{pmatrix}$$

where $P^{(i)}$ is defined on $C_i \times C_i$ and $[P]_{k\ell} = 0$ for $k \in C_i, \ell \in C_j$ and $j < i$ ($i, j = 1, \dots, r$). Define σ_i and ν_i as follows:

$$\sigma_i = \max \{ \sigma(P^{(i)}) \mid P \in M \} \quad ; \quad i = 1, \dots, r$$

$$\nu_i = \max \{ \nu(P^{(i)}) \mid P \in M, \sigma(P^{(i)}) = \sigma_i \} \quad ; \quad i = 1, \dots, r$$

where $\nu(P^{(i)})$ denotes the index of $P^{(i)}$ with respect to $\sigma(P^{(i)})$. Then there exist sequences of vectors of appropriate dimension, denoted by $\{x_j^{(i)}; j = 1, \dots, \nu_i\}$, for $i = 1, \dots, r$, which satisfy the following functional equations:

$$\max_{P \in M} P^{(i)} x_{\nu_i}^{(i)} = \sigma_i x_{\nu_i}^{(i)}$$

$$\max_{P \in M} P^{(i)} x_j^{(i)} = \sigma_i (x_j^{(i)} + x_{j+1}^{(i)}) \quad ; \quad j = 1, \dots, \nu_i - 1$$

while furthermore under the aperiodicity assumption, we have for $x(n)$, defined by (1):

$$x^{(i)}(n) = \binom{n}{\nu_i - 1} \sigma_i^n x_{\nu_i}^{(i)} + \dots + \binom{n}{1} \sigma_i^n x_2^{(i)} + \binom{n}{0} x_1^{(i)} + O(\rho_i^n); \quad n = 0, 1, 2, \dots$$

with $x^{(i)}(n)$ denoting the restriction of $x(n)$ to C_i , and with $\rho_i < \sigma_i$ ($i = 1, \dots, r$). Furthermore it holds :

$$\sigma_1 > \sigma_2 > \dots > \sigma_r.$$

□

With theorem 6 we completed the description of the asymptotic behaviour of $x(n)$, defined by (1), under the aperiodicity assumption, mentioned at the beginning of this section. It will be clear that we may weaken this assumption, e.g. by assuming aperiodicity only for those matrices which appear infinitely often as the maximizing one in the dynamic programming recursion. Also notice that by a simple data-transformation we can obtain a system in which every matrix becomes aperiodic (compare Schweitzer [9]), whereas the chain structure of the matrix remains completely the same. To be specific, define for all $P \in M$

$$\tilde{P} = \alpha P + (1 - \alpha)I \quad (0 < \alpha < 1) .$$

Then \tilde{P} possesses the same eigenvectors as P , and $\sigma(\tilde{P}) = \alpha\sigma(P) + (1 - \alpha)$. Furthermore, if P^* exists, then \tilde{P}^* exists and $P^* = \tilde{P}^*$.

We may even drop completely the aperiodicity assumption and obtain analogous results for certain subsequences of $\{x(n) ; n = 0, 1, 2, \dots\}$ (compare Sladký [11]).

Finally, the results obtained in this paper may be useful for estimating $\sigma_i = \max_{P \in M} \sigma(P_{i,i})$ and $x_j^{(i)}$ for $j = 1, \dots, v_i ; i = 1, \dots, r$ (compare Sladký [11], Zijm [13]).

Appendix

In this appendix we will treat some details, which were omitted in the proof of theorem 5, step 1 and step 3. The results will be based on a paper by Federgruen and Schweitzer [3], however we will treat a somewhat simpler case than they did. Suppose we have the following situation :

A Markov decision process with discrete time space, finite state space $S = \{1, \dots, N\}$ and finite action space K . In each state $i \in S$ we may take action k which moves the system to state j at the next decision epoch with probability p_{ij}^k while furthermore a reward r_i^k is earned. A policy f is a function from the state space to the action space with which we associated the matrix $P(f)$ and the column-vector $r(f)$, defined by

$$P(f) = \left(p_{ij}^{f(i)} \right)_{i,j \in S} \quad ; \quad r(f) = \left(r_i^{f(i)} \right)_{i \in S} .$$

Furthermore we assume

$$\sum_{j \in S} p_{ij}^k = 1 \quad (i \in S, k \in K) .$$

With each $r(f)$ we associate an approximating sequence $\{r(f,n) ; n = 0,1,\dots\}$ such that $\lim_{n \rightarrow \infty} r(f,n) = r(f)$ geometrically, i.e. there exist a $\rho < 1$ and $C > 0$ such that, for all f

$$\|r(f,n) - r(f)\| \leq C \cdot \rho^n .$$

Finally define the maximal gain vector g^* by

$$(a) \quad g^* = \max_f P^*(f) r(f)$$

and assume all $P(f)$ to be aperiodic. Let v_0 be a fixed vector and define v_n , $n = 1,2,\dots$ by

$$(b) \quad v_n = \max_f \{r(f,n-1) + P(f)v_{n-1}\}$$

From Federgruen and Schweitzer [3] we conclude:

There exists a vector w^* such that

$$(v_n - ng^* - w^*)$$

converges to zero geometrically fast, if n tends to infinity.

Notice that this Markov decision process describes precisely the situation which we encountered in the proof of theorem 5, step 1 and step 3, except for the fact that the matrices $P_{1,1}$, resp. $P_{k+1,k+1}$, were not necessarily (sub)stochastic. However, recall that there exist strictly positive vectors $\hat{\mu}$, resp. $\hat{\mu}_{k+1}$, such that (5) in lemma 1 holds. A transformation like the one in the proof of the lemma 2 then yields immediately the situation in which the matrices are (sub)stochastic. Furthermore, both in step 1 and in step 3 we have g^* , defined above, is equal to zero (in step 1 we have $\lim_{n \rightarrow \infty} x_0(n) = 0$ geometrically, in step 3 $g^* = 0$ follows from the last two formulas in the proof). We finally summarize the result:

Lemma A: Suppose we have a Markov decision as described above, except for the fact that the transition matrices are not necessarily (sub)stochastic; however for some $\mu > 0$ we have

$$\max_f P(f) \mu \leq \mu$$

Let furthermore all the matrices be aperiodic and suppose g^* , defined by (a), is equal to zero. Then there exists a w^* such that for $\{v_n; n = 0, 1, 2, \dots\}$ defined by (b) we have

$$\lim_{n \rightarrow \infty} v_n = w^* \text{ geometrically.}$$

□

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