MASTER

Equilateral triangle map folding

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Equilateral triangle map folding

Master Thesis

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Abstract

We study map folding for a map with an equilateral triangle grid. The map folding problem asks what the computational complexity is to decide whether a crease pattern on a regular grid can be flat folded, respecting all creases. A map is flat foldable if all creases can be folded flat without cutting the paper, moving through itself, stretching or shrinking the paper. In this thesis we introduce the equilateral triangle grid and explore the problem of deciding whether a given crease pattern can be flat folded under different folding models.

Different sizes of paper are considered and explored for flat foldability. We expand on recent findings for strip folding and show that an equilateral triangle strip always has a flat folded state. We provide insight in the simple foldability of an equilateral triangle strip and conjecture that an equilateral triangle strip is always flat foldable. We provide insight in a new reachability problem using equilateral triangle strip folding. We show the reachable locations and how we can reach each location. Using this result we use equilateral triangle strip folding to express a new variation of the Traveling Salesman Problem. We show that deciding flat foldability on an equilateral triangle map with assigned creases and polyiamond paper is NP-complete for the one-layer, some-layers and all-layers simple folding models. We generalize this result and show that deciding flat foldability on an equilateral triangle map with assigned creases and hexagonal paper is NP-complete for the some-layers and all-layers simple folding models. Furthermore, we show that an unassigned equilateral triangle map with polyiamond paper is also NP-complete for the some-layers and all-layers simple foldability models.

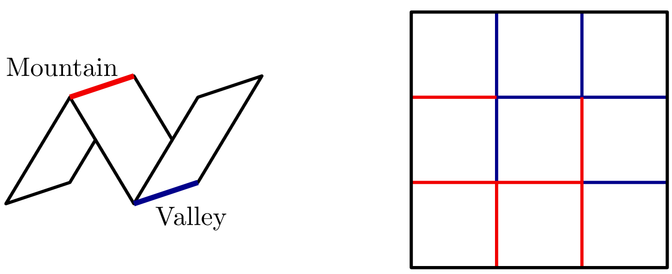
Contents

Chapter 1

Introduction

Origami has been studied in many different problem settings. Origami is the art of folding paper (derived from the words 'oru' which means folding and 'kami' which means paper). In origami, one tries to fold some shape from a single piece of paper without cutting the paper, moving through itself, stretching or shrinking the paper. Traditionally origami uses a square or rectangular piece of paper. We can generalize this by folding a polygonal piece of paper. Paper is formally defined as a connected polygon, possibly with holes. This two dimensional piece of paper is flat.

When folding the piece of paper, we fold creases into the paper. Repeatedly folding a piece of paper puts multiple creases into the paper. When a folded piece of paper is unfolded to its original unfolded state, we are left with some creases on the piece of paper. This is called a *crease pattern*. Fundamentally, a crease pattern is a collection of line segments (creases) on the piece of paper. These creases can be assigned either to be a *mountain fold*, folding the crease to a protruding ridge, or a *valley fold*, folding the crease to a indenting trough. The mountain folds are indicated by a red line segments and the valley folds are indicated by a blue line segment. A visual example of a mountain and valley fold is shown in Figure 1.1(a). An example of a crease pattern is shown in Figure 1.1(b).



(a) (b)

Figure 1.1: A mountain and valley fold (a) with an assigned crease pattern containing both mountain and valley folds (b). Mountain folds are shown in red and valley folds are shown in blue.

1.1 Related work

Origami is one part of the bigger field of folding algorithms. Within folding algorithms there are three main fields: Linkages, Paper, and Polyhedra [9]. These fields encompass topics from protein folding [12] to wrapping packages [11]. The field of folding is too large to give a complete overview. Therefor we focus on the relevant parts, which is paper folding.

Origami is part of the field of paper folding. Paper folding has roughly two main fields: design and foldability. The field of paper design asks whether a given piece of paper can be folded into an object with some property, like a desired shape. For example, this can be through committing a cardinal sin of origami folding: cutting the paper. The fold and cut problem asks whether we can fold a piece of paper such that we can obtain any desired shape by performing a single cut through the paper and unfolding it [8][7].

This thesis mainly focuses on the foldability field. Foldability asks which crease patterns can be folded using exactly the given creases. Any shape can be considered, whether it is in two dimensions or three dimensions. A particular problem which has been the focus of research is origami which folds from a two dimensional crease pattern into a two dimensional shape. This is also called flat-folding.

Problem statement 1 (**Flat folding**, [9])**.** *Given is a piece of paper with a crease pattern, possibly with a given mountain-valley assignment. Does there exist a flat folding which satisfies the given crease pattern?*

Flat foldability has been the main interest of research in foldability. A couple of interesting properties have already been proven for flat foldable crease patterns. The Kawasaki theorem is a very useful and simple condition for flat foldable origami. It is a condition on the angles and the amount of creases meeting in a single vertex.

Theorem 1 (Kawasaki, [20])**.** *A single-vertex crease pattern defined by angles* $\theta_1 + \theta_2 + \cdots + \theta_n = 360°$ *is flat foldable if and only if* n *is even and the sum of the odd angles* θ_{2i+1} *is equal to the sum of the even angles* θ_{2i}*.*

Theorem 1 is a condition which has to hold for any local vertex in a flat foldable crease pattern. This local flat foldability problem can be used for the general flat foldability problem. The general flat foldability problem is the main question for flat foldability: given a crease pattern, does it have a flat folded state? This general flat foldability problem is NP-hard [6].

There are variations of the flat foldability problem for which the complexity could be different. One such a variation is the map folding problem. Map folding is a sub problem of the general flat foldability problem where the paper is partitioned into an $m \times n$ regular grid of squares where the boundary of each square is a crease with either a mountain or valley assignment. An example of such a map is shown in Figure 1.1(b). Understanding map folding does not only help understanding the mathematics behind artistic origami, but can also be applied in sheet metal manufacturing or carton boxes for packaging where folding is employed to create shapes out of flat sheets of material.

The main question in map folding is still open.

Open Problem 1. *What is the computational complexity of deciding whether a square* $m \times n$ *grid has a flat folded state with a specified mountain-valley assignment?*

This problem is complex, but there has been progress on sub-problems of the map folding problem. These sub-problems divide the map folding problem in different paper shapes, folding models and crease directions. As we are looking at flat foldability, we define what a flat folded state is.

Definition 1 (**Flat folded state**, [13])**.** *A* flat folded state (f, λ) *of a piece of paper* P *is a function* $f : P \to \mathrm{R}^3$ *which maps* P *into Euclidean 3-space with a function* λ *specifying the local stacking order of pairs of points in contact.*

This definition basically states that when a piece of paper P is folded flat, there can be points where the paper overlaps with other pieces of paper. The order in which the pieces of paper are overlapping is called the stacking order.

The shape of the paper is important as to how restrictive any folds are. If we have a map of the shape $1 \times n$, it is linear to decide whether such a map is flat foldable [5]. There has been progress on maps of the shape $2 \times n$ where a polynomial algorithm was found to decide whether such a map is flat foldable [23][1]. The paper can also be some polygonal shape. For the square map folding problem, the polygon has to be orthogonal with creases which are parallel to the edges of the polygon. This problem was shown to be NP-complete [3].

With folding models we consider different kind of folding operations. There are *simple* folds and *general* folds. A simple fold is a fold which is non-intersecting in the folding motion. This means that the paper can not bend during the folding motion. The paper behaves like a rigid sheet. Furthermore, with simple folding only one (set of) crease(s) is folded at a time. General folds do not have these restrictions. Literature has been mostly focused on simple folds as it has practical applications in sheet-metal bending. With the simple fold model we can only fold a single crease along a line segment at once. Arkin et al. [5] introduced simple foldability and introduced a couple of different models for simple folding.

There are three simple folding models (see Figure 1.2). These are one-layer, some-layers and all-layers simple folds. In the one-layer model only a single fold can be folded at a time, even if there are multiple foldable creases on top of each other. In this model only paper on the top side of the paper can be folded, otherwise it is always blocked by some other piece of paper. In the some-layers simple fold model some of the paper can be folded. This model is the general case as any amount of layers can be folded, one layer, all layers or somewhere in between. The all-layers model is when all layers of a crease have to be folded. This case is most interesting to manufacturing, as this is the best to automate for heavy machinery.

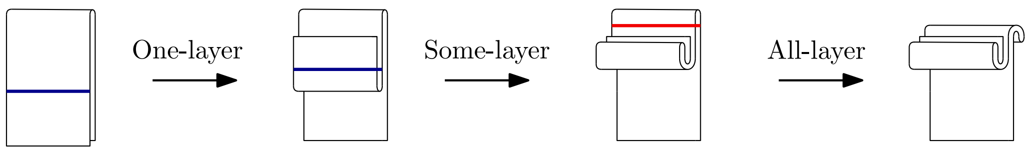


Figure 1.2: Three simple folding models. Folding sequence with the three folding models is shown.

Simple folding is still hard for different shapes of paper, but efficient results have been found for the traditional $m \times n$ map folding problem in particular simple folding models [5][3][2].

Other variations can be achieved with different crease directions. The traditional map folding has creases on only two axes, horizontal and vertical. What if we only add diagonals to traditional map folding? Such a crease pattern is called a box-pleating pattern. Recent research suggests that this is hard for $m \times n$ maps [1][3], but easier for $1 \times n$ maps [18]. Progress has also been made on $2 \times n$ maps with a box-pleating pattern where a linear time algorithm was found by showing that local foldability implies global foldability [19].

These different variations can be evaluated for different paper shapes, crease directions and the assigned or unassigned version. This can then be evaluated for the different folding models.

We define three different crease patterns and paper shapes which have been studied in the literature. We abbreviate the different paper shapes and crease patterns: Orthogonal paper with orthogonal creases ⧈, square paper with 45° creases ⊠, or rectangular paper with orthogonal creases ⊞.

Model	**Assigned**			**Unassigned**		
	⧈	⊠	⊞	⧈	⊠	⊞
One-layer	Strong [3]	Open	Poly [5]	Open	Open	Poly [5]
Some-layers	Strong [3]	Strong [3]	Poly [5]	Strong [3]	Strong [3]	Poly [5]
All-layers	Strong [3]	Strong [3]	Poly [5]	Strong [3]	Strong [3]	Linear [2]

Table 1.1: Current results for the complexity of two dimensional simple folding problems. Problems are either open, solvable in linear time (Linear), solvable in polynomial time (Poly) or strongly NP-complete (Strong).

[1]This is an unpublished result, but generally accepted.

The first research papers on map folding related problems were for specific one-dimensional maps, called strips. One of the first studies was called stamp folding where combinatorial results were shown for the amount of ways of folding the stamp [21]. A stamp is a strip with all creases. A stamp is trivially flat foldable for any crease assignment. These results are not for specific mountain or valley fold combinations, but are for the unassigned version of the problem.

For one-dimensional folding problems, the unassigned version is trivial as all creases can be assigned alternating mountain/valley to obtain a flat folding. Arkin et al. [5] showed a linear algorithm to solve orthogonal simple folding in the one-layer, some-layers and all-layers model by showing that local foldability implies global foldability for the assigned version of the problem.

When the creases are no longer orthogonal to the paper, the paper is folded in two dimensions. Demaine et al. [10] showed that strips with a zig-zag pattern where the consecutive creases have a non-acute angle are always foldable. A NP-completeness proof for one-dimensional strip folding with diagonal creases was published [18]. They argued NP-completeness by reduction from the Partition problem.

1.2 Contributions

We introduce a new map folding variation: the equilateral triangle map. This variation changes the type of grid. All previous map folding problems have been in a square grid. The equilateral triangle map uses a grid from equilateral triangles. A polyform with the equilateral triangle as its base form is called a *polyiamond*. For a polyiamond, all equilateral triangles are equally sized. An example of an equilateral triangle map is shown in Figure 1.3.

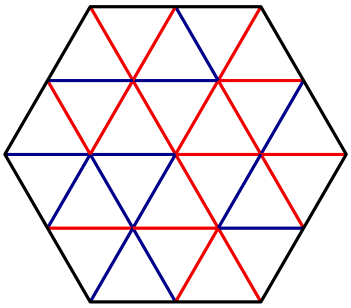


Figure 1.3: An example of an equilateral triangle map with an assigned crease pattern.

This type of map can still satisfy Kawasaki's theorem (Theorem 1) as each vertex of the crease pattern contains an even amount of creases. This new type of map poses a new open problem in map folding for equilateral triangle maps.

Open Problem 2. *What is the computational complexity of deciding whether an $m \times n$ equilateral triangle grid can be folded flat (has a flat folded state) with a specified mountain-valley assignment?*

In this thesis we distinguish between two types of maps: one-dimensional and two-dimensional. The map shown in Figure 1.3 is an example of a two-dimensional map. When we take a single strip from this two-dimensional map, we call this the one-dimensional equilateral triangle problem, or an equilateral triangle strip. An example of this is shown in Figure 1.4.

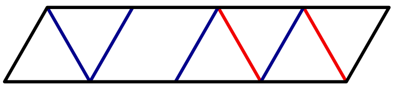


Figure 1.4: An example of an equilateral triangle strip with an assigned crease pattern.

Chapter 2 shows results and an exploration for one-dimensional equilateral triangle map folding. In Section 2.1 we look at the folding motions of a one dimensional equilateral triangle strip.

Because of the direction in which these creases are oriented, they have interesting behaviour in which locations the strip can reach. Because of the reachability of the strip, some routing problems emerge from this. We show a variation of the Traveling Salesman Problem (TSP) for folding an equilateral triangle strip. Furthermore, the equilateral triangle strip flat foldability problem is introduced in Section 2.2. We show that general equilateral triangle strip flat foldability is always possible. Simple foldability is more difficult, as we show intuition and results from computing a large amount of strips using brute force. We conjecture that simple flat foldability for equilateral triangle strips is always possible. Furthermore, we show that the $1 \times n$ NP-completeness argument from Jia et al. [18] is insufficient and conjecture, based on the results from equilateral triangle strip simple foldability, that the actual complexity is polynomial. Table 1.2 shows the current one-dimensional results and the problems which this thesis addresses.

Model	**Assigned**		
	□	⧄	▽
Simple flat foldability	Linear [5]	**Open**	Open
General flat foldability	Linear [5]	Yes [10]	**Yes**

Table 1.2: Results for the complexity of one-dimensional folding problems. Problems are either open, solvable in linear time (Linear), or always flat foldable (Yes). **Bold** results are new in this thesis.

Chapter 3 looks at two-dimensional flat foldability for equilateral triangle creases. We introduce the new problems for one-layer, some-layers and all-layers for different types of equilateral triangle paper shapes. We show strong NP-completeness for polygonal paper with equilateral triangle creases ⧖ and hexagonal paper with equilateral triangle creases ⎔. The unassigned version has a conjecture of NP-completeness with equilateral triangle creases.

Model	**Assigned**		**Unassigned**	
	⧖	⎔	⧖	⎔
One-layer	**Strong**	**Strong**	**Strong**	Open*
Some-layers	**Strong**	**Strong**	**Strong**	Open*
All-layers	**Strong**	Open	**Strong**	Open

Table 1.3: Results for the complexity of two dimensional equilateral triangle simple folding problems. Problems are either open or strongly NP-complete (Strong). **Bold** results are new in this thesis. Problems with an asterisk (*) have a conjecture of strong NP-completeness.

Chapter 2

One-dimensional equilateral triangle folding

One-dimensional map folding is often regarded as a simpler crease pattern. It simplifies the general map folding problem by only considering a single strip. A nice property is that all creases are locally simple foldable, or it would not be one-dimensional. A one-dimensional equilateral triangle map is a strip of alternating oriented equilateral triangles as shown in Figure 2.1. The length of the strip is the amount of equilateral triangles in the strip.

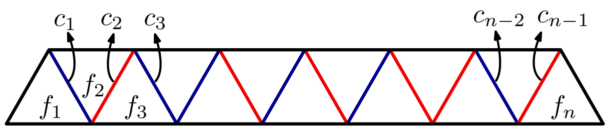


Figure 2.1: An example of an equilateral triangle strip of length n. It has $n-1$ creases $\{c_1, ..., c_{n-1}\}$ and n faces $\{f_1, ..., f_n\}$.

In this chapter we look at two different problems for one-dimensional equilateral triangle strips. In Section 2.1 we will look at reachability for equilateral triangle strips. We consider the question: Given some strip S, what is the set of reachable locations of S. Because the creases of the strip are not parallel to each other, the strip folds in different directions depending on which crease is folded. This means that the strip can reach a large range of different locations. We look at how we can reach the locations in the reachable set of locations S. Using this, we can express interesting routing problems with a folding problem. We look at the Traveling Salesman Problem (TSP) for its applications in robotics. We give lower and upper bounds for the folding TSP problem and conjecture the complexity of equilateral triangle folding TSP. Furthermore, we experimentally evaluate reachability.

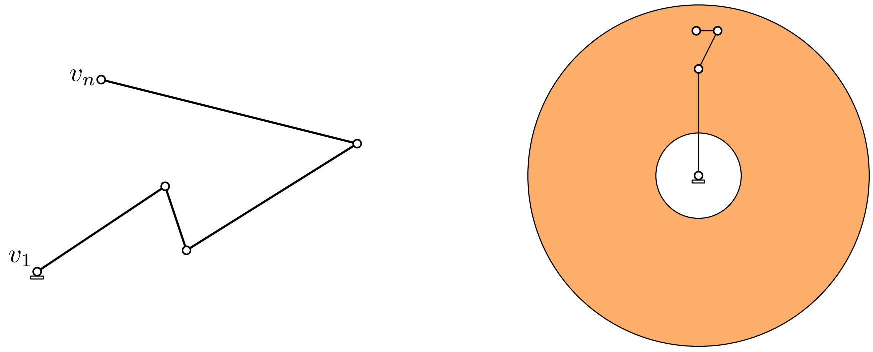
Section 2.2 looks at the flat foldability for the equilateral triangle strip. We ask the question: Given some strip S, does there exist a (simple) flat foldable state? We show a proof for the general folding model and provide a conjecture for the simple folding model. A large amount of strips was experimentally evaluated for simple flat foldability. From these results we show some combinatorial observations for the equilateral triangle strip.

2.1 Reachability

We consider the following problem statement for equilateral triangle reachability:

Problem statement 2 (Reachability). *Let S be an equilateral triangle strip of length n and $n-1$ creases without an assignment. What is the set of reachable locations of S? Show how we can reach the reachable locations.*

Reachability for folding algorithms has been explored for linkages [9][17]. A *linkage* is a collection of line segments joined at their end points. The line segments can have different lengths. The main type of linkages which have been part of the literature have been fixed-length linkages. Fixed-length linkages are linkages where the line segments are rigid and can not change their lengths. The line segments can rotate at any angle around the its joints. The applications of linkages are varied from robotics to protein folding [22][12]. A linkage is a collection of fixed-length one-dimensional line segments joined at their endpoints (like a robot arm). An example of a linkage is shown in Figure 2.2. The endpoints between each line segment are called *joints*. The line segments can rotate about their incident joints. A linkage which is fixed in place at one end is called an *arm*. The last endpoint in the linkage is called the *hand*. We use the same definitions for equilateral triangle strip folding reachability as has been defined for linkages.



(a) (b)

Figure 2.2: An example of a linkage. In (a), vertex v_1 is fixed at the origin of the coordinate system. The entire construction is called the *arm*. The final vertex v_n is called the *hand*. In (b) we can see the reachable locations of the illustrated strip indicated by the orange area.

Let S be an equilateral triangle strip of n evenly sized equilateral triangles $\{f_1, ..., f_n\} = S$ where each $f_i \in S$ is the same size (see Figure 2.1). The first triangle f_1 is fixed in place at the origin of a coordinate system. For reachability, we will use a Cartesian coordinate system as shown in Figure 2.3. In this coordinate system, moving right on the the horizontal axis increases the x-coordinates and going up the vertical axis increases the y-coordinate. Let there be $n - 1$ creases $c_1, ..., c_{n-1}$ between each pair of equilateral triangles. Comparing this structure to linkages, a crease is a joint, a single triangle is a segment and the entire strip is the arm. The final triangle f_n of the strip is the hand. Now we ask the following question: What are the reachable locations of the hand f_n?

Whenever a crease of the strip is folded, the hand moves to some location. A visual example of a strip folding to different locations is given in Figure 2.4. We are interested in the reachable locations and within how many folds a location can be reached. The folds can be along any crease line on the equilateral triangle strip. The fold direction (mountain or valley) of the crease is not prescribed.

This way of looking at reachability can give a new perspective on robotic arm technology. The strip moves between different discrete locations by folding creases. In the strip folding problem the locations are not separated by distance, but by folds. If we fold a crease on the strip, the hand of the strip will fold to a different location. The new location can be on the opposite side of the reachable locations, if the two locations are separated by a single fold. We can see this as a different path finding problem where locations are not separated by a euclidean distance, but by a number of folds. This is a novel way of looking at path finding problems.

Applications can range from assembly lines to PCB (Printed Circuit Board) constructions. These problems require an arm to place multiple components in certain locations as quickly as possible. This often relies on finding the shortest tour between multiple locations. This is the classical traveling salesman problem and is unfortunately NP-hard when locations are separated

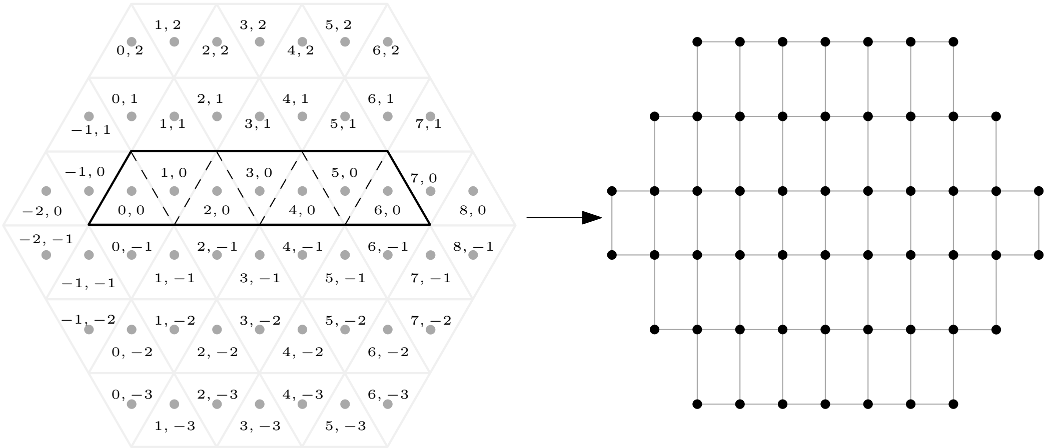


Figure 2.3: Illustration of the Cartesian coordinate system for equilateral triangles. The first triangle of the strip $f_1 \in S$ is fixed at the origin of the grid at $(0, 0)$. Each dot represents the coordinates of one of the triangles in the square Cartesian grid.

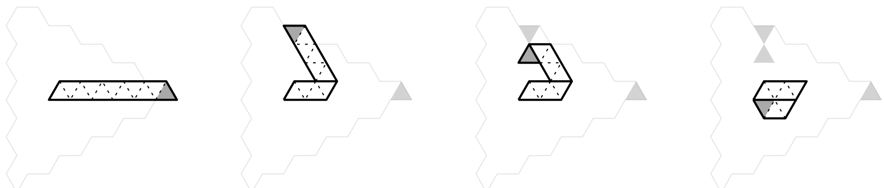


Figure 2.4: Example of a strip folding to different locations. The maximum reachable locations are indicated by the border around the strip. The previous locations are also marked.

by euclidean distances.

Before we look at any path finding problems, we look at the locations that can be reached by the equilateral triangle strip.

2.1.1 Strip folding reachability

In this section we will show the reachable locations for equilateral triangle strips. We show that each location can be reached with at most four folds.

We start by showing the amount of reachable locations of the equilateral triangle strip. We first look at a single line of equilateral triangle locations, which is perpendicular to one of the creases. We consider the folding locations in the Cartesian grid. We will first look at the amount of folds which are needed in order to reach a location along the same line perpendicular to the creases. This does not consider a strip, only a single location and the location it would fold to were it to fold along some crease.

Lemma 1. *Given a location p_0 and a line l through p_0 and perpendicular to one of the crease directions, we can reach any location along l by folding at most 2 creases c_i, c_j perpendicular to l.*

Proof. Given is the initial location p_0 and the perpendicular line l as illustrated in Figure 2.5. Folding along a crease c_i reflects p_0 to a new location reflected across c_i. For every c_i we get that folding c_i will reflect p_0 from $(0, 0)$ to $(0, -2i - 1)$. Thus from folding any single crease, we get a coordinate of odd parity. These locations are indicated in Figure 2.5 by p_1, p_3 and p_5. Now notice that when folding c_0, the coordinate shifts by one. This means that if we first fold c_0 and then another c_j as a second fold, we obtain a different parity as the second fold will reflect to $(0, -1 - 2j - 1) = (0, -2 \cdot (j - 1))$. This new location clearly has an even coordinate which covers the locations of even parity. These locations are indicated in Figure 2.5 by p_2 and p_4.

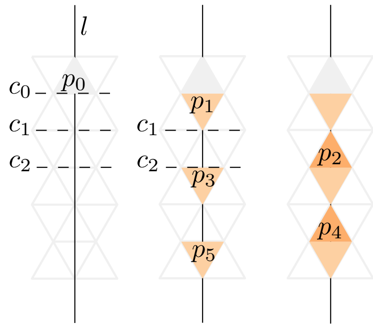


Figure 2.5: Folding some location along a perpendicular line l to the creases. We indicate the locations $p_0, p_1, p_2, p_3, p_4, p_5$ which have the coordinates $(0, 0); (0, -1); (0, -2); (0, -3); (0, -4); (0, -5)$ respectively. p_1, p_3, p_5 are the locations which can be reached from p_0 by folding c_0, c_1 or c_2. p_2 and p_4 are the locations which can be reached from p_1 by folding c_1 or c_2.

Thus we can reach all locations of odd parity within one fold and all locations of even parity within two folds. A similar proof holds when rotating the line l to be perpendicular with differently oriented creases. □

Lemma 1 showed that every location in one line is reachable within two folds. This was with any amount of creases up to i We now prove the amount of locations which can be reached given some amount of consecutive creases. This setting is derived from the normal strip folding instances which we encounter. This setting is shown in Figure 2.6.

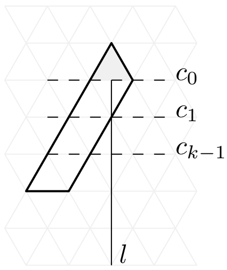


Figure 2.6: Reachability of a strip of length $k + 1$ folding along a line l with all possible parallel consecutive creases.

Lemma 2. *Given a location p_0, k parallel creases $\{c_0, ..., c_{k-1}\} = C$ where c_0 is adjacent to p_0 and the remaining $\{c_1, ..., c_{k-1}\}$ creases are along the next $k - 1$ grid lines on the same side of p_0 as c_0, and a line l through p_0 and perpendicular to all $c \in C$, we can reach $2k$ triangles along l.*

Proof. This setting is visualized in Figure 2.6. The reachable locations given any crease from a given location was shown in Lemma 1. We now consider the case where we have k creases $\{c_0, ..., c_{k-1}\}$ all adjacent to each other. An example is shown in Figure 2.5 with creases c_0, c_1 and c_2. Since we reflect along the lines, we can reach k new locations along l for each crease. These locations all have the same parity, thus we switch parity by folding c_0. This leaves $k - 1$ creases, which gives $k - 1$ new locations which we can fold to. With the original location of p_0, this yields $2k$ reachable locations along the perpendicular line l. □

Given Lemma 1 and Lemma 2, we obtained two tools in order to prove how many reachable locations an equilateral triangle arm has. The reachable locations depend on the length of the strip. We could shorten the strip by one by folding the end of the strip. This would give a strip of $n - 1$, which has its own reachable locations. We call the shortening of the strip an *end fold*.

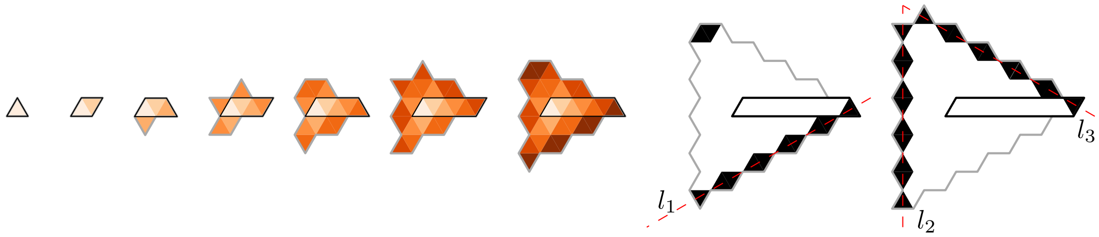


Figure 2.7: Reachable locations per strip length. New locations per strip length are indicated by a diverging color. The new locations per strip of length n compared to the strip of length $n-1$ are indicated.

We use end folds of the strip throughout the proof to cover reachable locations which could be reached by smaller strips. We show a proof by induction for the amount of reachable locations.

Lemma 3. *The amount of reachable locations of an equilateral triangle strip of length n is:*

$$a(n) = \begin{cases} n & \text{if } 0 \leq n \leq 2 \\ n+1 & \text{if } n = 3 \\ a(n-1) + n + 1 & \text{if } n \text{ is odd and } n > 3 \\ a(n-1) + 2 \cdot (n-2) & \text{if } n \text{ is even and } n > 3 \end{cases} \tag{2.1}$$

Proof. We show this with a proof by induction. Examples for different strip lengths are given in Figure 2.7. A strip of length zero is trivial. The strip of length one does not contain any creases, thus the hand of the strip is the strip itself. The only reachable location for a strip of length one is the location of the single triangle. A strip of length two only has a single crease. We add the new location of the hand of the strip to the reachable locations. Folding the end of the strip gives the location of the strip of length one, thus we can reach two locations. The strip of length three has four reachable locations. It has four reachable locations as shown in Figure 2.7.

There are two different cases for reachable locations of an equilateral triangle strip. These are those for a strip of odd length, and the reachable locations for an even length. These two different cases are illustrated in Figure 2.7 marked by the black triangles. These two cases emerge as a consequence of the alternating orientation of the hand of the strip.

For a strip of length n, we can clearly reach all locations of the strip of length $n-1$ by end folding. Now distinguish the different parity cases on the length of the strip.

For a strip of odd length, there is a new line of reachable locations along the bottom of the previous reachable locations indicated by l_1. Along this line there are $(n-1)/2$ perpendicular creases from the strip. By Lemma 2 we get $n-1$ new locations. At the top of the reachable locations there is only a single crease parallel to that segment, which means we get two new reachable locations at the top. This means that in total we have $n+1$ new reachable locations. Thus for a strip of length n where n is odd we get $a(n-1)+(n+1)$ reachable locations.

For a strip of even length, there are two new lines compared to the previously reachable locations which are indicated by l_2 and l_3. For the top line indicated by l_3, we have $(n-2)/2$ creases which are perpendicular to it, which means we get $n-2$ new locations by Lemma 2. The line l_2 uses the same creases after the first crease of the strip has been folded. This means that along this line there are $n-2$ new locations as well. This means that there are $2 \cdot (n-2)$ new reachable locations. Thus for a strip of length n where n is even we get $a(n-1) + 2 \cdot (n-2)$ reachable locations. □

We now look at how we can reach the reachable locations. We specifically look at how many folds it takes to get to a location. A visual representation of the minimum amount of folds which it takes to reach a location is given in Figure 2.8.

Theorem 2. *If some coordinate can be reached by strip folding, it can be reached with at most 4 folds.*

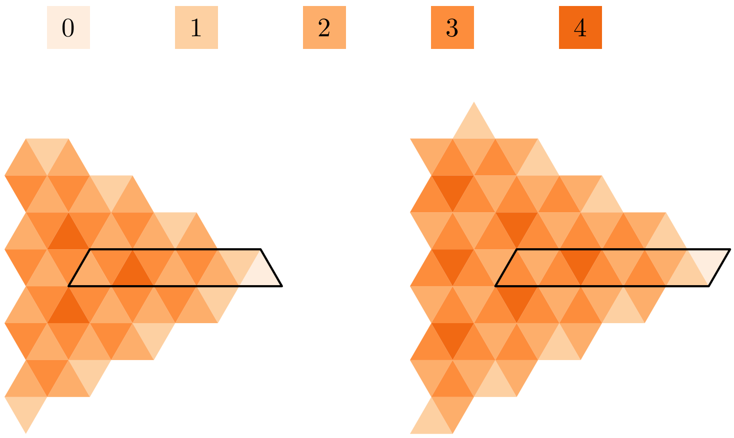


Figure 2.8: Reachable locations with the minimum amount of crease folds. The minimum required folds for each location is indicated with the color of the locations.

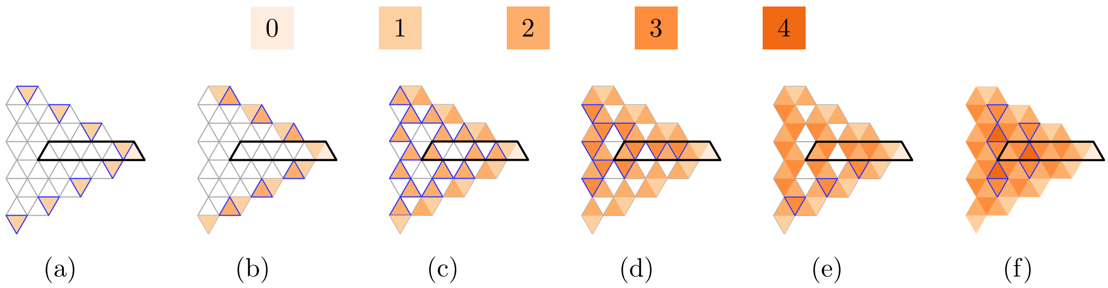


Figure 2.9: Reachable locations by category and how many folds it takes to reach the location. The triangles outlined in blue are added from the previous picture.

Proof. In this proof we will show for a couple of different categories of locations how they can be reached and in how many folds. This is done by using Lemma 1 and the 2-fold property proven here. We will show that we can reach a large amount of locations using only two folds. The remaining locations can be reached by changing the parity of the strip by end folding the last two creases c_{n-2} and c_{n-1}.

Figure 2.9 shows the different categories of locations and in how many folds they can be reached. There are some locations which can be reached in the same amount of folds but need a different argument, so they have their own category. We show how each location can be reached and how it can be scaled for any value of strip length n.

One-fold locations can be trivially reached by folding any of the creases of the strip. The reachable locations are shown in Figure 2.9(a).

Two fold locations are divided in two categories. The first category we consider is the two-fold locations next to the one-fold locations shown in Figure 2.9(b). This is done by diagonal folds from the one-fold locations as shown in Figure 2.10(d). Figure 2.9(c) shows the two-fold locations which can be reached by performing a horizontal fold from the one-fold locations.

The three-fold locations shown in Figure 2.9(d) can be reached by folding from the two-fold locations as described in Lemma 1 and shown in Figure 2.10(a) and (c). The perpendicular creases exists, since there has to be paper between the origin and the two-fold locations. This shows how we can use the parity of the locations to get the different locations. The three-fold locations shown in Figure 2.9(e) are reached by end folding twice from the one-fold locations, which shortens the strip by two. End folding takes a single fold, thus doing it twice gives us three folds.

After all these locations have been reached, we are left with some vertically aligned locations which do not yet fit in a category. The strip can be aligned with the vertical from the two-fold

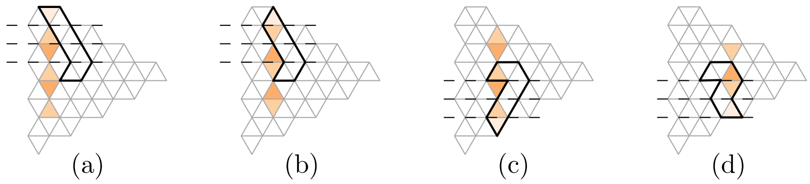


Figure 2.10: Reachable locations in two folds from the indicated locations. These strips can reach the color coded locations within two folds.

locations of Figure 2.9(b) as shown in Figure 2.10(b). End folding the strip once again from the two-fold location changes the parity within the vertical column. Folding a horizontal fold can reach the locations shown in Figure 2.9(f) with the technique described in Lemma 3.

As shown in Lemma 3, the one-fold locations will exist for as many creases in the strip. Since the other locations are relative to the one-fold locations, the same argument holds for any length strip. Since the locations with the most required folds is 4, no location needs more than 4 folds. □

2.1.2 Equilateral triangle folding TSP

In this section we will analyze the complexity of equilateral triangle TSP. We can express classical path finding problems using one-dimensional equilateral strip folding. Assembly lines often deal with TSP problems, thus it is of interest to look at the TSP for strip folding. TSP is defined as follows: given a graph $G(V, E)$ where each edge $e \in E$ has a weight, TSP asks to find a closed walk of minimum possible total weight that visits all vertices $v \in V$ of G. We define equilateral triangle strip folding TSP in a similar way. Given a set of reachable locations L, equilateral triangle TSP asks to find a closed walk of minimum total folds or unfolds that visits every location $l \in L$. With equilateral triangle strip folding, the weights of the edges are the amount of folds and unfolds between two folded states. Folded states means the folded creases of the strip. Every unique folded state maps to a single location, but not every location maps to a unique folded state. This means that there exist locations which may be reached by different folded states. Furthermore, different folded states means that we can reach different locations within a single fold or unfold. Thus there is not a set amount of neighboring locations, as this would depend on the folded state of the strip. For the TSP problem we only have to visit all locations, thus there are multiple folded states possible to traverse the locations.

We simplify the representation of the equilateral triangle folding problem. Since we have creases between each triangle of the strip, we can represent each crease as a bit in a bit string. In the bit string a 0 represents an unfolded crease, and a 1 represents a folded crease. For instance, the strip in Figure 2.9 has the following bit string: 00000000 as no crease out of the eight possible creases has been folded. When we fold a crease, the bit is flipped to represent the new state of the strip. Clearly each unique bit string maps to a single location. In order to find a optimal route for the TSP problem, we need to find a sequence which visits all locations with the least amount of folds or unfolds. Using the bit string representation, this would mean that we have to minimize the amount of bit flips.

This problem is very close to the Hamming distance TSP. The Hamming distance is the number of positions between two bits which are different. An illustration of a complete Hamming graph for three bits is given in Figure 2.11. For the optimizing Hamming distance TSP we have to find the minimum bit flips to traverse some set of bit strings. The Hamming distance TSP was proven to be NP-complete [14].

Equilateral triangle folding TSP is slightly different from Hamming distance TSP. The Hamming distance has to get through a couple of unique bit strings. The equilateral triangle folding TSP has to reach certain unique locations and some of these locations can be reached by multiple different bit strings (folded states). Any of these bit strings will satisfy the requirement of reaching the location. Note that not every location has multiple bit strings. Such TSP problems are TSP with neighborhood (TSPN) problems. In the TSPN problem, there are collections of multiple loc-

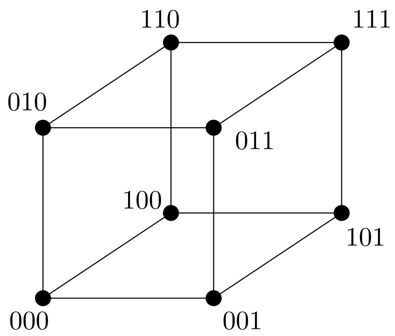


Figure 2.11: Illustration of the Hamming distance. Each node differs a single bit from its neighbors.

ations. The salesman has to visit at least one location in each collection (neighborhood) instead of all locations in the collection [4]. We have a similar generalization for the Hamming distance TSP, as we only need to visit one of the bit strings out of all possible bit strings which are mapped to a single location. The equilateral triangle folding TSP is a special case of the TSPN problem.

We will give the upper and lower bounds for the equilateral triangle folding TSP. We use the property proven in Theorem 2 for the upper bound.

Lemma 4. *Given a equilateral triangle strip of length k and n locations, the upper bound of the amount of folds to complete the tour for equilateral triangle folding TSP is $6n$.*

Proof. By Theorem 2 we can fold to every location in at most 4 folds. A trivial upper bound for the equilateral triangle folding TSP would then be $8n$, since we can fold to any location in 4 folds from the starting position and then unfold in 4 folds again to reach the starting position. Thus every location would take 8 folds. We can tighten this upper bound by showing that we can reach the 3-fold locations and 4-fold locations by folding the same folds, which means that we do not have to (un)fold those creases.

Any location reachable within one or two folds is trivially foldable in 6 folds. When we have to move from a location which is reachable with 3 folds to another location which is reachable with 3 folds, this is also trivially foldable in 6 folds. The cases we have to consider is moving from locations of 3 folds to locations of 4 folds, 4 folds to 3 folds, and 4 folds to 4 folds.

Case 4-fold location to 4-fold location We first consider folding from a 4-fold location to another 4-fold locations. The locations here are described in Figure 2.9(f). In this case, we used the parity of the column of the 4-fold locations for Theorem 2. This means that we had to change the parity by end folding the strip. Similarly to get into the column Theorem 2 performed another end fold. This means that for any 4-fold location, we have to end fold the strip twice. Thus going from a 4-fold location to another 4-fold location for a strip of length n is equivalent of going from a 2-fold location to another 2-fold location for a strip of length $n-2$. Thus the creases which folded the end of the strip do not have to be re-folded when going from a 4-fold location to another 4-fold location. This means we only need at most 2 unfolds and 2 folds to go from a 4-fold location to another 4-fold location.

Case 3-fold location to 4-fold location (and reverse) Now consider the case of folding from a 3-fold location to a 4-fold location. We consider the 3-fold locations from Figure 2.9(d). All the shown 3-fold locations are in the same column as the 1-fold locations, but with a different parity. The parity difference is obtained by end folding from the 1-fold location and performing a horizontal fold. The end fold means that again, the final crease of the strip is folded to reach these locations. The same holds for the 3-fold locations in Figure 2.9(f). The 3-fold locations show in Figure 2.9(e) are obtained by end folding twice, thus also folding the final two creases. Now combining this with the argument made before about the 4-fold locations, we can see that going from a 3-fold location to a 4-fold location, the final crease is always folded. This means that we have to unfold and refold one less crease when moving from the 3-fold to a 4-fold location. This means we do not have 3 unfolds and 4 folds, but 2 unfolds and 3 folds which is 5 folds in total. Similar reasoning holds for 4-fold to 3-fold.

Since we have already seen that all 3-fold locations fold the final crease, the same argument holds for folding from a 3-fold location to another 3-fold location.

For any other combination of folds we can always unfold all creases and fold all other creases within 6 folds. We can have for instance a 2-fold location and a 4-fold location which have no creases in common, thus we would need at 6 folds to fold between these locations.

Thus we have at most 6 folds between any two locations. This means that the upper bound for the equilateral triangle folding TSP is $6n$. □

For the lower bound there is nothing more than the trivial n folds. Whenever we fold a single crease, we fold to a new location, since we reflect the hand across the crease. Doing this n times trivially gives n locations which the hand moves over. We can do this for n different locations. Therefor we give the trivial lower bound for equilateral triangle folding TSP.

Lemma 5. *The lower bound of the amount of folds to complete the tour for equilateral triangle folding TSP is n.*

Despite the interesting lower and upper bounds on the total amount of folds which are needed to complete the tour no matter the size of the problem, finding the optimal path is still likely to be NP-hard since it is a version of the Hamming TSP with specific sets of neighborhoods.

Conjecture 1. *Equilateral triangle folding TSP is NP-hard.*

2.1.3 Experimental evaluation

In this section we experimentally evaluate one-dimensional folding reachability. We at the amount of folds which it takes to reach certain locations for two different types of strips. Furthermore, we look at the amount of ways we can reach a reachable location. This will give intuition for the size of the neighborhoods for the equilateral triangle folding TSP. We have simulated the reachable locations for two types of strips; equilateral triangle strips and square staircase strips.

Figure 2.12: The two different types of strips which we will consider in this section. (a) is the equilateral triangle strip and (b) is the square staircase strip.

Figure 2.12 illustrates the two types of strips which we consider. Both strips have a one-dimensionality to their structure, but move in two dimensions when a crease is folded. The staircase strip has creases in two directions, just like the equilateral triangle strip. The staircase is more intuitive to reason about whilst being in some ways similar to the equilateral triangle strip. A nice property of the staircase strip is that when a crease is folded, the direction of the other creases does not change. This means that it is easier to independently fold creases without changing the problem. This makes the staircase strip easier to simulate and gives a nice entry point to the reachability problem for folding.

We evaluate the minimum required folds to reach a location and the total amount of different folding sequences to reach a location for both different strip folding problems. We simulate both scenarios for both strips. The code used to simulate these problems can be found at https://github.com/gwvdh/triangle_folds.

Square staircase count The reachable locations for the staircase strip are shown in Figure 2.13. This figure suggest that the minimum amount of folds required to reach a reachable location is four. For the staircase we have the option to fold along two separated axes. Just like with the equilateral triangle strip, we can change the parity by folding the last creases.

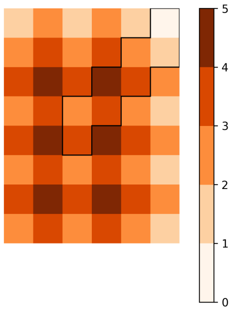


Figure 2.13: Square grid reachability. Colors indicate the minimum amount of folds it takes to reach a location.

The amount of locations for the staircase grid is equal to the amount of creases in each direction multiplied by 2. This is different whether the length of the staircase strip is even or odd. Thus we get Equation 2.2 for the amount of reachable locations of a staircase strip of length n.

$$f(n) = \begin{cases} \frac{n}{2} \cdot (\frac{n}{2} - 1) & \text{if } n \text{ is even} \\ \frac{n-1}{2} \cdot (\frac{n-1}{2}) & \text{if } n \text{ is odd} \end{cases} \tag{2.2}$$

More interesting for this type of strip is the number of different ways to reach some location. In the equilateral triangle folding TSP problem this would represent the size that a neighborhood can have. We do this by distinguishing between the different folded states where the hand is over the location. The different states with which we can reach some location is shown in Figure 2.14 for strips of length .

We can see a recurring pattern for the different locations. The center of the strip has to most amount of different states as expected. However, the origin itself does not always have the most amount of different states. There are distinguishable categories of similarly reachable locations which have a size of two by two.

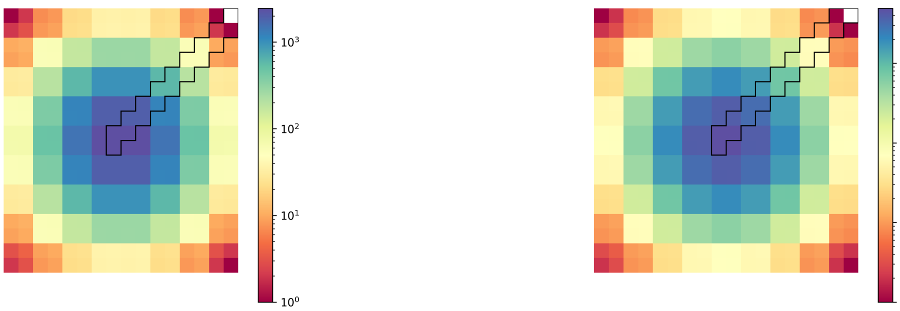


Figure 2.14: Square grid reachability for staircase strips of length 17 (left) and 18 (right). Colors indicate the amount of different folding states which reach the location.

A similar pattern can be seen for the triangle strips. Two examples for odd and even strip lengths are shown in Figure 2.15. The center again can be reached by the most different folded states. The similarly reachable locations are again partitioned in a couple of similar regions. This time they resemble a rhombuses close to each other circling the center.

Figure 2.15: Triangle grid reachability. Colors indicate the amount of different states which reach the location.

The consequence of this is that the TSP problem might have many different ways of getting to the locations which are more to the center. This visually shows the difference between the hamming TSP and the triangle strip fold TSP.

We present two sequences for the maximum amount of states with which we can reach some location for a strip of length n. In the equilateral triangle folding TSP problem this would represent the maximum size that a neighborhood can have.

n	Staircase strip	Triangle strip	n	Staircase strip	Triangle strip
3	3	3	10	39	46
4	4	4	11	63	82
5	5	5	12	103	137
6	7	7	13	203	271
7	9	11	14	403	501
8	12	17	15	703	868
9	21	24	16	1228	1734

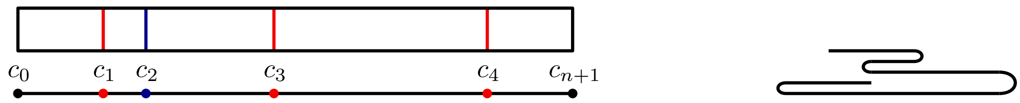
Table 2.1: Maximum amount of states with which we can reach some location from the reachable locations of a strip of length n.

2.2 One-dimensional Flat-foldability

In this section we will explore flat foldability for one-dimensional equilateral triangle strip folding. We will show one case of equilateral triangle strip folding where we only have parallel creases is always flat foldable by results from the literature. Furthermore we will show that equilateral triangle strips are always flat foldable in the general folding model. We present our observations for flat foldability in the simple folding model and give direction for future research.

The one-dimensional case for map folding removes some of the complexity from the flat-foldability decision problem. It is one of the simplest types of crease patterns as all creases are parallel to each other and perpendicular to the strip. This means that we can imagine the paper as a line segment with points marking the creases.

We consider a piece of paper in a strip with creases perpendicular to the strip (refer to Figure 2.16). This piece of paper can be represented using a line. Each crease is represented by a point and with the ends of the strip also represented as points. This line segment is represented by



(a) (b)

Figure 2.16: Flat-foldability for one-dimensional strips. (a) a one-dimensional strip with a crease pattern and its one-dimensional representation. (b) a valid flat-folded state of the strip of (a).

the creases and end points $c_0, ..., c_{n+1}$ where $c_1, ..., c_n$ denote the creases and c_0 and c_{n+1} are the end points of the strip. Each crease $c_1, ..., c_n$ has an assignment of either mountain or valley. The question we are concerned with is: Given a mountain-valley pattern, does there exists a flat-folding satisfying the specified mountain and valley assignments?

For strips which have parallel creases which are perpendicular to the strip, there exist strips which are not flat-foldable. An example of an unfoldable crease pattern is two consecutive creases c_i, c_{i+1} which have the same assignment direction and where the two line segments to the pair of creases which are longer than $|c_i - c_{i+1}|$. This case is shown in Figure 2.17.

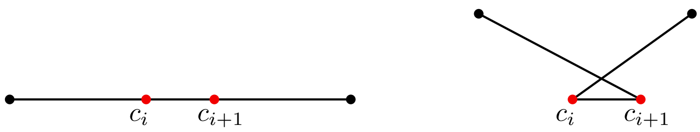


Figure 2.17: Crease pattern for a strip which is not flat-foldable.

This decision problem was shown to be $O(n)$ solvable in worst-case time (on a machine supporting arithmetic on the input lengths) by Arkin et al. [5]. This was done by showing that any flat-folded state can be reached by a sequence of *crimps* and *end folds*. These are two local operations on the one-dimensional strip and are shown in Figure 2.18.

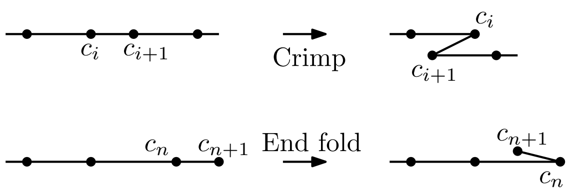


Figure 2.18: Crimping and End fold operations on a one-dimensional strip. Crimping removes the creases c_i and c_{i+1} and retains the flat-foldability decision problem for the remaining creases. Similarly, the end-fold removes the crease c_n.

Formally we say that a pair of consecutive creases (c_i, c_{i+1}) is *crimpable* if c_i and c_{i+1} have opposite assigned directions and the adjacent segments are at least as long as the segment between c_i and c_{i+1}:

$$|c_{i-1} - c_i| \geq |c_i - c_{i+1}| \leq |c_{i+1} - c_{i+2}|$$

A line has a *foldable end* if the end of the line is at most as long as the segment adjacent to the end. Here c_{n+1} is the foldable end and crease c_n the crease which folds:

$$|c_{n-1} - c_n| \geq |c_n - c_{n+1}|$$

Both these operations reduce the problem whilst preserving flat-foldability for the remainder of the strip. Arkin et al. [5] showed that every flat-foldable one-dimensional assigned pattern in fact

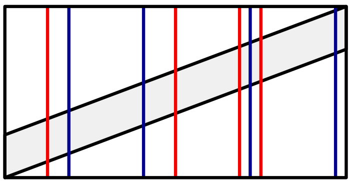


Figure 2.19: Instance of the strip folding with creases at an angle with the universal sheet as the traditional one-dimensional map folding problem.

contains a crimpable pair or a foldable end. This result holds for these very basic one-dimensional problems as all creases are perpendicular to the strip.

Demaine et al. [10] looked at an extension of the one-dimensional strip folding problem by varying the angle at which the creases are. All creases still have to be parallel to each other, but can be positioned at an angle with respect to the strip. Demaine et al. define a strip folding instance I and transform it into a traditional one-dimensional strip folding problem called the universal sheet of I. This transformation is shown in Figure 2.19. Clearly if the universal sheet of I is flat-foldable, I is flat foldable. However, there are cases where I is flat-foldable, but the universal sheet of I is not. This occurs when the creases have a sufficiently small angle for the faces to fold next to each other as shown in Figure 2.20.

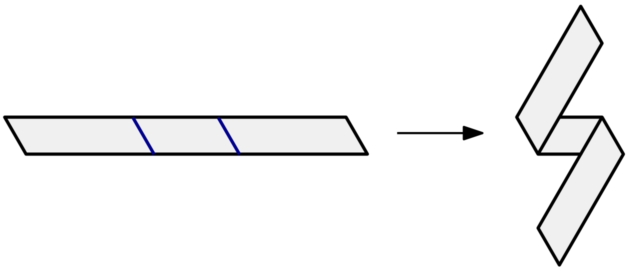


Figure 2.20: Folding a strip folding instance I with parallel creases which is flat foldable, but the universal sheet of I is not.

For an equilateral triangle strip, the angle and parallel creases are such that the faces fold next to each other. This gives a simple lemma for this specific case of equilateral triangle strip folding.

Lemma 6. *Given an equilateral triangle strip instance I of the strip flat folding problem with parallel creases and mountain-valley assignment, I is always flat-foldable.*

Proof. Without loss of generality, assume that the equilateral triangle strip has a width of 1. Thus parallel creases are separated by $d = \frac{2}{\sqrt{3}}$ by the definition of an equilateral triangle strip and equilateral triangle. By the definition of an equilateral triangle, the angle of the crease is $\theta = 60°$. As shown by Demaine et al. [10, Lemma 6], the strip is flat-foldable if $d \geq \frac{1}{\sin 2\theta}$. We have $\frac{2}{\sqrt{3}} = \frac{1}{\sin 2 \cdot 60°}$, thus the strip is flat-foldable. □

The equilateral triangle strip folding problem is not restricted to parallel lines as there are two different line directions. Demaine et al. [10] have looked at the zigzag case for strip flat-foldability. They considered general flat-foldability for strips with non-acute zigzag creases as shown in Figure 2.21. The zigzag creases had to be non-acute, which means that $\theta_2 - \theta_1 \geq 90°$ with θ_1, θ_2 as shown in Figure 2.21. Strips with these properties are always flat-foldable (see Theorem 3). This was shown by separating the strip into *flakes*, and putting every flake into a different layer. This is shown in Figure 2.22. Then each each flake can be stitched back together at the parts of the paper which have no overlap with any other part of the paper, as they are in different layers. The parts which do not overlap with any part of the paper are called *ribs* (see

Figure 2.21). We can stitch the flakes back together because the ribs are exposed both above and below.

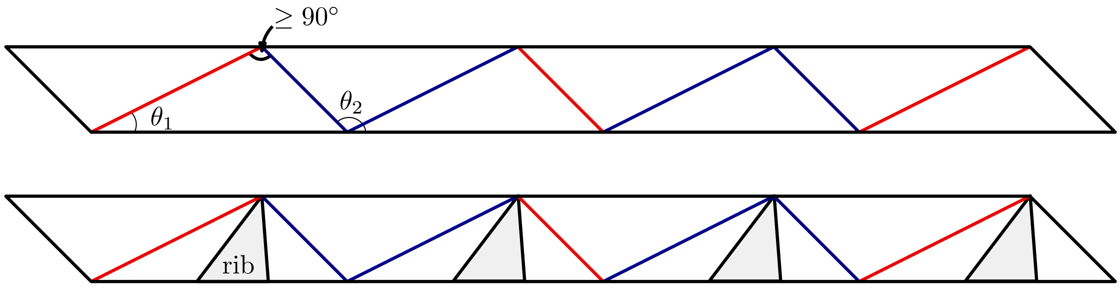


Figure 2.21: A strip instance with non-acute zigzag creases where $\theta_2 - \theta_1 \geq 90°$.

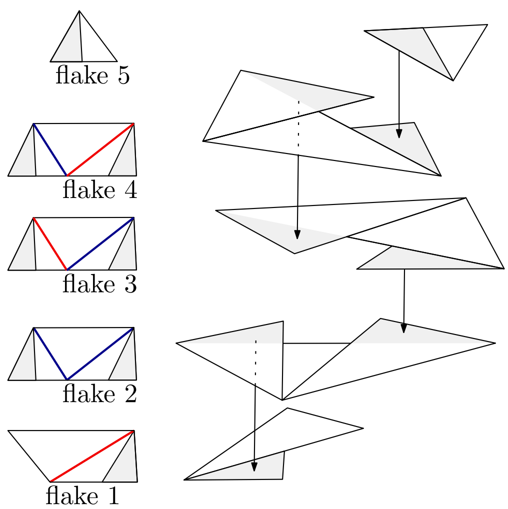


Figure 2.22: Flakes of the strip from Figure 2.21. The flakes are all in their own layer.

Theorem 3 ([10, Theorem 3])**.** *Given a strip instance I with non-acute zigzag creases and a M/V assignment, I is flat-foldable in the general folding model.*

We notice two properties of the proof structure of Theorem 3. Nowhere in the proof the property of the creases meeting in a single point is needed. This means that the creases do not have to intersect in a single point for the proof of Theorem 3 to work. Thus the creases can be further apart or one of the creases of the zigzag is missing, as long as they adhere to the other properties. Thus the ribs could be lengthened without it affecting the proof structure and each face can be a trapezoid instead of a triangle. Secondly, the proof did not require each flake to be exactly the same. Thus the ribs might differ in size, but ribs which have to be connected can still be fused as they are the same.

2.2.1 General flat foldability of equilateral triangle strips

We look at general foldability for one-dimensional equilateral triangle strips.

Problem statement 3 (**Equilateral triangle strip flat foldability**)**.** *Given is an equilateral triangle strip with a any crease pattern with a mountain-valley assignment. Is the specified mountain-valley crease pattern flat foldable?*

For the equilateral triangle strip we have $\theta_1 = 60°$ and $\theta_2 = 120°$ which means that $\theta_2 - \theta_1 \ngeq 90°$. This means that we can not use the same proof as Theorem 3 to prove general foldability for equilateral triangle strips, because the ribs of the flakes would not be guaranteed to be exposed above and below. However, we can utilize a similar idea. We show that general flat-foldability does exist for any equilateral triangle strip. We show the end fold and crimp operations for equilateral triangle strips. We use these for the proof of general foldability. The end fold operation basically folds a single triangle at the end of the strip back onto the strip without affecting the flat foldability of the remaining strip. The crimp operation shortens the strip through three folds without affecting the flat foldability of the other creases.

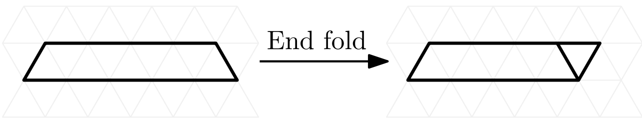


Figure 2.23: The end fold for an equilateral triangle strip. We fold the final crease of the strip to reduce the length of the strip by one.

Lemma 7. *Folding a foldable end preserves flat-foldability.*

Proof. See Figure 2.23 for an example of an end fold. Folding the final triangle of the strip clearly reduces the length of the strip by one. We take the flat folding of the original crease pattern. We remove the end face f_n from the layer it is in the flat folding and double up the number of layers for the second to last face f_{n-1}. This is clearly possible, as there is no paper between face f_n and f_{n-1}. This gives us a new valid flat folding of the crease pattern. □

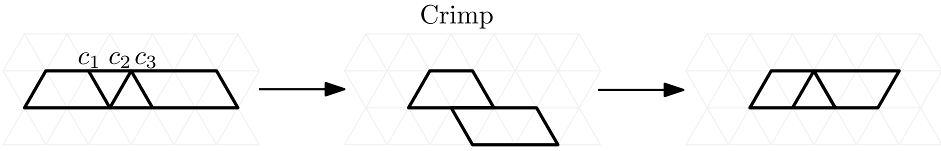


Figure 2.24: The crimp operation for an equilateral triangle strip. We fold the parallel creases first. Then we can fold the third crease no matter the crease assignment. This crimps the strip by two triangles.

Lemma 8. *Crimping a crimpable crease pattern preserves flat-foldability.*

Proof. The working of the proof is shown in Figure 2.24. We have three consecutive creases c_1, c_2, c_3 which are a crimpable triplet. Since the three creases are consecutive it follows that c_1 and c_3 are parallel. They can have any crease assignment. Folding the c_1 and c_3 first is always possible without intersecting any other parts of the strip by Lemma 6. Folding the crease c_2 folds the two parallel pieces of paper on to each other as shown in Figure 2.24. The remaining strip is crimped by two triangles. These folds do not overlap with any other folds as they are single triangles. Therefor we can take the crease pattern of the crimped strip and fold it flat. Then add the crimped layers to the layer ordering according to the crease pattern. Now we have a valid flat folding of the strip. □

With these lemmas we can prove flat foldability for equilateral triangle strips in the general folding model. We consider equilateral triangle strips with any crease pattern and any mountain/valley assignment.

Theorem 4. *An assigned equilateral triangle strip with any crease pattern can be flat folded in the general folding model.*

Proof. In this proof we will use a similar technique of ribs and flakes as Demaine et al. [10] as shown in Figure 2.21 and Figure 2.22. We look to find appropriate ribs as shown in Figure 2.21 for the equilateral triangle strip. We first show how to simplify the strip. Then we show a couple of different crease patterns which we can flat fold and identify as different flakes. These flakes can then be combined to create a flat foldable order.

Whenever a crease between two faces f_i, f_{i+1} is folded, there is a single triangle of overlap between f_i and f_{i+1}. An example of this is shown in Figure 2.25. Any triangle after this could function as a rib for our purposes. With this observation, we need to address the cases where there exist faces of length one in the strip.

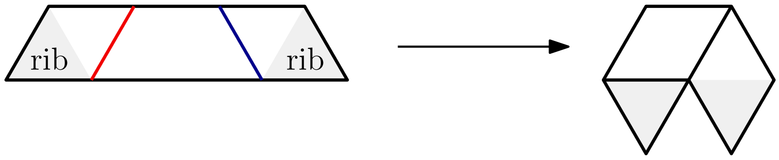


Figure 2.25: An example of a flake for equilateral triangle map folding. faces overlap by exactly one triangle. The triangles which represent the ribs are indicated by the gray background.

Given is a strip S with an assigned crease pattern. Not all possible creases have to be present. We can simplify the strip by using the Crimp and End folds. We end fold and crimp the strip S to obtain strip S' according to Lemma 2.23 and Lemma 2.24. Clearly, there can not exist any two consecutive faces $f_i, f_{i+1} \in S'$ of length 1, as they would be crimpable according to Lemma 2.24. This leaves strips where there is a face of length one between faces of lengths greater than one.

Let f_i be the face of length one. The ribs for the case where we have a face of length one between faces of a length greater than one is shown in Figure 2.26. We can see that in this case the overlapping parts of the paper is of length two. Thus if the faces next to face f_i have a length of greater than two, we have another valid flake.

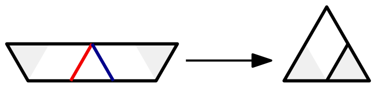


Figure 2.26: An example of a flake for equilateral triangle map folding which contains a face of length one. Instead of a single triangle overlapping, two triangles are overlapping. The triangles which represent the ribs are indicated by the gray background.

In case that one of the faces next to face f_i is not of length greater than two, we could get the situation in Figure 2.27. In this case the face of length two does not contain a rib. There could be multiple faces of length two concatenated, in which case the paper traces the length of the larger strip as shown in the figure.

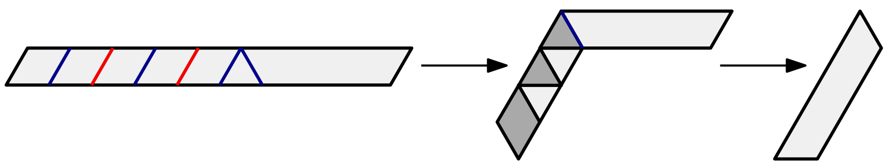


Figure 2.27: An example of a flake with concatenated faces of length two and a longer strip adjacent to the face of length one. The longer face overlaps with the folded faces of length two.

We will show that we can create a flat foldable flake for similar patterns as shown in Figure 2.27. As shown in Lemma 6 we can always fold concatenated even length strips. Thus we fold these parallel crease parts first. We fold these parts first, obtaining the flake in Figure 2.28. We will still be able to fold the creases c_i and c_{i+1} since the two faces f'_1 and f'_2 consist of faces of length two, which means that there are gaps in each face which we can weave through to get above the other. This means that ribs of the flakes can be exposed as they are needed.

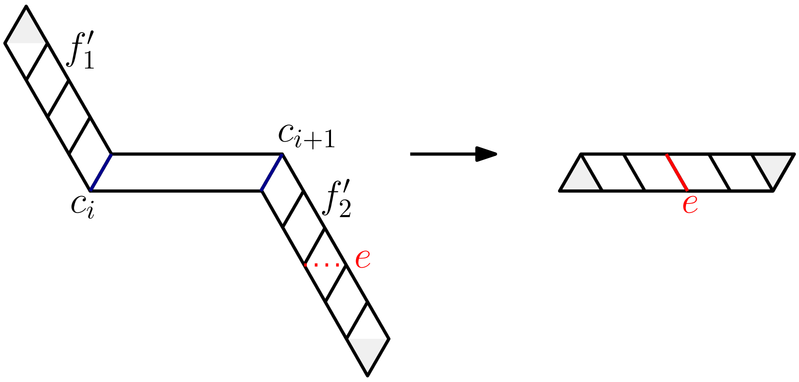


Figure 2.28: A flake where the parallel creases have been folded first into new faces f_1' and f_2'. There are two unfolded creases: c_i and c_{i+1}. The face f_1' can weave through f_2' and come out at edge e of face f_2'.

If any of the faces next to a face of length two is greater than two, we immediately obtain a new triangle which can function as a rib, thus obtaining a new flake.

This gives us different flakes with ribs for each possible crease pattern. Now we join the different folded flakes together, as we put each folded flake in its own layer, similar to Figure 2.22, and fuse the corresponding ribs. Since we have shown that there is always a rib exposed for every possible crease pattern, we can fuse the ribs together.

Therefor, we can flat fold any crease pattern in the general folding model. □

2.2.2 Simple equilateral triangle strip flat foldability

In this section we will look at some approaches which were promising and gave some insights into simple equilateral triangle strip flat foldability. We show partial results, however a complete resolution to this problem is open. We are concerned with the following problem statement.

Problem statement 4 (Equilateral triangle strip simple flat foldability). *Given is an equilateral triangle strip with a any crease pattern with a mountain-valley assignment. Is the specified mountain-valley crease pattern flat foldable in the simple folding model?*

We first look at folded states which might be problematic in the simple folding model.

Cyclic layer orders In the general folding model there was no restriction on the folding motion. This is the case for the simple folding model. Since in the simple folding model the paper can only rotate around a single axis, we can no longer just put parts of the strip in different layers like in Theorem 4. There exist valid flat-folded states of equilateral triangle strips which can not be flat folded using simple folds. Some of these states have a cyclic layer ordering. An example of such a cycle is shown in Figure 2.29. This strip is cyclic as each face of the strip is above and below another. This strip is clearly not simple unfoldable, thus we can not reach this state using simple folds.

The existence of a cycle in a flat folded state does not mean that it is not simple foldable. There do exist strips which can fold into a layer ordered cycle with simple folds. The folding sequence for one of those strips is given in Figure 2.30. These strips rely on folding a crease which moves part of the paper down and other parts of the paper up. In Figure 2.30 this is done by folding the crease c last, which folds part of the strip up and other parts of the strip down. This allows one face to get above one strip and below another.

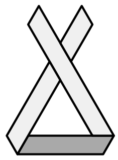
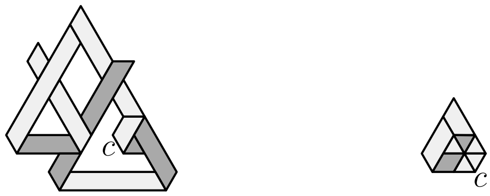


Figure 2.29: An example of a valid folded state containing a cycle. This state is not reachable using simple folds. The strip itself is simple foldable to some other flat-folded state.



(a) (b)

Figure 2.30: Example of a simple folded state containing a cycle. An example which clearly shows the cycle is shown in (a) and the smallest found example is shown in (b). The folding sequences for both strips can be seen in Appendix A.

This shows that even cyclic layer orders might still be simple flat foldable. Thus the layer ordering of a folded state might not help to verify whether the state is reachable through simple folds.

Unfolding from the convex hull A different way of looking at folded states is to look at the shape of the strip when it is folded. If we can simple unfold the strip from the folded state, we can clearly also simple fold the strip. We will try to use the convex hull to find a crease which is simple unfoldable from the folded state of the strip.

No matter in which order the creases are folded, the eventual shape of the folded strip is the same. Thus we are only concerned with the geometric shape of the folded state. The shape of the folded shape is called the *silhouette*. An example of a silhouette of a strip is given in Figure 2.31. This means that the silhouette might be able to give clues about a valid folding order as it is a consequence of folding all creases. We show that there exists a silhouette for every strip.

Lemma 9. *Given a strip S with some crease pattern P, there exists a silhouette of S.*

Proof. By Lemma 4 there exists a flat-folded state of the strip. Thus we can project the flat-folded state on the plane. □

The silhouette can also be obtained by folding the creases alternating mountain/valley, which is always flat-foldable.

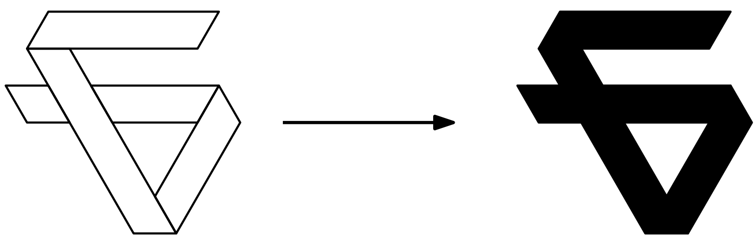


Figure 2.31: An example of a folded strip and its silhouette.

Since we have the flat-folded state, we look to unfold all creases in order to obtain the unfolded strip and crease pattern. Creases which unfold without intersecting other parts of the paper are the best candidates for simple unfolding. Therefore we look at the convex hull of the silhouette. A nice property of the convex hull is that any straight line along the convex hull of the silhouette will not intersect the silhouette itself. This means that the entire silhouette is on one side of that line. Folding some parts of the strip across this line would separate the strip at the folding line, which means that the two parts can not overlap. Thus if there exists a crease of the silhouette which is also on the convex hull, we can simply unfold that crease without any of the paper intersecting other parts of the paper.

Lemma 10. *Given a strip S with silhouette $\mathcal{S}$ and an assigned crease c on $\mathcal{CH}(\mathcal{S})$, we can simply fold c.*

Proof. Let l_c be the tangent of $\mathcal{CH}(\mathcal{S})$ through c. By the definition of the convex hull, the entirety of $\mathcal{S}$ is on one side of l_c. Thus if we unfold c to create a new silhouette $\mathcal{S}'$, there is no new overlapping surfaces on the side of $\mathcal{S}$. Thus unfolding crease c was possible according to the assignment, as it did not influence the remaining foldability of the other creases. □

Thus by Lemma 10, we can always unfold a crease if it is on the convex hull, whilst preserving unfoldability of other creases. This means that if there always exists a crease on the convex hull of the silhouette, we can inductively unfold all creases and thus obtain a simple folding of the strip. However, there does not always exist a crease on the convex hull. There are plenty examples of folded strips which do not have a crease on the convex hull. Figure 2.32 illustrates an example of a strip which does not have a crease on the convex hull.

Figure 2.32: A small example of a silhouette which does not have a crease on the convex hull. The convex hull is made to be axis aligned, as this is the only lines which are interesting for our problem.

The creases in Figure 2.32 are close to the convex hull, but not on it. These creases can still simply unfold, as do other examples of creases which lie close to the convex hull. There also exist examples where it is less easy to see if the creases are unfoldable. One such an example is given in Figure 2.33.

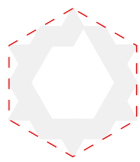


Figure 2.33: Silhouette of a strip which does not have a crease on the convex hull. Unfolding any crease will make the silhouette fold within its own convex hull.

An alternative to get the creases on some hull is to consider another version of the convex hull, named the $\mathcal{O}$-hull. The $\mathcal{O}$-hull hits the creases which are on the outside of the silhouette. These creases are probably more likely to be simple unfoldable since there is less paper onto which the crease fold can create new layers. Thus we might want to consider an $\mathcal{O}$-hull with $\mathcal{O}$-lines at multiples of 60° [15].

Whenever a crease is unfolded from this silhouette, there can be paper which falls within the convex hull of the original silhouette. This means that some kind of order is forced on the remaining silhouette and thus the next unfold would have additional restrictions, namely the stacking order of the overlapping bits of paper. The problem is not properly simplified, which makes reasoning about it very difficult.

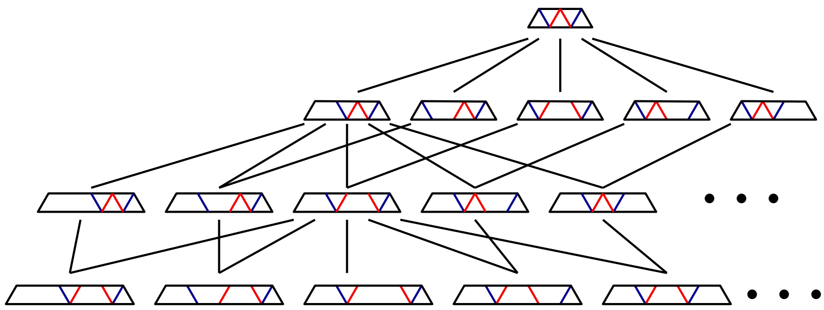


Figure 2.34: Lattice ordered strips by their length. Each strip can be made from a its child by adding additional distance between creases.

Lattice ordered strips Some strips have a very similar crease pattern, but different lengths. The crease directions of different strips are the same and the assignment are the same, but the strip length is different. There is an additional distance between creases. We can order these different strips as a lattice as shown in Figure 2.34. The strip with all creases is the root of the entire lattice and that strip is always foldable by end folding (Lemma 7). Thus the hope would be that we can find some common order to fold similar creases between these strips. By experimentally brute force calculating all simple foldable states of all strips up to length 10 (see Section 2.2.3), there was a common order for all strips on a chain of the lattice. Do note that because we maintain the crease directions we have to add two to the length of the next strip in the chain, thus the total length of each chain through the lattice in the database is at most four. This suggests that perhaps induction on the length of the strip might be a promising approach of proving simple foldability.

Even though this might suggest an inductive proof, the small number of strips tested might have been the reason for the positive experimental result. There are problems with trying to proof simple foldability by induction on the length on the strip. A shorter strip is not necessarily a subset of a longer strip. There can be more overlap between different faces and thus the shorter strip has a completely different set of restrictions than the longer strip. This kind of induction is devoid from the geometry of the problem, which is vital to understanding simple folding conflicts. Therefor it seems that this approach to decide simple flat foldability would probably not be very fruitful.

2.2.3 Experimental evaluation of flat-foldability

In this section we look at the flat-foldability for many different strips. We look at all possible strips of a length of 10 or less. These strips have been evaluated on simple flat-foldability by brute force. All valid folding sequences have been calculated and stored in a database. From that database, we can look at patterns and similarities between different strips. When simulating simple folding motions we use a triangular coordinate system.

Triangular coordinate systems The Barycentric coordinate system is a very popular coordinate system for triangular grids. This coordinate system is widely used in computer graphics and other geometric systems using triangles because of the way it represents the centers of the triangles. However, this coordinate system is most useful for continuous coordinate system and thus is not applicable for our grid.

In the Cartesian grid, simulating a fold is cumbersome. When folding at an angle, the new location does some diagonal motion which is not easily expressed in x and y movements. However, the Cartesian grid is often nice for other applications, thus we do show a transformation from a new grid coordinate system to the Cartesian grid.

The new triangle coordinate system is shown in Figure 2.35(a) with additional global fold lines along side with the dual Cartesian coordinate system. When we fold some crease of the strip, it

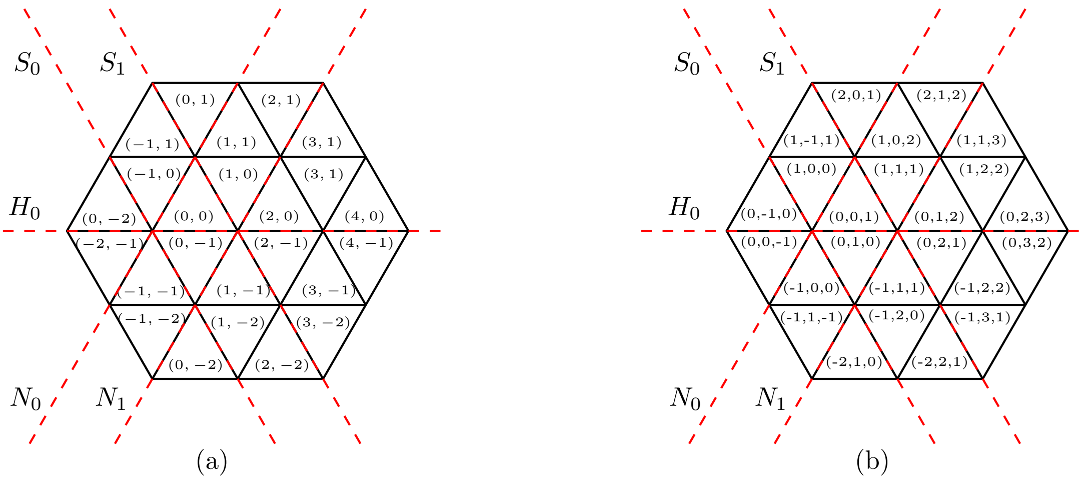


Figure 2.35: The Cartesian grid coordinates in the left figure (a). The new triangle coordinates are shown in the right figure (b).

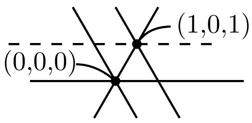


Figure 2.36: Two different coordinates with S_0 shifted to S_1.

lies on one of these global fold lines. All triangles which are folded will reflect around this global fold line. We label each of the global fold lines by their direction and the index of the line. For the S and N global fold lines, the index increases as the line is to the right. For the H global fold line, the index increases as the is above. When we fold about a fold line in the Cartesian grid, it is not very clear what the reflected coordinate is. We make reflecting calculations easier by introducing a new coordinate system. Note that for our purposes a discrete coordinate system suffices.

We base each coordinate of a triangle on the adjacent global fold lines. An illustration of the grid coordinates is shown in Figure 2.35(b). Since we have three fold directions, we get a triplet for each coordinate. The order in which the global fold lines appear in the triplet is chosen as (i, j, k) for H_i, N_j, S_k. The three lines H_0, N_0 and S_0 come together in a single point. Consider point $(0, 0, 0)$. Shifting S_0 to S_1 gives coordinate $(0, 0, 1)$, which is represented by a triangle of side length 1 (See Figure 2.36). Shifting to S_2 gives a triangle of side length 2, and so on. This of course also holds for shifting other fold lines. From this observation we can give the equation (Equation 2.3) for the side length s of the enclosed equilateral triangle by three fold lines H_i, N_j, S_k.

$$s = k - j - i \tag{2.3}$$

Note that when $s < 0$, the triangle is upside down. A point where all three fold lines come together is a triangle of side length 0. Thus we obtain a convenient equation for calculating one of the fold lines out of two others. Let (i, j, k) be the coordinate for an intersection between three fold lines. Then Equation 2.4 holds.

$$i = k - j \tag{2.4}$$

We use Equation 2.4 to calculate the reflected location of a given triangle location about some fold line. Figure 2.37 illustrates an example of the reflection of a triangle location. Given is the reflection line N_0 and the location $(i, j, k) = (1, 0, 0)$. We can calculate the points of intersection with the reflection line from the triangle coordinate as shown in Equation 2.4. This gives us two sides of the reflected location of the triangle. All that remains is to reflect N_i about $N_0 = N_{i'}$,

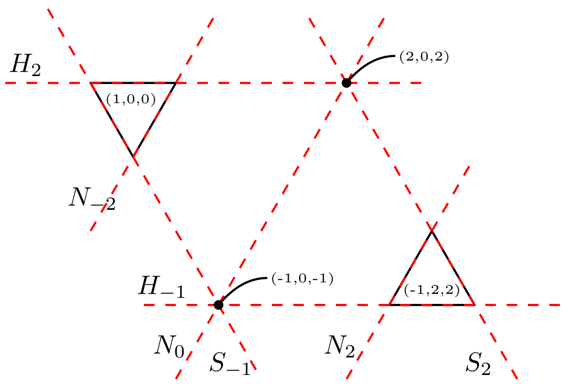


Figure 2.37: Reflecting a triangle location across reflection line N_0.

which is easily done by $2 \cdot i' - i$. We can represent this in matrix form as shown in Equation 2.5 for a given horizontal fold line H_a with location (i, j, k), Equation 2.6 for a given fold line N_b with location (i, j, k), and Equation 2.7 for a given fold line S_c with location (i, j, k).

$$f_H(a, i, j, k) = \begin{pmatrix} a & i & j & k \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{2.5}$$

$$f_N(b, i, j, k) = \begin{pmatrix} b & i & j & k \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.6}$$

$$f_S(c, i, j, k) = \begin{pmatrix} c & i & j & k \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \tag{2.7}$$

We can transform these fold line coordinates to Cartesian coordinates and the other way around. This can be convenient when working with visualizations which are often for Cartesian coordinates. Given a fold line coordinate (i, j, k) Equation 2.8 shows the transformation from fold line coordinates to Cartesian coordinates. Given a Cartesian coordinate (x, y) Equation 2.9 shows the transformation from Cartesian coordinates to fold line coordinates. It is important to check whether the triangle is upside down in order to properly translate it between the two coordinate systems. This can be done by checking if the result from Equation 2.3 is negative (thus upside down) or not.

$$g(i, j, k) = \begin{cases} \begin{pmatrix} i & j & k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix} & \textbf{if } k - j - i > 0 \\ \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \end{pmatrix} & \textbf{if } k - j - i < 0 \end{cases} \tag{2.8}$$

The transformation from Cartesian coordinates to the triangle coordinates is given in Equation 2.9.

$$g'(x,y) = \begin{cases} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} & \textbf{if } (x+y) \mod 2 = 0 \\ \left(\begin{pmatrix} x & y \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix}\right) \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} & \textbf{otherwise} \end{cases} \tag{2.9}$$

2.2.4 Experimental observations

To get more intuition about strip flat-foldability, we have evaluated all strips of up to a length of 10 triangles for flat-foldability. Each possible crease assignment and crease pattern was evaluated. All of the strips had a sequence of creases which could flat-fold the strip.

Notes on the code

The code for simulating one-dimensional simple folds for the equilateral triangular map can be found at `https://github.com/gwvdh/strip-folding`.

With this simulation we are mostly interested in the foldability of strips with some assigned crease pattern. In total we have evaluated 29524 strips, which are all strips of length smaller or equal to ten. Each strip was tested for foldability for all different crease patterns and crease assignment. All strips were simple flat foldable. All valid folding orders were calculated. The flat folded states were also stored for each folding order, from which we can find the unique flat folded states of the strip. Two equal flat folded states are flat foldable states (see Definition 1) where the stacking order for all points is the same.

The number of possible strips of length n is equivalent to 3^{n-1}. For a strip of length n we have at most $n-1$ possible creases. For each number of creases i we have $\binom{n-1}{i}$ possible combinations of creases. Each of these combinations has 2^i mountain valley assignments. Thus combining this and using the binomial theorem, we get Equation 2.10 for the number of different strips of length n.

$$\sum_{i=0}^{n-1} \binom{n-1}{i} \cdot 2^i = 3^{n-1} \tag{2.10}$$

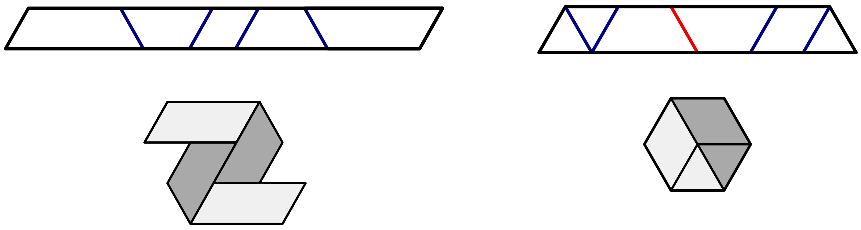
Length n	Number of strips	3^{n-1}	Length n	Number of strips	3^{n-1}
1	1	1	8	2187	2187
2	3	3	9	6561	6561
3	9	9	10	19683	19683
4	27	27	11	117147	117147
5	81	81	12	531441	531441
6	243	243	13	1594323	1594323
7	729	729	14	4782969	4782969

Table 2.2: Strip lengths to the number of possible strips of length n. This is equivalent to 3^{n-1}

An approach to foldability could be to fold the ends of the strip iteratively. However, using the database, we found counter examples where folding either the first or last crease first meant that the strip was no longer simple flat foldable. This case is shown in Figure 2.38. The main reason behind it being unfoldable is that there are two faces which would align with each other which means that neither of the other creases can be folded anymore.

To give a overview of strip simple flat foldability, we present some plots from the database.

In Figure 2.39 we can see the amount of folded states against the strip length and the amount of creases. Only unique states are plotted, so if two folding orders fold to the same state, they are only counted once. This plot uses bins to better visualize the different strips which have similar results. The colors indicate the amount of strips per bin. There is a a large amount of strips



(a) (b)

Figure 2.38: A strip where neither end can simple fold first (a). (b) shows where neither end can simple fold last.

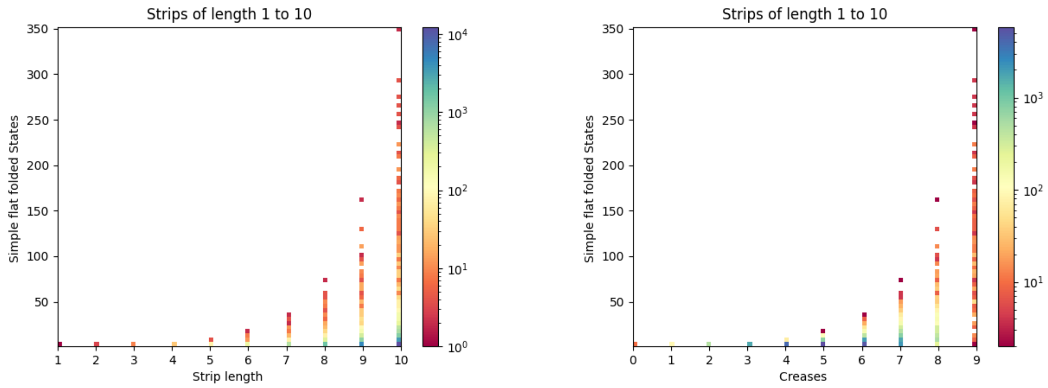


Figure 2.39: Length and amount of creases of the strip against the number of flat folded states per strip.

which have very few states. This is because there are many states where only a single state is possible. These are strips which have all parallel creases and strips which have an alternating mountain-valley crease assignment. The plot with the amount of states against the amount of orders shows little some peaks similar to the strip length. The peaks in these plots are the strips with all creases with all creases having the same mountain-valley assignment.

2.2.5 Foldability conjecture

We have evaluated many small strips and found no strip which did not have a simple flat-foldable sequence. There were many promising approaches to this problem, but no proof has been found as of yet. For any proof it would probably be important for it not to be devoid of the geometry of the folded state of the strip. Even seemingly unfoldable states, like cyclic layer orderings, have examples which are simple foldable. Because of the different observations and the experimentally evaluated strips, we present the following conjecture.

Conjecture 2. *Any 1D strip with an assigned equilateral triangle crease pattern can be simple flat folded.*

2.2.6 A quick note on the one-dimensional map folding with diagonals

The strip folding for $1 \times n$ map folding with diagonals was proven to be NP-hard by Jia et al. [18] using a similar PARTITION structure as Arkin et al. [5] used. In this problem we are given a strip with a square grid pattern with diagonals. The diagonals can turn the paper in order to make the PARTITION construction as shown in Figure 2.40(a). The construction by Arkin et al. [5] for the PARTITION problem only holds if the folding order of the diagonals is enforced. To enforce the order in which the diagonals fold, each diagonal could only fold if a long parallel crease along the strip is folded as shown in Figure 2.40(b). These parallel strips make the strip to no longer be a $1 \times n$ strip. Therefor, this reduction does not hold for the $1 \times n$ strip. This means that the argument for NP-hardness by Jia et al. [18] is insufficient, as the folding order is not enforced.

(a) (b)

Figure 2.40: Two essential parts of the PARTITION construction from the proof of Jia et al. [18] and Arkin et al. [5] for strip folding with diagonals.

It seems that the greater the angle between creases, the easier the problem becomes, as some folding problems are always flat foldable [10]. Our observations for equilateral triangle strip folding and Conjecture 2 all suggest that the problem where the strip only has diagonals might always have a simple flat folding. $1 \times n$ strip folding with diagonals has as sub problem the one-dimensional folding problem which was linear. We therefor conjecture that $1 \times n$ strip folding with diagonals is solvable in polynomial time.

Conjecture 3. $1 \times n$ *strip folding with diagonals is polynomial time solvable.*

Chapter 3

2-Dimensional equilateral triangle simple flat-foldability

In this chapter we look at the flat-foldability of 2-dimensional equilateral triangle map folding. In Section 3.2 we show NP-completeness for a polyiamond structure with assigned creases in the one-layer, some-layer, and all-layer models. In Section 3.3 we show a stronger NP-completeness result for assigned creases for hexagonal paper. For the unassigned creases with polyiamond paper, a proof of NP-completeness for the some-layer and all-layer is presented in Section 3.4. Intuition and gadgets for the unassigned problem with any paper are presented in Section 3.5.

This chapter is concerned with the following general question:

Problem statement 5 (Equilateral triangle simple flat-foldability). *Given is a crease pattern on a equilateral triangle map. Is there a simple flat-folding satisfying the specified creases pattern? If there exists a flat-folding, construct such a simple folding.*

To prove NP-completeness we use similar proving structures as the literature used for crease patterns with 45° creases. Arkin et al. [5] proved weak NP-completeness for square paper with paper aligned creases and creases at multiples of 45° with respect to the axes. To prove this they reduced from the PARTITION problem, which results in a *weakly* NP-complete result. This result was improved by Akitaya et al. [3] result by using 3-PARTITION. Since they used 3-PARTITION, the result was *strongly* NP-complete. We will draw inspiration from the results by Akitaya et al. to prove strong NP-completeness for equilateral triangle map folding.

Map folding with paper-aligned creases at multiples of 45° seems to be similar to equilateral triangle map folding. As is the case with 45° creases, when an equilateral triangle strip is folded it moves in two dimensions as seen in Chapter 2. Thus many of the properties of 45° maps can compare to equilateral triangle map folding. This might suggest that similar ideas from 45° folding can be used in equilateral triangle folding.

First we give an introduction to the proving structure 3-PARTITION and the definition of strong-NP-completeness which we will use throughout this chapter and in the proofs.

3.1 3-Partition

The class of NP-complete problems is a class of algorithms for which there is no known algorithm which can solve any of these problems in (sub)polynomial time unless P=NP. If there does exist a polynomial time algorithm for any of the problems in the NP-completeness class, there exists a polynomial time algorithm for all problems in the NP-completeness class. The time complexity of these NP-complete algorithms is often expressed only by the input length n. However if an algorithm runs in polynomial time in the input length but not polynomial in the magnitude of the representation of the input, it still belongs to the NP-completeness class. An example of this is the PARTITION problem. Given is a set S of positive integers. Can we split (partition) S into two subsets $S_1, S_2 \subseteq S$ and $S_1 \cap S_2 = \emptyset$ such that $\sum_{s_1 \in S_1} s_1 = \sum_{s_2 \in S_2} s_2$? This can be done using dynamic programming in polynomial time in the input length. Let $\sigma = \sum_{s \in S} s/2$, which is the total sum of one of the sets. Then for each $s_i \in S$ we either add it to a new set S_1 or not. If we add it, we reduce σ by s_i, otherwise we no longer consider s_i. Using this recurrence, we can solve the PARTITION problem in polynomial time in the length of S. However, we maintain a sum σ, which could have an exponential amount of bits. This would mean that we have to subtract an exponential amount of bits, which is no longer polynomial. This is called *weakly NP-complete*. *Strong* NP-completeness is defined as a set of problems for which there does not exist an algorithm solving them in time bounded by a polynomial in the input length *and* the magnitude of the largest number in the given problem instance [16].

Proving strongly NP-completeness can be done by reducing from 3-PARTITION. 3-PARTITION is different from the PARTITION problem as instead of partitioning in two subsets with some amount of integers in it, we partition into $\frac{n}{3}$ subsets with exactly three integers in each subset which sum to the same number. This means that the input has to contain a multiple of 3 positive integers. Formally we are given a set of n integers $A = \{a_1, ..., a_n\}$. A has to be partitioned into $\frac{n}{3}$ sets such that each set sums to t. Thus $t = (\sum_{a \in A} a)/\frac{n}{3}$. 3-PARTITION was used by Akitaya et al. [3] for square paper with paper aligned creases at multiples of 45° creases, resulting in a strongly NP-complete reduction.

We will use a similar proof structure as the construction by Akitaya et al. [3] for the 2-dimensional equilateral triangle map folding.

3.2 Polyiamond paper with assigned equilateral triangular creases

In this section we will prove strong NP-completeness for equilateral triangle flat-foldability with a polyiamond paper and assigned creases. For this proof we use 3-PARTITION in order to obtain a strong NP-completeness result for this problem. We reduce from 3-PARTITION by constructing a gadget which can only be flat-folded if and only if the 3-PARTITION instance has a solution. In order to make the proof easier to understand, the construction will be a single path. Thus there are no holes in the construction.

The gadget is given in Figure 3.1. With this gadget we prove Theorem 5 which proves strong NP-completeness in the one-layer, some-layers and all-layers models of simple folding on equilateral triangular map folding with polyiamond paper and assigned creases. We prove strong NP-completeness by reduction from 3-PARTITION. We construct a polyiamond with mountain and valley assigned axis-aligned creases. We are given a 3-PARTITION instance with n integers $A = \{a_1, ..., a_n\}$. We require that all integers $a_i \in A$ are a multiple of $2n$ and close to $t/3$. This is to make sure that it is not possible that there are any other number than 3 faces sum up to t. If for some $a_i \in A$ this is not the case, we multiply every integer $a \in A$ by $2n$ and add a large number ∞ ($\infty = 10nt$) to all integers such that the requirement holds.

There are five main functional sections of the polyiamond; the Hook, Staircase, Wrapper, Column and Cage (see Figure 3.1). The integers $A = \{a_1, ..., a_n\}$ from the 3-PARTITION instance are encoded in the Staircase by faces $\{f_1, ..., f_n\} = S$ of length $|f_i| = a_i \in A$. Each face $f_i \in S$ is surrounded by a mountain and a valley crease. Each $a_i \in A$ is even because we required that each $a_i \in A$ is a multiple of $2n$. This means that the mountain and valley crease surrounding each $f_i \in S$ have the same direction. Because all creases in the Staircase have the same direction, there is a small face of 2 triangles between each $f_i \in S$. These small faces should not be able to influence the sum to t. This is prevented by the large number and the $2n$ divisibility requirement for each $a_i \in A$, since the total sum of all small faces can not be larger than $2n$. Folding the creases around three faces $f_i, f_j, f_k \in S$ should simulate putting three integers into a set. Lemma 11 shows what happens when we fold the creases around three faces.

Lemma 11. *Given are distinct $a_i, a_j, a_k \in A$ and Staircase $\{f_1, ..., f_n\} = S$. Folding the creases around f_i, f_j and f_k moves the point Q' up by $3t/2$ if and only if $a_i + a_j + a_k = t$.*

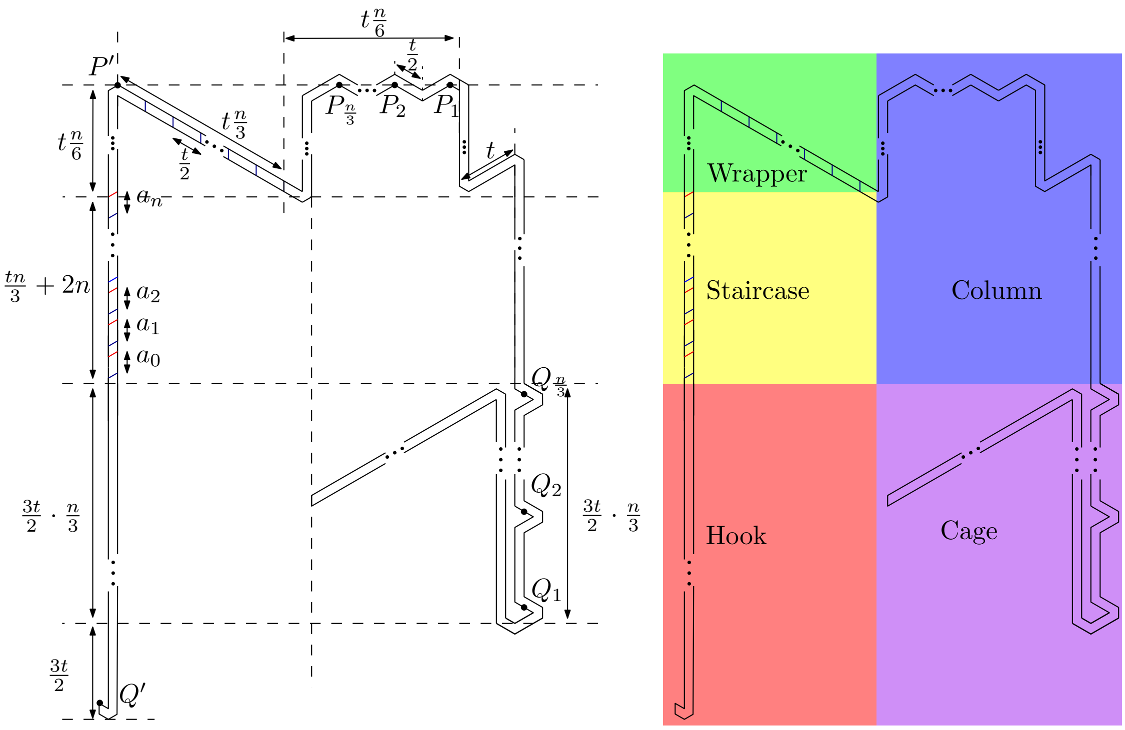


Figure 3.1: A simple polyiamond with equilateral triangular creases with assigned mountain and valley creases. The construction simple folds if and only if the corresponding 3-PARTITION instance has a solution.

Proof. An illustration of folding three staircase faces is given in Figure 3.2. We fold the creases around f_i. Because the creases are parallel, f_i folds upward to the left. Because the strip faces upward and not down, P moves at least a_i. Since the strip faces upward, this adds another $a_i/2$ to the upward movement. Thus, folding the creases around f_i moves P upward by $a_i + a_i/2 = 3a_i/2$. The same holds for f_j, f_k thus folding the creases f_i, f_j and f_k moves P upward by $3(a_i + a_j + a_k)/2$. Since $a_i + a_j + a_k = t$, folding all three faces moves P upward by $3t/2$. Since we defined $|f_i| = a_i$, the implication also holds the other way. □

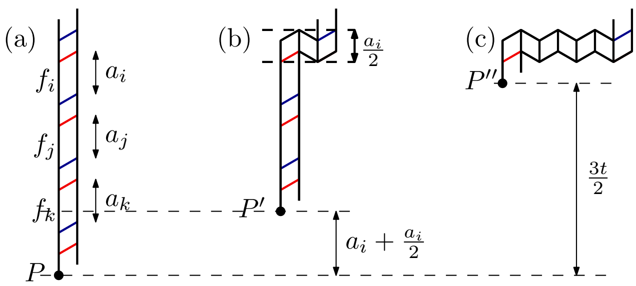


Figure 3.2: Folding the Staircase. (a) shows the unfolded staircase. (b) illustrates the staircase with a single face folded. (c) shows all three faces folded

There are creases along the Wrapper which are $t/2$ apart from each other. The Wrapper creases have to fold from right to left for the construction to work.

Lemma 12. *The Wrapper is flat foldable if and only if all Wrapper creases fold from right to left.*

Proof. Consider the two folded cases illustrated in Figure 3.3. When we fold a crease where the Wrapper does not cross the Column, we get the situation in Figure 3.3(b). Here c_1 is folded and the vertical part of the Wrapper is past crease c_2. c_2 is not foldable any more as the vertical part would have to go through the Wrapper. When we fold a crease where the Wrapper does cross the Column, we get the situation in Figure 3.3(c). Here c_3 is folded, folding the Wrapper past the Column. c_4 can no longer fold, or the Wrapper would fold through the Column. Folding other creases other than c_4 is the same as one of these two cases. Thus the Wrapper is not flat foldable when we do not fold the Wrapper creases from right to left.

Folding c_4 first would give the situation in Figure 3.3(d). This would allow crease c_3 to fold. This can be repeated with the next crease. Thus the Wrapper is only flat foldable if we fold from right to left. □

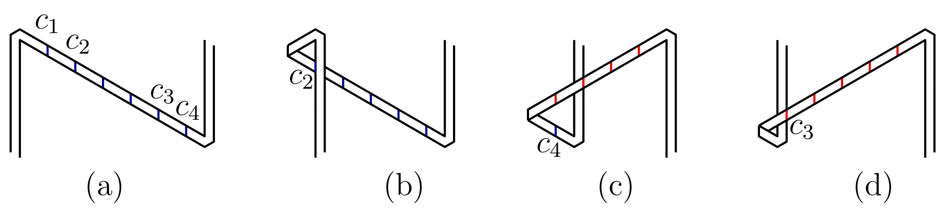


Figure 3.3: Folding the Wrapper. (a) shows the unfolded wrapper. (b) shows the Wrapper when c_1 is folded. (c) shows the Wrapper when c_3 is folded.

Theorem 5. *The assigned simple flat-foldability decision problem for polyiamond paper with equilateral triangle creases is strongly NP-complete in the one-layer, some-layers and all-layers models.*

Proof. We combine the properties induced from Lemma 11 and Lemma 12 to prove the theorem. We consider the entire construction from Figure 3.1. The process of checking a 3-Partition solution is visualized in Figure 3.4.

As shown in Lemma 11, when we fold the creases around three faces $f_i, f_j, f_k \in S$ for which $a_i + a_j + a_k = t$, the Hook is raised by $3t/2$. This means that by construction the point Q' and Q_1 are aligned. Folding a crease of the Wrapper flips the Hook and Staircase towards the Cage as shown in Figure 3.4. When a Wrapper crease has been folded, the point Q' and Q_1 are now touching as the Hook is forced inside the Cage. Because the Wrapper is folded from right to left by Lemma 12, the Hook is forced to pass through the Cage $n/3$ times sequentially from Q_1 to $Q_{\frac{n}{3}}$. Each Q_{i+1} is positioned exactly $\frac{3t}{2}$ above Q_i by construction and thus the Hook can only move through that slot if the correct height is reached by the correct Staircase folds. If the sum is not correct, the Hook would overlap parts of the Cage after the first wrapper fold. Thus if the Staircase does not have the correct sum, the Wrapper can not fold. Since all creases of the Wrapper have the same assignment, the Hook is forced through the Cage. After the Wrapper has folded twice, the Hook is moved back to its original position, only moved upwards by $3t/2$ which means we can repeat the steps with the next triplet summing to t.

Since the Wrapper contains $2 \cdot n/3$ creases, it forces the Hook $n/3$ times through the Cage. If the Wrapper tries to fold without a proper subset of Staircase folds, the Hook will either intersect with the Column if the sum of the subset is too large, or intersect with the Cage if the sum of the subset is not large enough. Folding only one Staircase crease means that the Wrapper can not fold since the Hook or Staircase will intersect the Column as the Staircase is now facing upward and sideways. As the Hook is reset after every cycle, it will stay within range of the Cage indents and will not be able to fold in front of the Cage but has to fold through it.

If the 3-Partition instance is valid, then the polygon has a simple folding, since we can fold the creases in the Staircase for a triplet and fold two Wrapper creases and repeat until all creases are folded. As argued, all Wrapper creases have the same crease assignment, thus the Hook passes through the Cage. For completeness we also prove that if the polyiamond has a simple folding, the 3-Partition instance has a valid solution. Since the Staircase has to fold for both creases

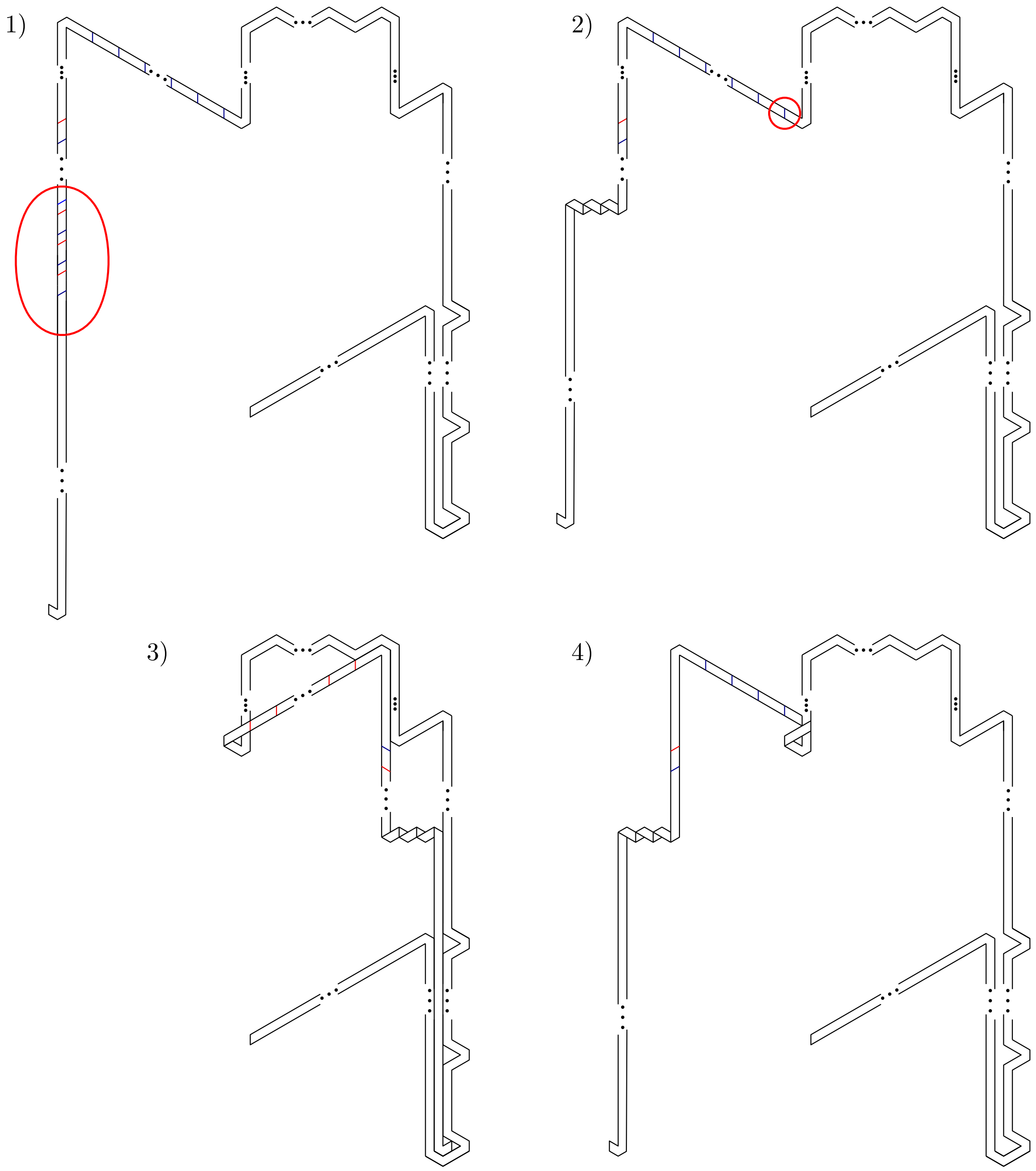


Figure 3.4: A simple polyiamond with equilateral triangular creases with assigned mountain and valley creases. The construction simple folds if and only if the corresponding 3-Partition instance has a solution. Creases which are folded next are shown by the red circles.

between each a_i for exactly three a_is in order to fit through the Cage with each wrap. All of the stairs in the Staircase are close to $t/3$ such that we need to shift by the right amount we need to have folded exactly three a_is. Because each a_i corresponds to a number in the 3-PARTITION instance, folding three sums to t.

Because each crease is folded sequentially, the construction holds for the one-layer and some-layer model. Since none of the creases only ever exist in a single layer, the all-layer model also holds.

As shown in Figure 3.1, the dimensions of the construction is bounded by a $O(tn) \times O(tn)$ rectangle which makes the reduction polynomial. □

3.3 Assigned equilateral triangle creases

The proof in Theorem 5 only considered a polyiamond shape. We can generalize this to fold from a more general kind of paper. For the traditional paper the general shape is *square* paper [3]. This is difficult in the equilateral triangle shape, thus we choose a different general shape; the hexagonal. In this section we are concerned with the following problem statement:

Problem statement 6 (**Assigned equilateral triangle simple flat-foldability**)**.** *Given is a crease pattern with mountain and valley assignments on a hexagonal equilateral triangle map. Is there a simple flat-folding satisfying the specified mountain and valley creases? If there exists a flat-folding, construct such a simple folding.*

We prove strong NP-completeness for the hexagonal equilateral triangle map folding with assigned creases. We construct a similar shaped construction as Section 3.2 from the hexagonal equilateral triangle map. In order to do this, we first fold the hexagonal into a long strip. From this long strip we will use some turn gadgets to create the shape of the polyiamond from Section 3.2.

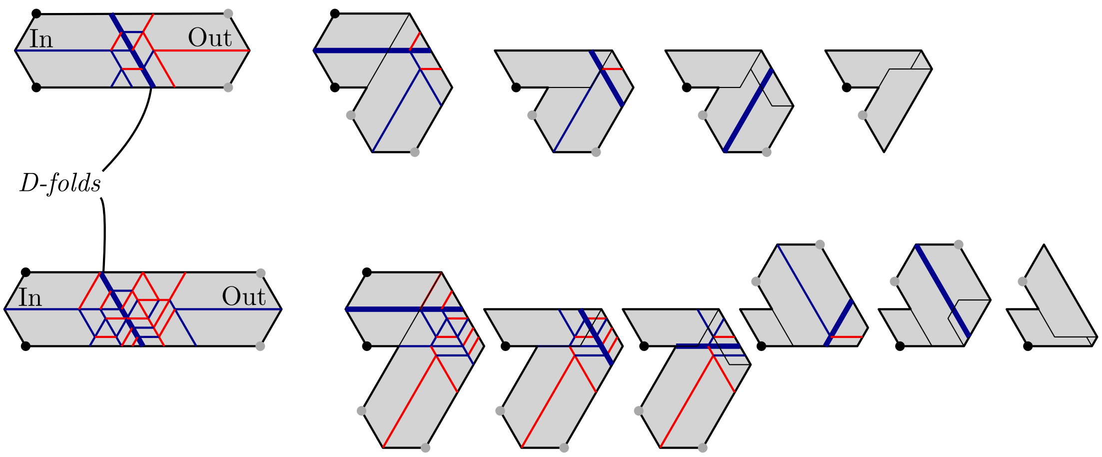


Figure 3.5: Assigned turn gadgets *Same* (top) and *Flip* (bottom) in an equilateral triangular grid. Red lines represent a mountain crease and blue lines represent a valley crease. The next crease to fold is indicated by the thicker line. *D-folds* are diagonal creases which span the width of the strip and turn the strip in some direction. The edges of the paper are labeled with black and gray dots.

We have two turn gadgets; *Same* and *Flip*. These two gadgets are shown in Figure 3.5. The Same turn gadget turns the paper and keeps the edge of the paper on the same side as the original edges. The Flip turn gadgets turns the paper and flips the edge of the paper to the other side than the original edge.

Lemma 13. *The turn gadgets* Same *and* Flip *can only be flat folded in the sequence as shown in Figure 3.5.*

Proof. Initially the highlighted D-folds are the only creases which can be folded. At any stage of the folding process of either of the gadgets, only a single crease can be simple folded by construction. Thus the turn gadgets can only be folded flat if the creases are folded in the sequence shown in Figure 3.5. □

Lemma 14. *We can construct the polyiamond in Figure 3.1 with some sequence of* Same *and* Flip *turn gadgets shown in Figure 3.5.*

Proof. Both the *Same* and *Flip* gadgets are necessary, as the edges of the paper between two gadgets have to align with each other. The edges of the paper have been marked with black and gray dots. The folded states of these gadgets are the turn direction of the paper, which means that we can not mirror or flip a gadget outright after some turn since we have to align the edges of the paper. Each turn gadget has to be folded in the specified order as indicated in Figure 3.5 as shown in Lemma 13. Consequently, we get the two horizontal creases marked *In* and *Out* which propagate the signal to the next turn gadget. For two concatenated turn gadgets, the *In* crease of the first gadget is connected to the *Out* crease of the other gadget. An example of two connected turn gadgets is shown in Figure 3.6. Because we are working with equilateral triangle creases, the directions in which the paper can go after a turn is an angle of 60°. The polyiamond has some turns which are at an angle of 120°. These turns can be achieved by performing multiple consecutive 60° turns as shown in Figure 3.6.

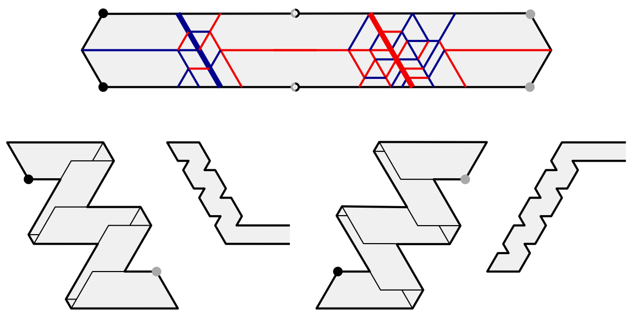


Figure 3.6: Zigzag gadget from a Same gadget and Flip gadget chained in a sequence. Strips in different direction from the standard direction can be emulated in this manner. The Flip part of this gadget can still be folded with only valley folds after the Same gadget has been folded.

In order to retain functionality of Staircase creases of the polyiamond, we must ensure that the Staircase creases are available to be folded all at once. The same holds for the Wrapper creases. Thus these parts of the polyiamond can not be constructed using the zigzag pattern. To construct the polyiamond from Figure 5, we start the strip at the end of the Cage. This part of the strip goes in a direction which is be achieved by a zigzag as shown in Figure 3.6. This means that the Wrapper and Staircase creases are on a long stretch of paper. All gadgets can be folded using valley folds only, which means that the paper will never intersect with itself.

After the Wrapper has been folded, the Wrapper creases can not fold until the Staircase creases are in place, otherwise it would intersect with the Column. The Wrapper creases can also not fold before the Hook has been folded. This is because the extra length would intersect with the bottom of the Cage. Thus the entire construction has to be folded before any of the creases from the polyiamond construction can be folded. Thus we obtain the similar polyiamond construction using the given *Same* and *Flip* turn gadgets. □

With Lemma 14 we can obtain the polyiamond from some long strip. This remains to show that we can fold the strip from some basic paper shape. For the equilateral triangle case we choose the hexagonal shaped paper as the basic shape.

Lemma 15. *Given a hexagonal shaped paper P and a strip P' with crease pattern Σ' consisting out of turn gadgets, there exists a crease pattern Σ which folds P into P' and all creases $c \in \Sigma$ have to be folded before all diagonal D-fold creases of the turn gadgets can be folded.*

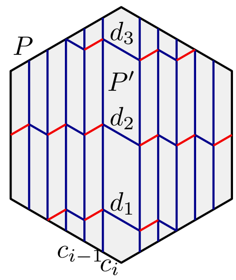


Figure 3.7: Initial *I-folds* to obtain the strip from a hexagonal piece of paper.

Proof. An illustration of the situation is shown in Figure 3.7. All creases in the crease pattern Σ are called initial folds, or *I-folds*. These vertical folds fold the paper P to the smaller strip P' in the middle. The creases have valley assignments. The crease pattern Σ' contains D-folds, which intersect with the I-folds. The crease $c_i \in \Sigma$ must fold before any D-fold can fold, since every D-fold intersects c_i. Because all $c \in \Sigma$ have the same crease assignment, c_i can not flat fold unless c_{i-1} has folded, or it would overlap with the paper P' and no longer able to flat fold. The same holds for the other half of the hexagon. This means that no D-fold can be folded before all of the I-folds are folded. □

We construct a similar shaped construction as Section 3.2, but create the shape of the polyiamond with turn gadgets. These turn gadgets will enforce the order of construction while only making folds which are local to the gadget. Since the order of folding requires overlapping creases, this gadget will only work for the some-layers and all-layers model.

We provide two turn gadgets; the *Same* and *Flip* gadget. These two are both needed in order to turn in different directions. We need these two different gadgets because the paper has to be aligned with the previous turn gadgets.

Theorem 6. *The assigned simple flat-foldability decision problem for hexagonal paper with equilateral triangular grid map folding is strongly NP-complete in the some-layers and all-layers models.*

Proof. This proof is by reduction from the decision problem in Theorem 5. We reconstruct this polyiamond from a strip of some thickness using concatenated turn gadgets with assigned creases. We prove that this thick strip of paper is simple flat-foldable if and only if the original polyiamond crease pattern is simple flat-foldable.

By Lemma 15 we can construct a strip of turn gadgets from a hexagonal paper. By Lemma 14 we can reconstruct the polyiamond using the turn gadgets from Figure 14. Thus if the assigned hexagonal equilateral triangle map is simple flat foldable, the polyiamond construction is also simple flat foldable. Since we construct the exact polyiamond from the turn gadgets, the converse also holds.

Since the turn gadgets require some folds of multiple layers, the one-layer model does not hold in this case. The turn gadgets always fold all layers, thus this reduction does hold for the all-layers model.

The size of the input paper is bounded by the number of input integers and the size of those integers of the 3-Partition instance. Thus the strip size is bounded by $O(nt)$ and the number of creases is bounded by the original piece of paper, thus $O(mn)$. □

3.4 Polyiamond paper with unassigned equilateral triangular creases

In Section 3.2 we presented the construction of a polyiamond with assigned equilateral creases. We look to prove similar strong NP-completeness for the unassigned case.

Problem statement 7 (Unassigned equilateral triangle simple flat-foldability). *Given is a crease pattern with mountain and valley assignments on an equilateral triangle map with polyiamond paper. Is there a simple flat-folding satisfying the specified mountain and valley creases? If there exists a flat-folding, construct such a simple folding.*

This construction of the assigned polyiamond does not translate well to the unassigned case. The Wrapper has to fulfill a requirement for the construction to function: folding the Hook through the Cage $\frac{n}{3}$ times. In order to do this, the Wrapper creases must fold from right to left and all fold with the same assignment. The crease assignment was not a problem for the assigned case, as we could just assign all creases the same mountain-valley assignment, which would trivially satisfy the requirement. Folding from right to left was also satisfied in the assigned case by the mountain-valley assignment of the Wrapper creases as shown in Lemma 12. For the unassigned case this is clearly not that easy. Thus in order to meet the requirements, we have to adjust the construction.

The complete construction is shown in Figure 3.8. We adjust the Wrapper and add an Arm. The adjusted Wrapper direction is to be able to force the Wrapper to fold from right to left. This is shown in Lemma 16. The Arm is to enforce the crease assignment. The remaining construction remains the same.

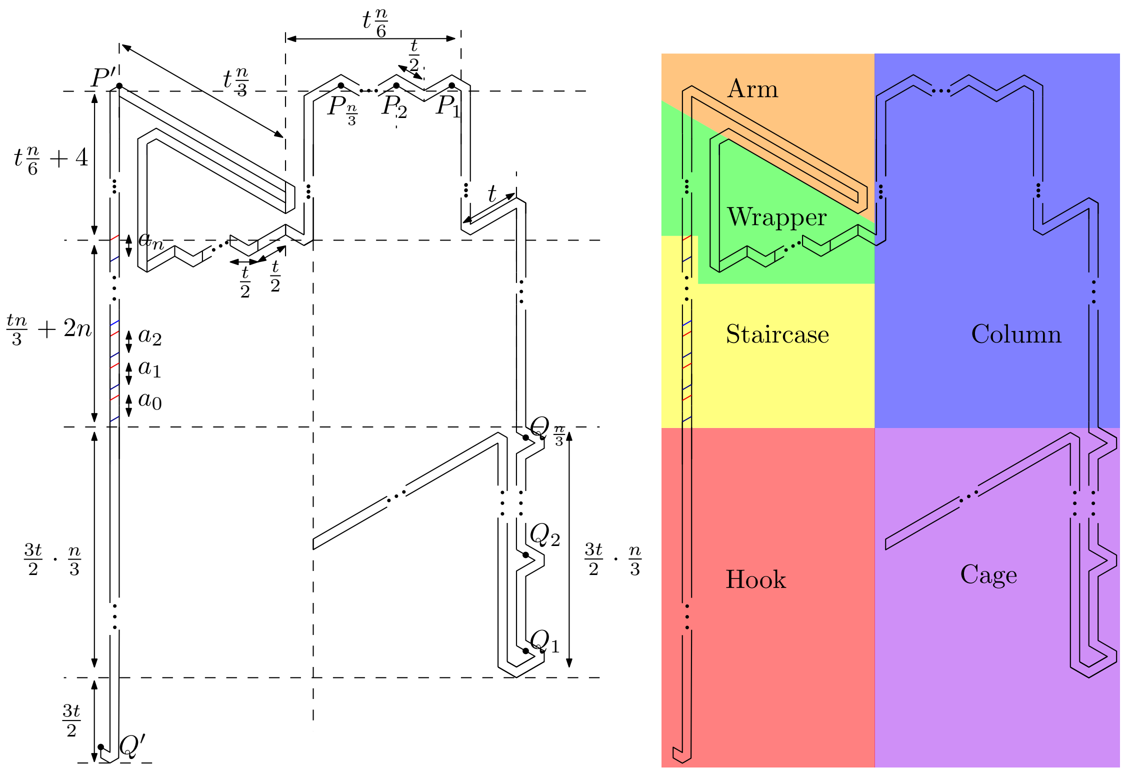


Figure 3.8: A simple polyiamond with equilateral triangular creases with assigned mountain and valley creases. The construction simple folds if and only if the corresponding 3-PARTITION instance has a solution.

We first look at the requirement of folding from right to left. We show that given the crease pattern in Figure 3.9, the wrapper has to fold from right to left, whilst maintaining simple foldability. Because the Wrapper in the assigned polyiamond shown in Figure 5 faces up, the Wrapper folds around the Column in different places. This means that any crease pattern forcing the order in which to fold the Wrapper creases will overlap different parts of the column. To still adhere to the all-layers model, the Column would have to include creases as well and fold along with the Wrapper, otherwise only the some-layers simple folding model would hold. Because of these difficulties, the wrapper is now a zigzag downwards in order to prevent overlap with the Column.

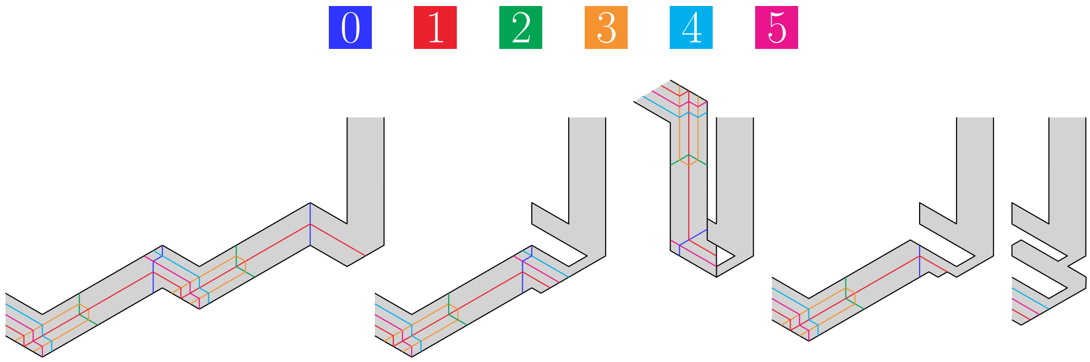


Figure 3.9: Unassigned crease pattern for the Wrapper. Creases are colored by their folding order. Repeating colors means a similar repeating folding order.

Lemma 16. *Given the crease pattern in Figure 3.9, the Wrapper must simple fold from right to left.*

Proof. We construct a crease pattern which forces the order of the creases. This crease pattern is shown in Figure 3.9. As the figure shows, the entire constructions folds up partly through the folding process at folding crease 4 or 5. If crease 5 is folded first, crease 4 only has one direction to fold which is in the opposite crease assignment. However, in the all-layers model this is not possible since crease 1 overlaps and is not yet foldable. Folding crease 4 first makes crease 5 foldable in the all-layers model. Since the entire construction is now overlapping itself, crease 5 has to have the opposite crease assignment as crease 4 or the Wrapper would move through itself. Thus crease 4 and 5 have to have the opposite crease assignment, which means that the Arm will be put back in its correct layer ordering after folding crease 4 and 5. Furthermore, at some point the Arm overlaps with the Column which means that the crease can only fold one way. Since these two folds have to be folded before any other crease of the Wrapper can be folded, this upward fold does not influence the folding of other creases in the Wrapper in any way. Thus this crease pattern properly enforces folding the Wrapper from right to left. ☐

It remains to show that the crease assignment for the Wrapper creases are all forced to be the same. This is enforced by the additional construction of the Arm in the polyiamond construction.

Lemma 17. *Consider the construction shown in Figure 3.8. The Wrapper creases all have to fold according to the same crease assignment.*

Proof. The addition of the Arm in Figure 3.8 is to make sure that every Wrapper crease is either all mountain or all valley. The Wrapper creases fold from right to left as shown in Lemma 16. When a Wrapper crease is folded, the Arm will overlap with the Column. If the next crease of the Wrapper does not fold in the same direction, the Arm would intersect with the Column. This interaction is shown in Figure 3.10. As the Wrapper creases fold, the Arm will overlap further along the Column, which keeps enforcing the same crease assignment. Thus all Wrapper creases have to fold in the same direction. ☐

With these adjustment we prove NP-completeness for the unassigned polyiamond.

Theorem 7. *The unassigned simple-foldability decision problem for polyiamond paper with equilateral triangle creases is strongly NP-complete in the some-layers and all-layers models.*

Proof. The reasoning is still very similar as the construction from Theorem 5. We have shown in Lemma 16 that the Wrapper has to fold from right to left. Lemma 17 showed that all creases

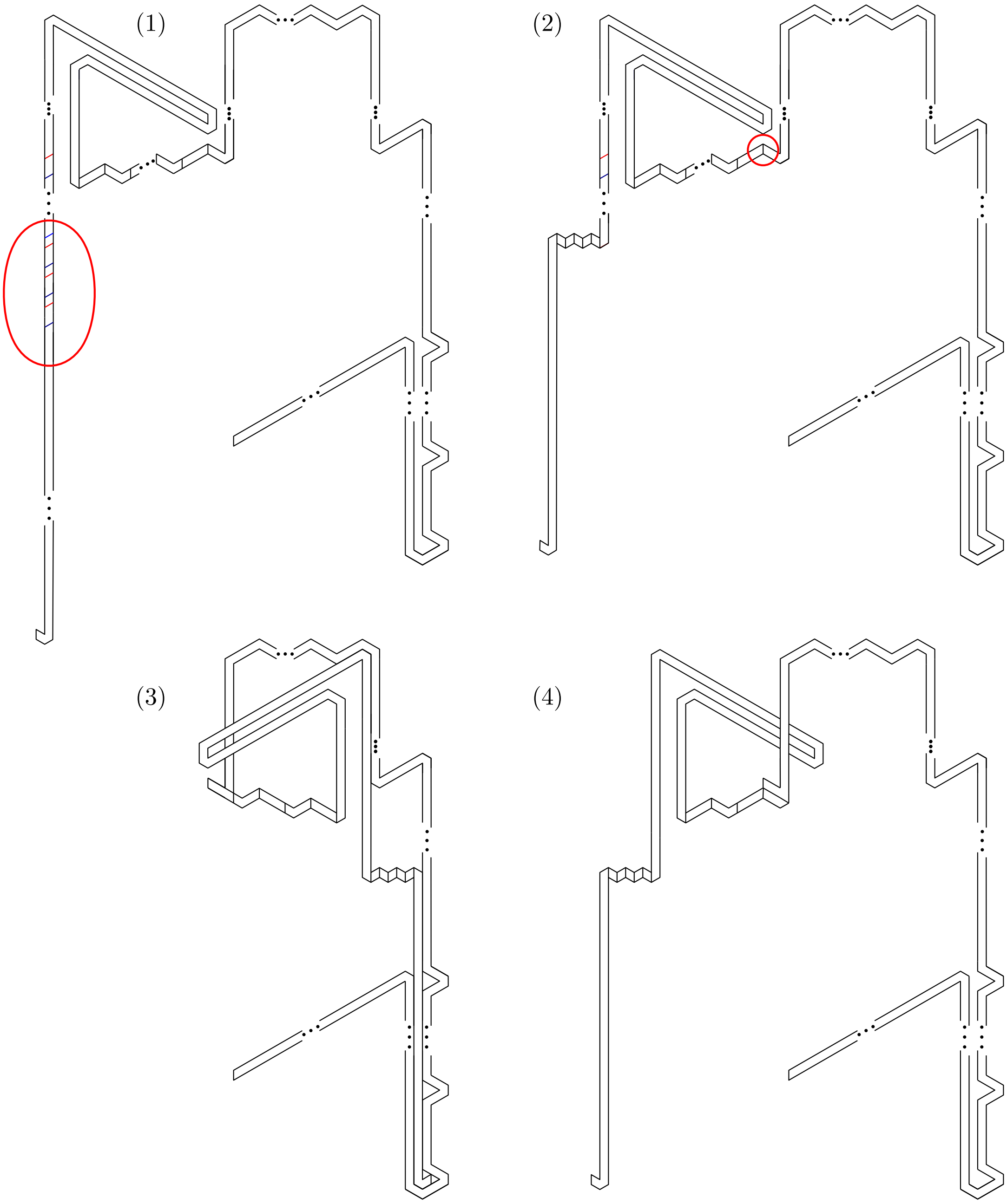


Figure 3.10: A polyiamond with equilateral triangle creases with unassigned creases. The construction simple folds if and only if the corresponding 3-Partition instance has a solution. Creases which are folded next are shown by the red circles.

have to be folded in the same direction. This forces the Hook through the Cage. Thus the same arguments from Theorem 5 directly apply under both the some-layers and all-layers models. Since we have the Wrapper folding multiple creases in order to force the folding order, the one-layer model will not hold. □

3.5 Unassigned equilateral triangle creases

In order to proof for the more general case of unassigned equilateral triangle folding, we construct some gadgets from which we can fold the polyiamond from Figure 3.8. However, this construction is not easily translated from a equilateral triangle strip. The Wrapper of the construction has angles of 120°. This means that the gadgets have to zigzag in order to replicate this angle. The problem with this is that what was in the construction a straight line between two turns, it now goes through multiple turn gadgets. If we draw that straight line, it will create some creases which go through the entire side of a turn gadget, thus is locally simple foldable. This will cause quite some unwanted behaviour, especially in the Some-layer simple folding model.

A construction would need to avoid turns of 120° as much as possible. At least extended lines through those turns are a problem. The main concern is that the functional creases for the 3-Partition problem have to be as much on the same face as possible.

For the unassigned version of the problem, we need to ensure that the entire construction is in place before any of the functional creases start folding. This means that we will have to propagate the signal first forward and then backward again in order to activate the functional creases after the construction is made. This each turn to fold twice, which means we have to create more gadgets in order to handle the extra case of propagating the signal backward.

The gadgets for the unassigned problem is shown in Figure 3.11. The gadgets from the assigned version are replicated for the unassigned version. Furthermore, there are four new gadgets with inputs and outputs called 2-Way turns. The folding order is indicated by the different colors. In order to prevent any of the functional creases to fold before the entire construction has been made, the 2-Way turns propagate the signal back after the forward signal has propagated through all other gadgets. Depending on which turn we make, we need a different gadget. This is not only the case for the forward propagating signal, but also for the backward propagating signal. This means that a gadget might have to flip twice; on the forward propagation and the backward propagation.

The signal propagates forward first and then backward. This is indicated in Figure 3.11 by In (input) and Out (output). The forward propagating signals are colored black and the backward return signals in red. The output signals can only be folded if the input signals have been folded. The turn is made after the forward signal has been completely folded. The return signal ensures that the gadgets remain aligned correctly. The signals determines the order in which the chained gadgets completed.

Because we propagate the signal throughout the construction, we need some extra gadgets. The side which the edge of the paper folds is important in order to properly chain the gadgets together. We capture all possible combinations with the four gadgets *Same-Same*, *Same-Flip*, *Flip-Same* and *Flip-Flip*. For instance Same-Flip means that on the forward propagating signal the edges of the paper are at the same side, and on the backward propagating signal the edges of the paper are on the other side.

The creases of the turn gadgets are shown in Figure 3.11. For each turn gadget, the folding sequence is shown in Appendix B. All creases fold can fold in the same direction, which means that it is simple foldable.

Conjecture 4. *The unassigned simple-foldability decision problem for equilateral triangular grid map folding is strongly NP-complete in the some-layers and all-layers models.*

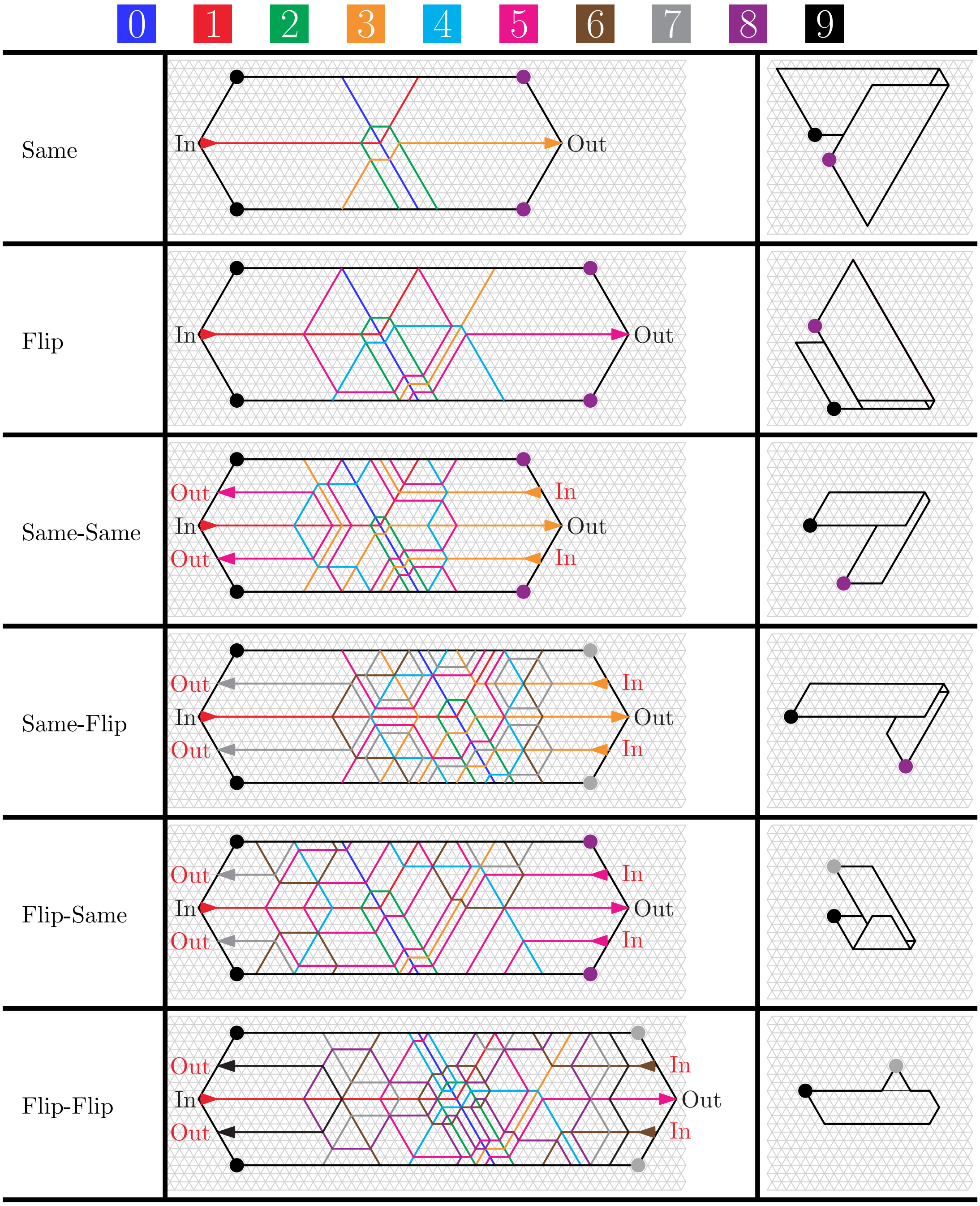


Figure 3.11: Turn gadgets for the unassigned case. Creases must be folded according to the color order at the top. The arrow heads indicate whether the crease is an input or output.

Chapter 4

Conclusion

In this thesis we have introduced a new variation of the map folding problem: Equilateral triangle map folding. As this is a new variation of the map folding problem, many different aspects were explored in both for the one dimensional equilateral triangle strip and the two dimensional equilateral triangle map.

In Section 2.1 we introduced the reachability problem for strip folding. We defined a strip as an arm and showed the reachable discrete locations. Furthermore, we showed that we can reach any reachable location within four folds. Using these properties, we defined a new TSP problem for strip folding. We showed the upper and lower bounds of the equilateral triangle strip TSP using the reachability results. Despite the upper and lower bounds, we conjectured that the equilateral triangle strip TSP is still NP-hard (Conjecture 1). More work has to be done on equilateral triangle TSP, which can likely be reduced from some version of TSP with neighborhoods.

In Section 2.2 we explored flat foldability for equilateral triangle strips in two different folding models. We proved that equilateral triangle strips can always be flat folded in the general folding model. Flat foldability for equilateral triangle strips in the simple folding model remains open. Many different approaches were presented, but no proof was found. An example of a simple folded state which does contain a cycle was presented. This shows that even states which might not seem to be simple foldable can be. We showed an approach using unfolding from the convex hull. This approach requires that there exists a crease on the convex hull in order to prove flat foldability. This approach looks promising as it utilizes the geometric shape of the folded state of the equilateral triangle strip. Counter examples were shown of folded state which do not have a crease on the convex hull. Lastly, we showed an approach which relies on induction on the length of the equilateral triangle strip. Through brute force calculating all possible folding orders, we found that a every chain of strips which have a similar crease pattern have at least one similar folding order. This was only calculated for strips up to a length of 10. This means that this method relies on a relatively small amount of strips which are similar. Even though this approach looks promising, we conclude that this approach is unlikely to be fruitful since it does not take into account the geometric properties of the entire strip.

In order to evaluate simple flat foldability for equilateral triangle strips, we calculated all foldable states of all possible equilateral triangle strips of length up to 10. All of these strip had a simple flat folding. To make calculations easier, we introduced a new coordinate system. This coordinate system makes reflecting in an equilateral triangle grid more convenient.

Using all previous results, we conjectured that equilateral strip folding in the simple folding model is always possible (Conjecture 2). Using the observations we made, we also conjectured that simple flat foldability for a $1 \times n$ strip with diagonals is polynomial time solvable (Conjecture 3).

In Chapter 3 we showed that various two-dimensional equilateral triangle map folding problems are NP-complete. We used a reduction from 3-Partition to show that equilateral triangle map flat foldability for polyiamond paper with some mountain valley assignment is strongly NP-complete. Furthermore, we showed that equilateral triangle map flat foldability for hexagonal paper with some mountain valley assignment is also strongly NP-complete by reduction from the equilateral triangle map folding for polyiamond paper. Using a similar gadget, we showed strong NP-completeness for unassigned equilateral triangle map flat foldability with polyiamond paper. Lastly, we presented some useful gadgets for a proof for unassigned equilateral triangle map flat foldability for some convex shaped paper. We conjectured that this problem is strongly NP-complete (Conjecture 4). Future research is encouraged to find a different gadget than the one presented in this thesis.

Future research could look at the main question for equilateral triangle map folding: what is the computational complexity of deciding whether an equilateral triangle can be folded flat with a specified mountain-valley assignment (Open problem 2)? This is similar to the long standing open problem of map folding (Open problem 1). We do not expect the equilateral triangle map folding problem to be easier than the square map folding problem.

Equilateral triangle strip simple foldability is an open problem which we conjectured has the very positive result of always foldable. Our computational method only calculated all strips up to the length of 10. This search space could be increased with more time and resources.

There is interest on the combinatorial side of research to find the possible valid flat folded states of stamps of some length [21]. This can be extended for equilateral triangle stamps as no formula has been found for this.

This thesis has not addressed the infinite simple folding models [3][2]. This is an interesting variation and requires completely different gadgets than those presented in this thesis.

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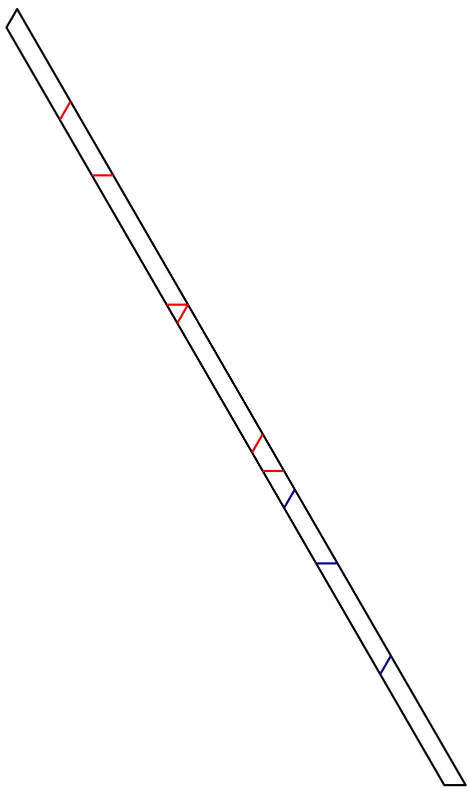
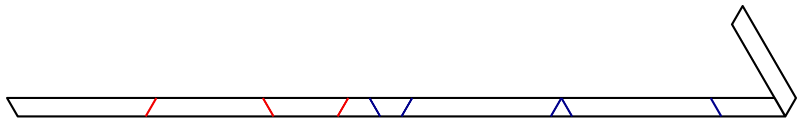
Appendices

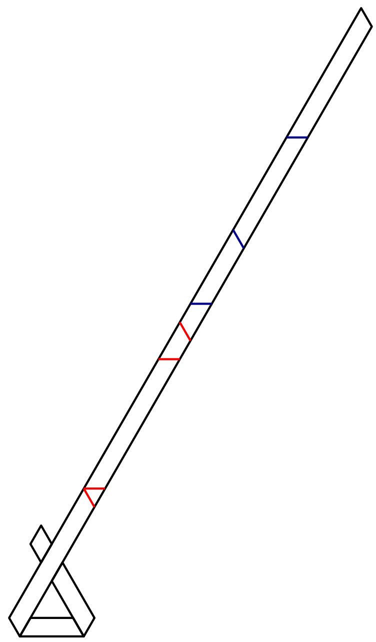
Appendix A

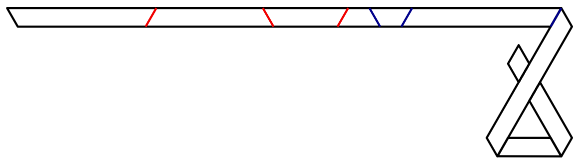
One-dimensional folding sequences

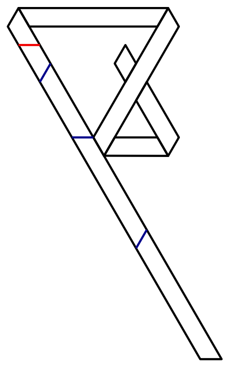
A.1 Simple foldable cycle

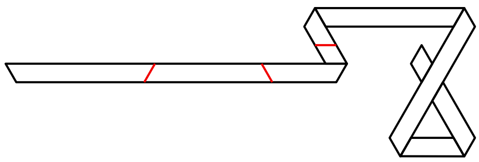


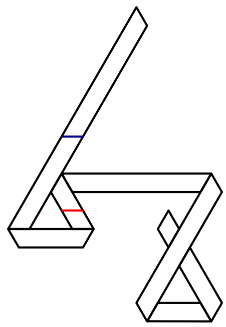


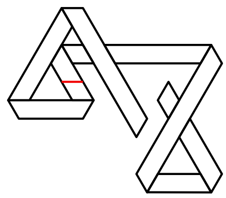


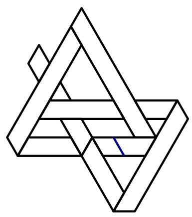


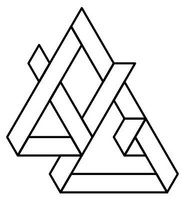












A.2 Smallest simple foldable cycle

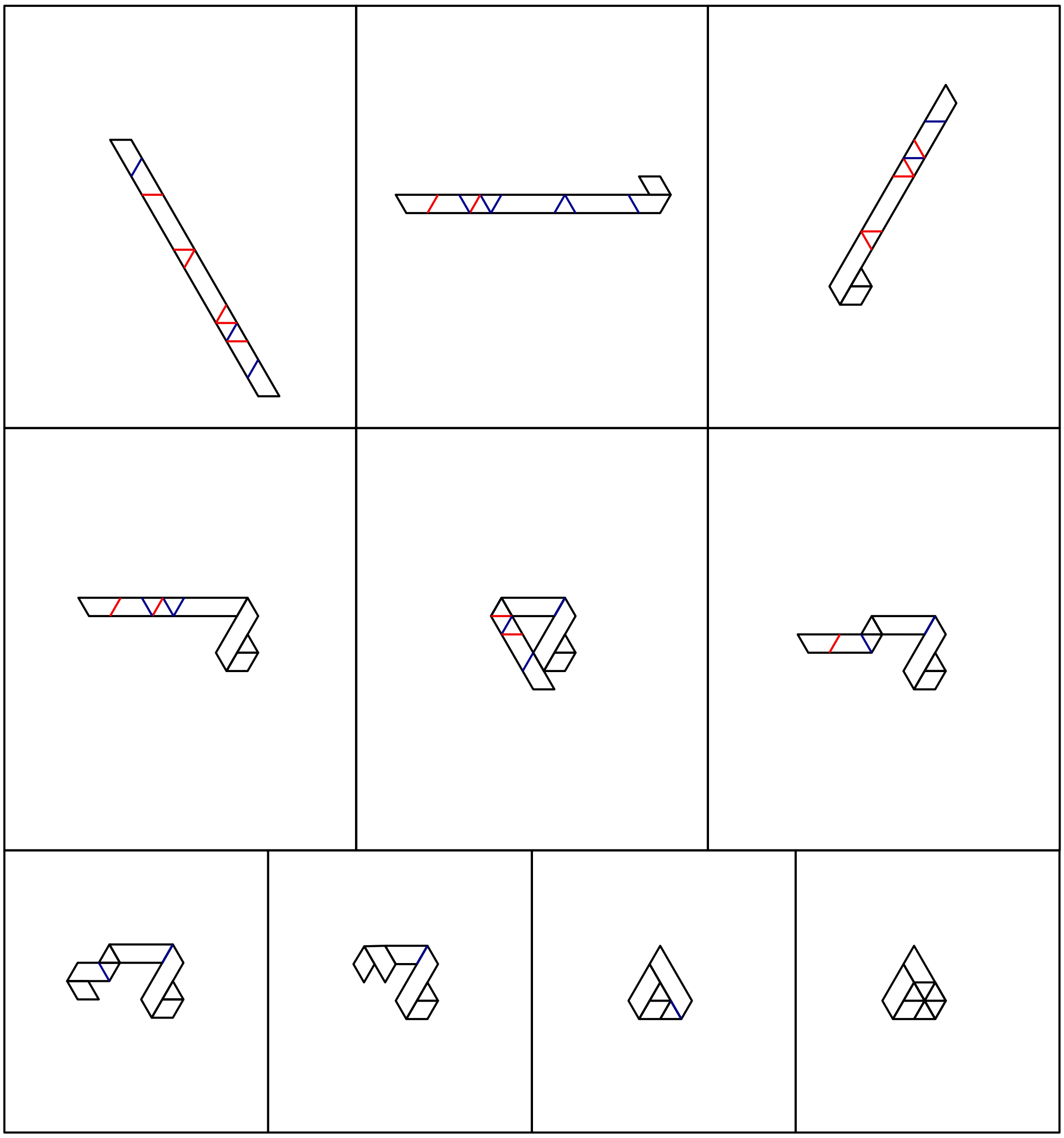


Figure A.1: Caption

Appendix B

Two-dimensional folding sequences

B.1 Unassigned turn gadgets

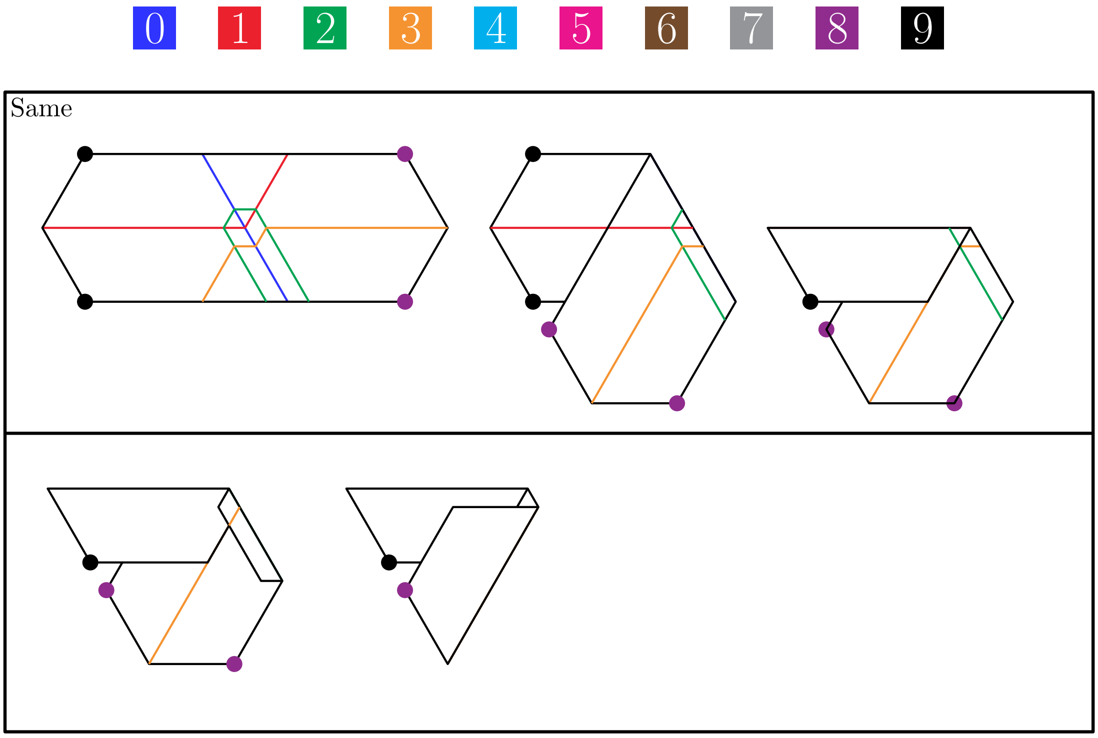


Figure B.1: The folding sequence for the unassigned Same turn gadget.

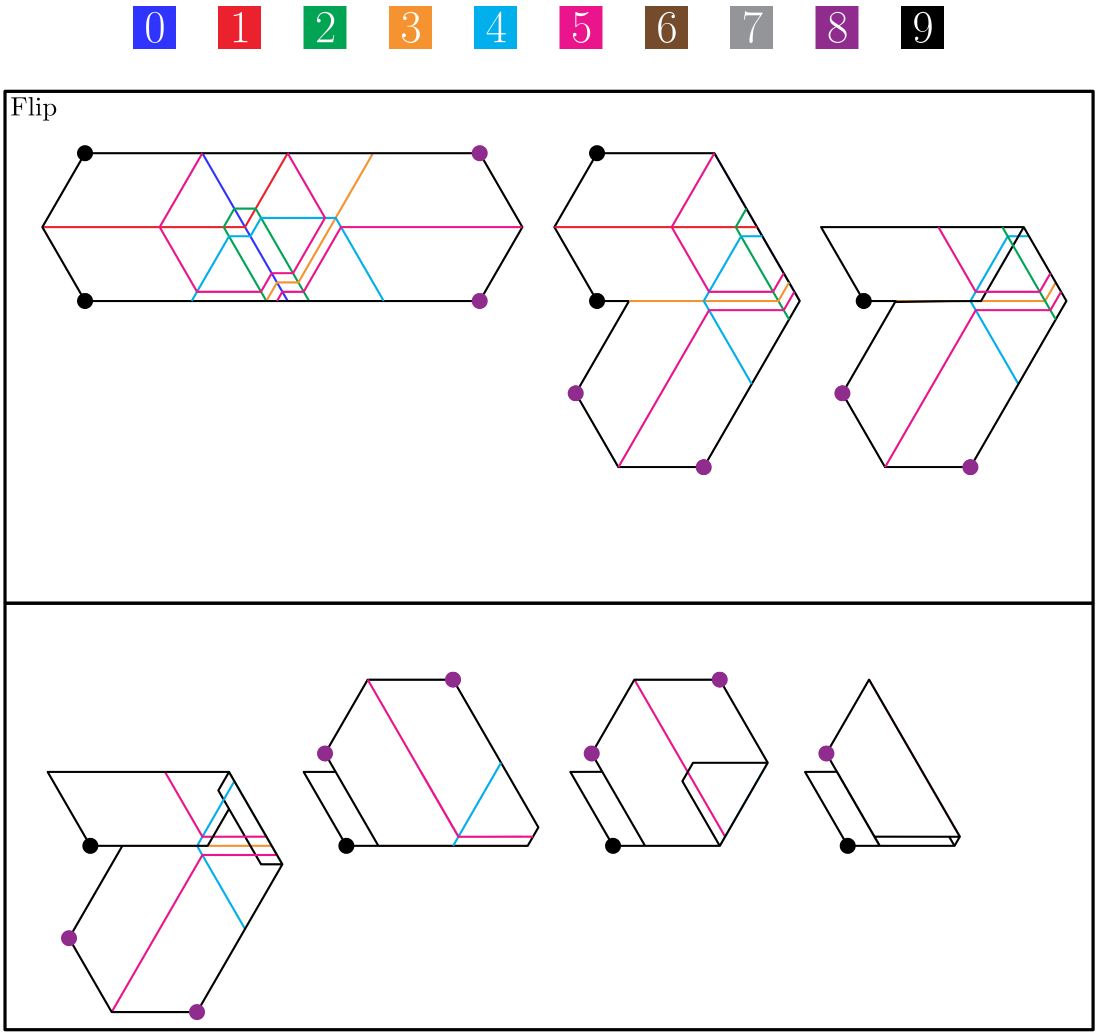


Figure B.2: The folding sequence for the unassigned Flip turn gadget.

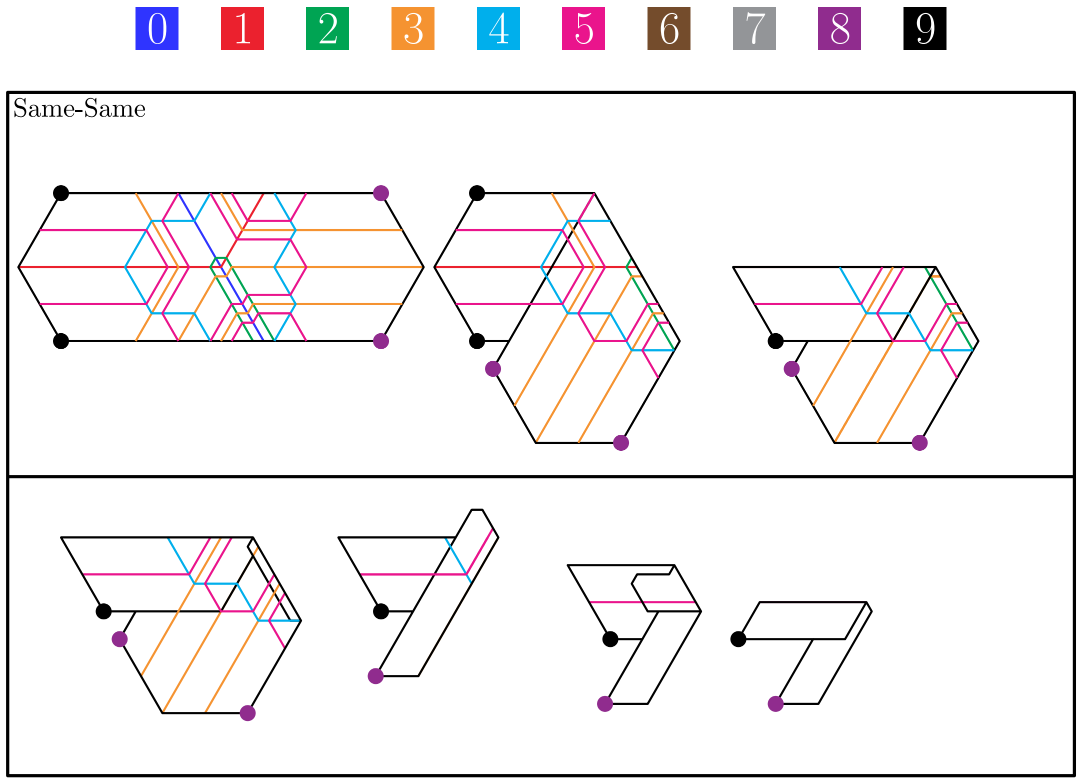


Figure B.3: The folding sequence for the unassigned Same-Same turn gadget.

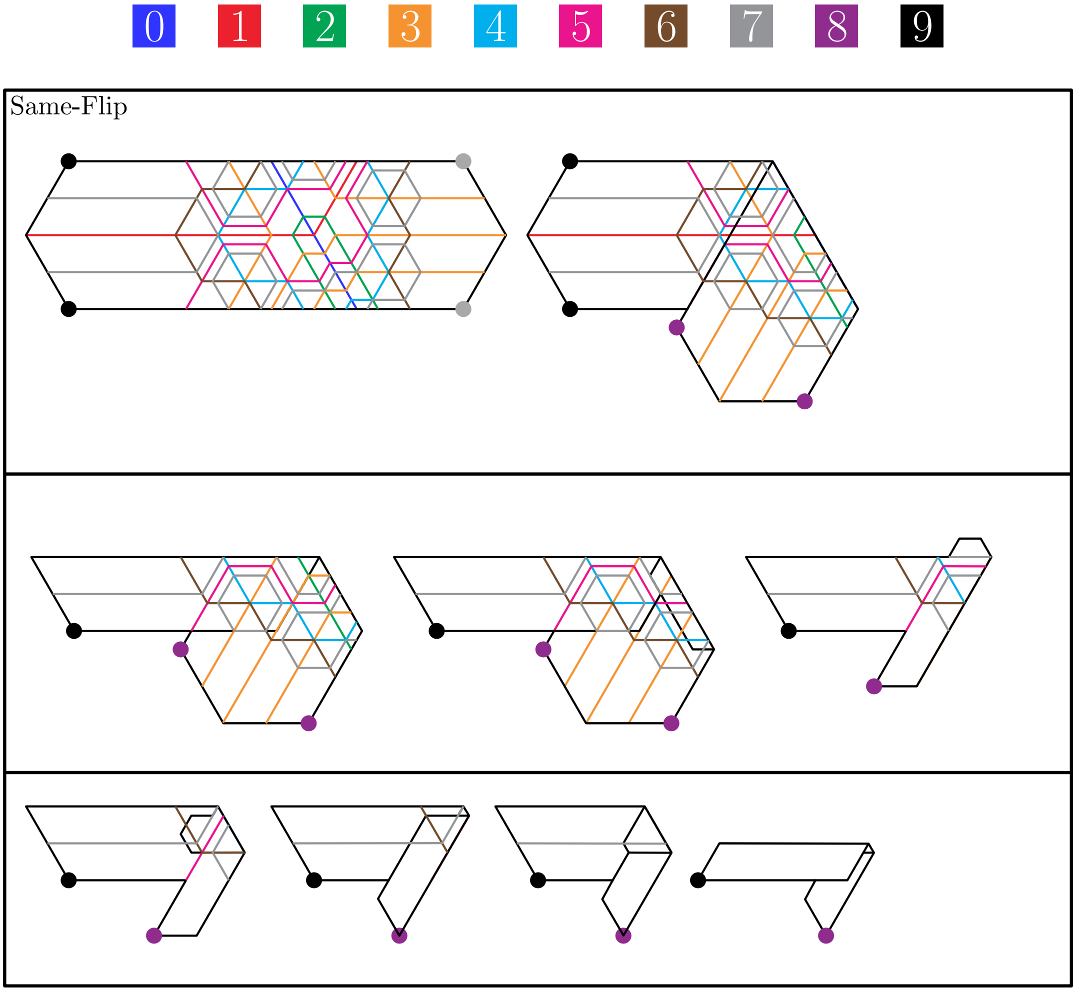


Figure B.4: The folding sequence for the unassigned Same-Flip turn gadget.

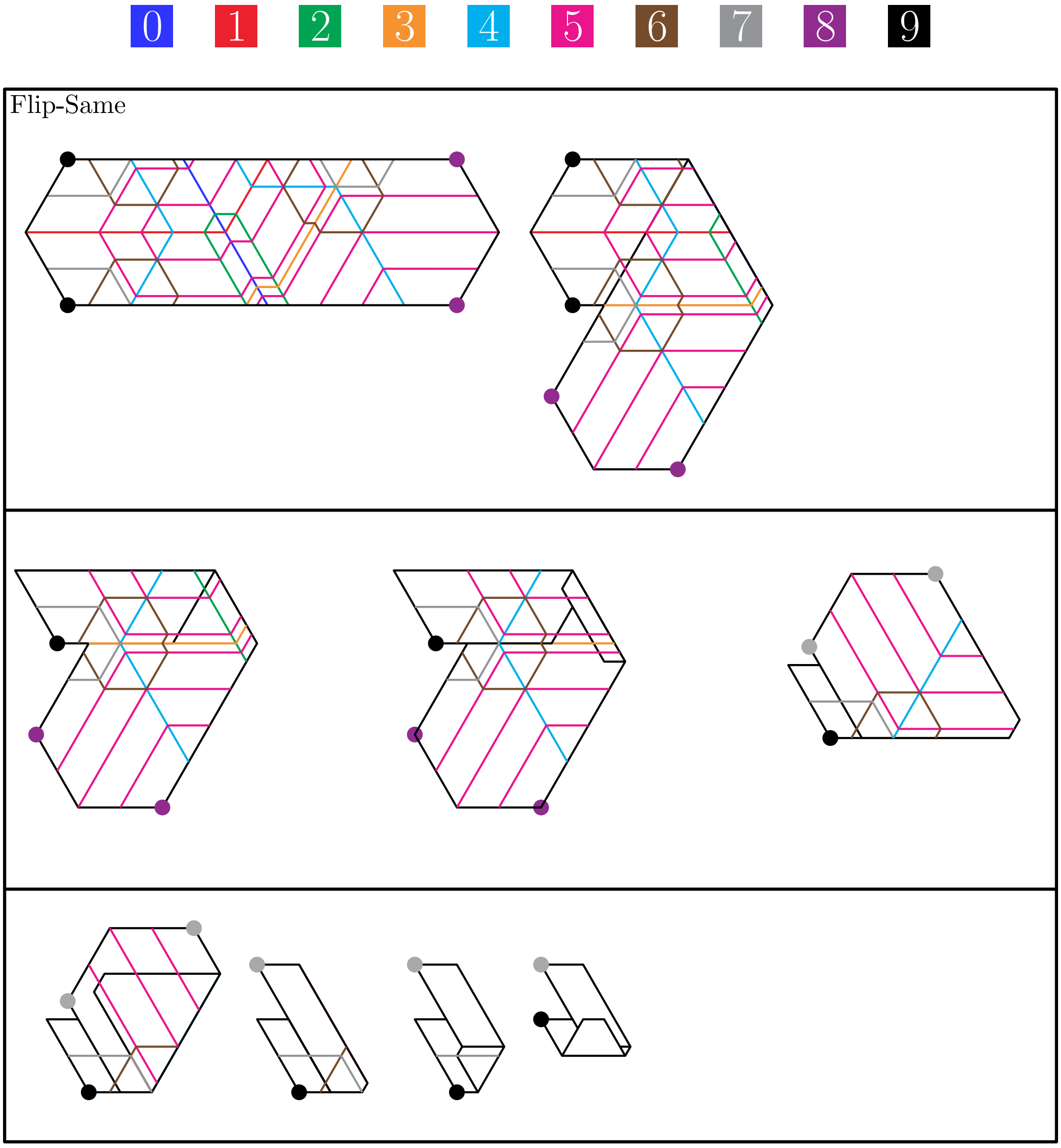


Figure B.5: The folding sequence for the unassigned Flip-Same turn gadget.

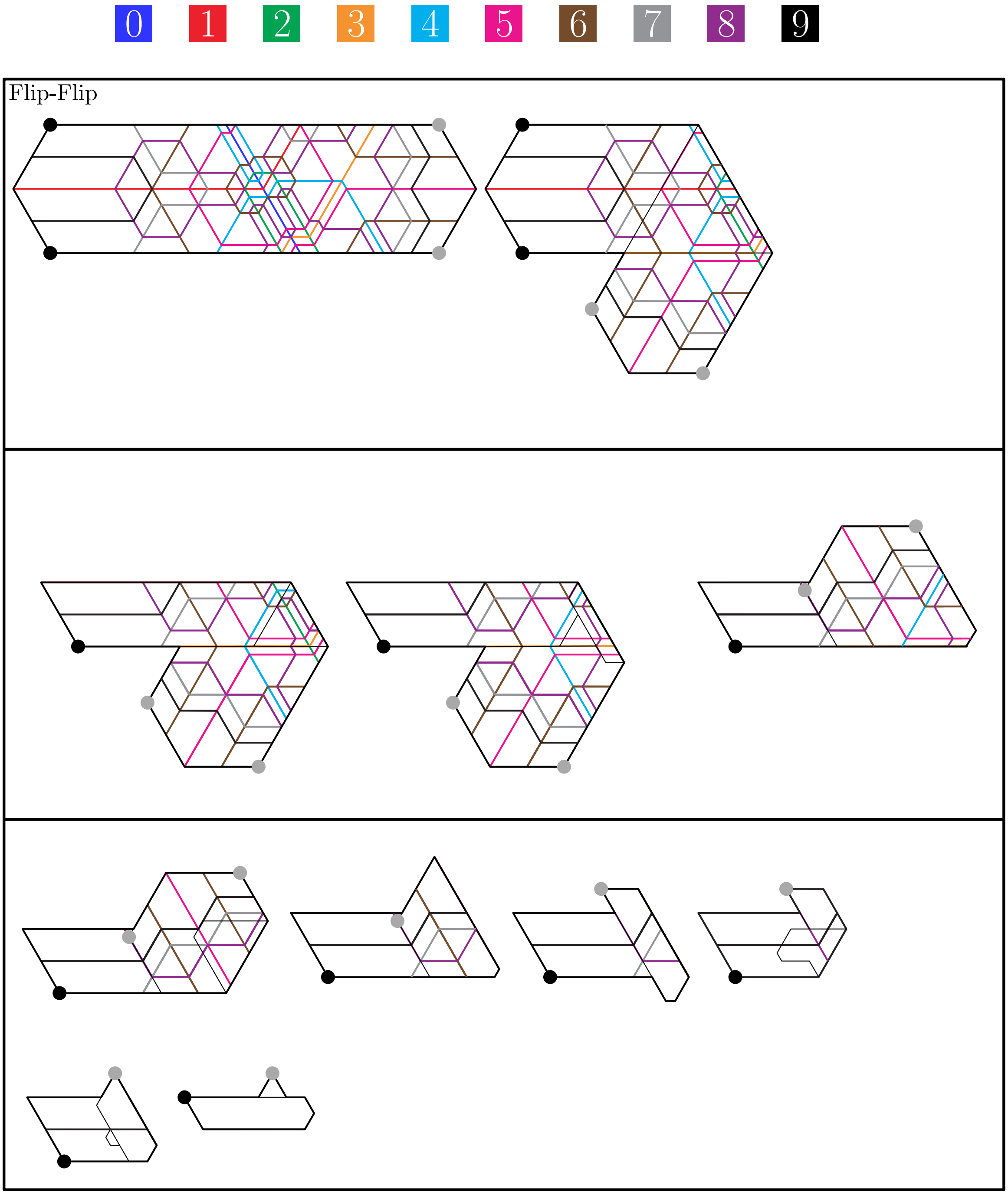


Figure B.6: The folding sequence for the unassigned Flip-Flip turn gadget.