Low order realizations for 2-D transfer functions

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Abstract

In this note a realization result for causal 2 - D transfer functions with separable numerator or seperable denominator is generalized. Not only a possible dimension of the local state space will be determined but also system matrices will be given.

1. Introduction

Recently some results concerning state space realization of a 2 - D transfer function have appeared in the literature, see [1], [2], [3]. These papers also give some results for the multivariable case. The obtained state space equations can be written in a form as was proposed by Roesser in [4]. This note will be concerned with the scalar case. Consider therefore the causal 2 - D transfer function

1.1. $$T(s,z) = \frac{p(s,z)}{q(s,z)} = \frac{p_0(s) + p_1(s)z + \quad + p_n(s)z^n}{q_0(s) + q_1(s)z + \quad + q_n(s)z^n}\ .$$

Here $p(s,z)$ and $q(s,z)$ are coprime two variable real polynomials, $p_i(s)$, $q_i(s)$ are polynomials in s only for $i = 0,\ldots,n$ and $q_n(s)$ is a monic polynomial $\neq 0$.

Causality means that the following degree conditions are satisfied

$$\deg_s(q_n(s)) \geq \deg_s(q_i(s)) \qquad i = 0,\ldots,n-1$$

$$\deg_s(q_n(s)) \geq \deg_s(p_j(s)) \qquad j = 0,\ldots,n\ .$$

We will now assume that

$$\deg_s(q_n(s)) = m\ .$$

The state space equations as proposed by Roesser can be written as follows

1.2. $$\begin{bmatrix} R_{k+1,h} \\ S_{k,h+1} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} R_{kh} \\ S_{kh} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{kh}$$

$$Y_{kh} = [C_1 \mid C_2] \begin{bmatrix} R_{kh} \\ S_{kh} \end{bmatrix} + D\, u_{kh}$$

$$k,h = 0,1,\ldots$$

$\begin{bmatrix} R_{kh} \\ S_{kh} \end{bmatrix}$ is the local state vector, u_{kh} is the (scalar) input and Y_{kh} is the (scalar) output. The matrices have appropriate dimensions. Initial conditions will be taken zero.

A well known result is

1.3. Theorem

Every scalar real 2 - D transfer function has a real realization which can be written in the form (1.2) where A_1 is an $n \times n$ matrix and A_4 is an $2m \times 2m$ matrix. Furthermore, if $p(s,z)$ or $q(s,z)$ is a separable polynomial (i.e. can be written as the product of a polynomial in s and a polynomial in z) then there exists a real realization of the form (1.2) where A_1 is an $n \times n$ matrix and A_4 is an $m \times m$ matrix.
By reversing the role of s and z one can obtain a realization of the order $(m + 2n)$ instead of $(n + 2m)$.

2. The result

We will now consider *first level realizations* $(A(s),B(s),C(s),D(s))$ of $T(s,z)$, see [2]. Here $A(s)$, $B(s)$, $C(s)$, $D(s)$ are proper 1 - D transfer matrices such that

2.1. $$T(s,z) = C(s)[zI - A(s)]^{-1}B(s) + D(s) .$$

Suppose, that $p(s,z)$ is not a primitive polynomial i.e., that $p_0(s),\dots,p_n(s)$ have a nontrivial common factor $\varphi(s)$. Let $\deg_s(\varphi(s)) = \ell \geq 1$. This factor is called the *content* of $p(s,z)$, and $\overline{p(s,z)}$, defined by $p(s,z) = \varphi(s)\overline{p(s,z)}$, is called the primitive part of $p(s,z)$. See also [5].
We factorize $q_n(s)$ as follows

$$q_n(s) = \psi_1(s)\psi_2(s)$$

2.2. such that

$$\deg_s(\psi_2(s)) \geq \deg_s(\varphi(s)) \text{ and } \deg_s(\psi_1(s)) \geq \max_i \deg_s(p_i(s)) - \ell .$$

If a factorization as in (2.2), satisfying the degree condition, does not exist such that $\psi_2(s)$ is a real polynomial, we proceed as follows (observe that this can happen only in the case where $\varphi(s)$ is a polynomial with odd degree). Let $\bar{\varphi}(s)$ be a common factor of $p_0(s),\dots,p_n(s)$ such that $\deg_s(\bar{\varphi}(s)) = \ell - 1$. Now a factorization, as in (2.2), in real polynomials such that $\deg_s(\psi_2(s)) = \deg_s(\bar{\varphi}(s))$ does exist and we can use $\bar{\varphi}(s)$ instead of $\varphi(s)$ in the following.

Let $p_i(s) = \varphi(s)\bar{p}_i(s)$, $i = 0,\dots,n$. Now it is clear that $(A(s),B(s),C(s),D(s))$ where

$$A(s) = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ \frac{-q_0(s)}{q_n(s)} & \cdots & & \frac{-q_{n-1}(s)}{q_n(s)} \end{bmatrix}, \quad B(s) = \begin{bmatrix} 0 \\ \vdots \\ \frac{\varphi(s)}{\psi_2(s)} \end{bmatrix}$$

2.3.

$$C(s) = \begin{bmatrix} \frac{\bar{p}_0(s)}{\psi_1(s)}, \dots, \frac{\bar{p}_{n-1}(s)}{\psi_1(s)} \end{bmatrix} + \frac{\bar{p}_n(s)}{\psi_1(s)} \begin{bmatrix} \frac{-q_0(s)}{q_n(s)}, \dots, \frac{-q_{n-1}(s)}{q_n(s)} \end{bmatrix}$$

$$D(s) = \frac{p_n(s)}{q_n(s)}$$

is a first level realization of $T(s,z)$.

We will use the notation

$$\bar{A}(s) = \begin{bmatrix} \frac{-q_0(s)}{q_n(s)}, \dots, \frac{-q_{n-1}(s)}{q_n(s)} \end{bmatrix}, \quad \bar{B}(s) = \frac{\varphi(s)}{\psi_2(s)}$$

$$\bar{C}(s) = \begin{bmatrix} \frac{\bar{p}_0(s)}{\psi_1(s)}, \dots, \frac{\bar{p}_{n-1}(s)}{\psi_1(s)} \end{bmatrix}, \quad \bar{D}(s) = \frac{\bar{p}_n(s)}{\psi_1(s)} .$$

We can realize these 1 - D transfer matrices and obtain the following realizations

$(\overline{AA},\overline{AB},\overline{AC},\overline{AD})$ for $\bar{A}(s)$

$(\overline{BA},\overline{BB},\overline{BC},\overline{BD})$ for $\bar{B}(s)$

$(\overline{CA},\overline{CB},\overline{CC},\overline{CD})$ for $\bar{C}(s)$

$(\overline{DA},\overline{DB},\overline{DC},\overline{DD})$ for $\bar{D}(s)$

(DA,DB,DC,DD) for $D(s)$

It is clear that we can do this in a way such that

$$\overline{AA} = \overline{BA} = DA\,, \qquad \overline{CA} = \overline{DA}$$

$$\overline{AC} = \overline{BC} = DC\,, \qquad \overline{CC} = \overline{DC}$$

(for instance by using observable canonical forms).

These realizations can be "tied together" to form a realization of the type (1.2) in the following way

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ & & \overline{AD} & & \end{bmatrix} \quad (n \times n)\,, \qquad A_2 = \left[\begin{array}{ccc|ccc} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & & & \\ & \overline{AC} & & 0 & \cdots & 0 \end{array}\right] \quad (n \times (2m-\ell))$$

$$A_3 = \begin{bmatrix} \overline{AB} \\ \overline{DB}\,\overline{AD} + \overline{CB} \end{bmatrix} \quad ((2m-\ell) \times n)\,, \qquad A_4 = \begin{bmatrix} \overline{AA} & 0 \\ \overline{DB}\,\overline{AC} & \overline{CA} \end{bmatrix} \quad ((2m-\ell)\times(2m-\ell))$$

2.4.

$$C_1 = [\overline{CD} + \overline{DD}\,\overline{AD}] \quad (1 \times n)\,, \qquad C_2 = [\overline{DD}\,\overline{AC} \mid \overline{CC}] \quad (1 \times (2m-\ell))$$

$$B_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \overline{BD} \end{bmatrix} \quad (n \times 1)\,, \qquad B_2 = \begin{bmatrix} \overline{BB} \\ \overline{DB}\,\overline{BD} \end{bmatrix} \quad ((2m-\ell) \times 1),\ D = DD,$$

where $\ell = \deg_s(\varphi(s))$.
Observe that if $p(s,z)$ is a primitive polynomial then the construction gives a realization of the order $n + 2m$ (the known result) and in case $\varphi(s,z) = \varphi(s)\varphi_1(z)$ (separable numerator) we obtain a realization of the order $n + m$.

It is a matter of straightforward verification that

$$A(s) = A_1 + A_2[sI - A_4]^{-1}A_3, \qquad B(s) = B_1 + A_2[sI - A_4]^{-1}B_2$$

$$C(s) = C_1 + C_2[sI - A_4]^{-1}A_3, \qquad D(s) = D + C_2[sI - A_4]^{-1}A_3$$

as is required. See [2].

Summarizing we have

2.5. Theorem

Let $T(s,z) = p(s,z)/q(s,z)$ be a causal 2-D transfer function. Suppose $\varphi(s)$ is the content of $p(s,z)$ and $\deg_s(\varphi(s)) = \ell$. Then there exists a realization of the form (1.2) of the order $n + 2m - \ell$. This realization is possibly complex (depending upon the factorization (2.2)) but there exists always a real realization of the order $n + 2m - \ell + 1$. □

Remark

By interchanging s and z the same kind of result can be obtained and one can take the minimum of the two for the order of a realization.

We will now derive an analogous result for the denominator case. Suppose $\psi(s)$ is the content of $q(s,z)$ and let the degree of $\psi(s)$ be $r \geq 1$. Let $q_i(s) = \psi(s)\bar{q}_i(s)$.
A first level realization of $T(s,z)$ is then $(A(s),B(s),C(s),D(s))$ where

$$A(s) = \begin{bmatrix} 0 & 1 & & \\ \vdots & & & \\ 0 & & & 1 \\ \dfrac{-\bar{q}_0(s)}{\bar{q}_n(s)} & \cdots\cdots & & \dfrac{-\bar{q}_{n-1}(s)}{\bar{q}_n(s)} \end{bmatrix} \qquad B(s) = \begin{bmatrix} 0 \\ \\ 0 \\ 1 \end{bmatrix}$$

$$C(s) = \begin{bmatrix} \dfrac{p_0(s)}{q_n(s)}, \ldots, \dfrac{p_{n-1}(s)}{q_n(s)} \end{bmatrix} + \frac{p_n(s)}{q_n(s)} \begin{bmatrix} \dfrac{-\bar{q}_0(s)}{\bar{q}_n(s)}, \ldots, \dfrac{-\bar{q}_{n-1}(s)}{\bar{q}_n(s)} \end{bmatrix},$$

$$D(s) = \frac{p_n(s)}{q_n(s)}.$$

Let

$$\tilde{A}(s) = \left[\frac{-\bar{q}_0(s)}{\bar{q}_n(s)}, \ldots, \frac{-\bar{q}_{n-1}(s)}{\bar{q}_n(s)}\right], \qquad \tilde{B}(s) = 1$$

$$\tilde{C}(s) = \left[\frac{p_0(s)}{q_n(s)}, \ldots, \frac{p_{n-1}(s)}{q_n(s)}\right], \qquad \tilde{D} = \frac{p_n(s)}{q_n(s)}.$$

We can now proceed in completely the same way as in (2.4) and obtain an analogous result.

Observe that the realization, we obtain in this way, is always real.

We now have

2.6. Theorem

Let $T(s,z) = p(s,z)/q(s,z)$ be a causal 2 - D transfer function. Suppose $\psi(s)$ is the content of $q(s,z)$ and $\deg_s(\psi(s)) = r$. Then there exists a real realization of the form (1.2) of the order $n + 2m - r$. □

This is a generalization of the separable denominator result.

Remark

Again, by interchanging s and z the same kind of result can be obtained and the minimum of the two can be taken as the order of a realization.

Remark

The result of this note can also be obtained by using a McMillan degree argument as is done in [3].

3. Conclusions

In this note a generalization of the realization result (1.3) has been obtained. The order of a realization can be taken to be the minimum of the numbers given by theorem (2.5) and theorem (2.6). If the numerator and the denominator polynomial both are primitive the usual "n + 2m" result is obtained. If the numerator or the denominator polynomial is separable then the "n + m" result is obtained.

4. References

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