A continuous version of the prisoner's dilemma

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by

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Abstract

The Prisoner's Dilemma is a non-zero-sum discrete two-player game. It is often used to study social phenomena like cooperation. In this article we describe and analyze a continuous version of the Prisoner's Dilemma. The continuous version may provide further insights in the phenomenon of cooperation because it allows new types of strategies.

Contents

1 Introduction

The Prisoner's Dilemma (PD) is a two-player game, explained below. It has been studied extensively, both in an empirical context and a theoretical context. In [1], Axelrod gives a very readable account of the PD and its relevance to everyday life. He draws from insights obtained through two tournaments for computer programs that play the iterated PD. Hofstadter summarizes these results and philosophizes about them in [4]. In [7], the authors report on numerous laboratory experiments conducted with human subjects in PD-like game settings. Davis treats the Prisoner's Dilemma among other mathematical games in [3]. A recent result concerning the PD is presented in [6].

In Sections 2 and 3 we describe the Prisoner's Dilemma and its iterated version. We introduce a continuous version of the PD in Section 4 and analyze it briefly in Section 5. Section 6 concludes this note. Some technical details have been collected in the appendices.

2 The Prisoner's Dilemma

The Prisoner's Dilemma (PD) is a game for two players, say, A and B. In an *encounter* or *move* of the PD, each player chooses either to *cooperate* (C) or to *defect* (D). Let us call the respective choices a and b. The profits $p_A(a,b)$ and $p_B(a,b)$ of A and B respectively are determined by the following *payoff matrix*:

p_A, p_B	$b = C$	$b = D$
$a = C$	R, R	S, T
$a = D$	T, S	P, P

(1)

where

S is the *sucker's payoff* (for a forsaken cooperator),
P is the *punishment* (for mutual defection),
R is the *reward* (for mutual cooperation),
T is the *temptation* (for defecting on a cooperator),

satisfying the *PD-condition*

$$S < P < R < T. \quad (2)$$

The objective of the players is to maximize their own profit; not just to do better than the other player. Note the symmetry $p_B(a,b) = p_A(b,a)$.

Typical values for the payoffs are

$$S, P, R, T = 0, 1, 3, 5. \quad (3)$$

If in this case, for instance, A cooperates and B defects, then A gains zero points and B gains five points. The dilemma arises because of the following two conflicting consequences of PD-condition (2).

1. No matter what B does, it is better for A to defect, since

$$\begin{aligned} p_A(C,C) &= R < T = p_A(D,C) \\ p_A(C,D) &= S < P = p_A(D,D) \end{aligned}$$

2. However, if B—who can be expected to reason like A—is going to do the same as A, then it is better for A to cooperate, since

$$p_A(D,D) \;=\; P \;<\; R \;=\; p_A(C,C)$$

The name *Prisoner's Dilemma* derives from the interpretation where the players are crime suspects awaiting their trial in separate prison cells. They cannot negotiate. The option to cooperate (with the other prisoner, not the justice department) corresponds to keeping one's mouth shut, not implicating the other. The option to defect corresponds to squealing. If both prisoners keep silent, they both get a mild sentence for lack of evidence. If both confess, they both get a more severe punishment. But if one talks while the other keeps quiet, then the tempted talker is acquitted and the silent sucker is sentenced maximally. This "payoff" scheme satisfies PD-condition (2), resulting precisely in the Prisoner's Dilemma.

Another interpretation is that where the players are trade partners. One of them will bring a box of rice, the other a box of beans. A move (transaction) consists of exchanging boxes. Cooperation corresponds to bringing a box filled with the promised merchandise. Defection corresponds to bringing an empty box. Again the "payoffs" satisfy the PD-condition.

Note that the Prisoner's Dilemma is a *non-zero-sum game*, because the profit that one player makes on a move does not necessarily equal the loss of the other player on that move. In a *zero-sum game* one would have $p_A(a,b) + p_B(a,b) = 0$ for all a and b. If the aim of the game would have been to earn more than the other player (i.e., to maximize the *profit difference*), then the game would not change if both components of any pair in the payoff matrix would be increased or decreased by the same amount. In that case, one can shift all payoffs to obtain a zero-sum payoff matrix. This results in an entirely different and less interesting game, since always defecting ensures that one does no worse than one's opponent.

3 The Iterated Prisoner's Dilemma

The *iterated* Prisoner's Dilemma consists of a *sequence* of PD-moves. We also call it a *PD-game*. The choices of A and B on move k ($k \geq 0$) are denoted by a_k and b_k respectively. In the analysis of the iterated Prisoner's Dilemma some new complications arise.

First, a player can adopt quite a complex *strategy* to choose between cooperation and defection on each move. The choice may involve the entire game history, that is, a_k may depend on all (a_i, b_i) with $0 \leq i < k$. It may also involve stochastic variables. Here are two examples of simple strategies.

RND_q (Random): On each move cooperate with probability q, defect otherwise.

TFT (Tit-for-Tat): On the first move cooperate, on each subsequent move do as your opponent did on the preceding move, that is, $a_0 = C$ and $a_{k+1} = b_k$ for $k \geq 0$.

Second, consider the *joint profit* $p_A + p_B$ on a move:

$p_A + p_B$	$b = C$	$b = D$
$a = C$	$R + R$	$S + T$
$a = D$	$T + S$	$P + P$

(4)

In order for the original dilemma to persist in the iterated PD, it is necessary (and sufficient) that the maximal joint profit is obtained for $a, b = C, C$ (yielding $2R$). Otherwise it would be possible for the players to earn the same or even more by cooperating and defecting on alternate moves, one player starting with cooperation, the other with defection. This gives rise to the additional condition

$$S + T \quad < \quad 2R. \tag{5}$$

A third complication in the iterated PD concerns the *number of moves*. In computer tournaments it is a practical necessity to limit the number of moves. Also in real life the number of encounters is limited. But usually it is not known in advance when the game ends. Axelrod takes the following approach in [1]. The probability to meet again after any move is assumed to be w with $0 < w < 1$, independent of the game's history. Because Axelrod's presentation is not sufficiently formal, we explain his approach in more detail in Appendix B.

The probability w can also be interpreted as a *weight* or *discount parameter*, which expresses how important potential future profits are for the cumulative profit over the whole game. A small value of w means that the future carries little weight, whereas a large value means that the future is likely to contribute considerably. Given w, Axelrod computes the *expected cumulative profit* $V(A|B)$ of strategy A playing a PD-game against strategy B by

$$V(A|B) \quad = \quad \sum_{k=0}^{\infty} V_k w^k, \tag{6}$$

where V_k is A's expected profit on move k ($k \geq 0$), given that this move occurs. For example, the expected cumulative profit of Tit-for-Tat playing against itself is

$$V(TFT|TFT) \;=\; \sum_{k=0}^{\infty} Rw^k \;=\; R/(1-w),$$

because Tit-for-Tat always cooperates with itself.

In Appendix C we show that when the future is discounted (i.e. $w < 1$), condition (5) is still sufficient—but no longer necessary—to exclude optimal profit by out-of-phase alternation of cooperate-defect choices.

4 A Continuous Prisoner's Dilemma

The Prisoner's Dilemma as described above is *discrete*, in the sense that each player chooses among *two* options: cooperate or defect. We now consider a *continuous* variant where each player chooses a real number in the closed interval $[0, 1]$. One can think of 0 as *total defection* and of 1 as *total cooperation*. The payoff functions can, for instance, be obtained from the discrete payoff matrix by linear interpolation:

$$\boxed{\begin{array}{rcl} p_A(a,b) & = & abR + a\bar{b}S + \bar{a}bT + \bar{a}\bar{b}P, \\ p_B(a,b) & = & baR + b\bar{a}S + \bar{b}aT + \bar{b}\bar{a}P, \end{array}} \tag{7}$$

where

$$\boxed{\bar{x} \;=\; 1 - x} \tag{8}$$

Note again the symmetry $p_B(a,b) = p_A(b,a)$ in (7). Also note that the discrete PD is embedded in this continuous version, since taking $C = 1$ and $D = \overline{C} = 0$ yields

$$\begin{array}{rclrcl} p_A(C,C) & = & R, & p_A(C,D) & = & S, \\ p_A(D,C) & = & T, & p_A(D,D) & = & P. \end{array}$$

In Appendix A we discuss efficient evaluation of the payoff functions.

Continuous versions of the Prisoner's Dilemma appear to be less well known than the discrete PD. For instance, they are not mentioned in the survey article [2], which does cover other extensions such as noise, i.e., a non-zero probability of misimplementation or misperception of choices. In [7], the authors consider discrete games with more than two choices per move, but they do not include continuous games.

One can argue that the continuous version models reality more faithfully, since real-life PD-like encounters hardly ever restrict the players to the two extreme behaviors of total cooperation or total defection. Consider, for example, the interpretation in terms of trade partners. Instead of bringing a full or an empty box, a player might also consider bringing a partially filled box (maybe reasoning that "the other will not notice a few beans less"). Naturally, in such intermediate cases, the payoffs will vary accordingly. This is nicely captured in our continuous version of the Prisoner's Dilemma.

We expect that this continuous Prisoner's Dilemma will provide further insight in the phenomenon of cooperation. Axelrod explains in [1] that a "good" strategy should be

1. *nice* (defect only to punish the other's defection),
2. *provokable* (indeed punish defection by somehow retaliating),
3. *forgiving* (restrain punishment once the other cooperates again), and
4. *clear* (easy to "understand" for other players).

In the discrete PD there are only limited possibilities for retaliation. Tit-for-Tat always punishes the other's defection by defecting itself on the very next move and immediately forgetting about it afterwards. Other retaliation schemes are incorporated in the following two strategies.

> $TFT_{m,n}$ (m-Tits-for-n-Tats): Cooperate, unless the other defects n times (in a row), then defect m times (I admit, this is a vague description).
>
> $GTFT_q$ (Generous Tit-for-Tat): Cooperate, unless the other defects, then once cooperate with probability q (defect with probability $\bar{q}$).

Observe that $TFT = TFT_{1,1} = GTFT_0$. In the discrete PD, players can only vary the duration and the probability of punishment when retaliating. In the continuous PD they can also vary the size of each punishment. Here are two examples of (parameterized) continuous strategies.

> ALL_x (Always-x, $x \in [0,1]$): For all k, $k \geq 0$, take $a_k = x$.
>
> $DTFT_r$ (r-Damped Tit-for-Tat, $r \in [0,1]$): Start with total cooperation and continue with an r-weighted average of 1 and the opponent's preceding choice, that is, $a_0 = 1$ and $a_{k+1} = r \cdot 1 + \bar{r} b_k = \overline{\bar{r}\bar{b}_k}$ for $k \geq 0$.

Retaliation by *DTFT* is not abrupt but "damped" with factor r. For $r = 0$ (no damping), however, we have $a_{k+1} = b_k$, which can be viewed as the continuous counterpart of Tit-for-Tat. And for $r = 1$ (total damping) we have $a_k = 1$, which is the same as ALL_C. Note that, in general, $a_{k+1} \geq b_k$, and that $a_{k+1} > b_k$ if and only if both $r > 0$ and $b_k < 1$. We will return to *DTFT* in the next section.

The continuous version of the PD given by (7) is one out of an infinite class. The only reason for considering this particular member is that it has such a simple definition.

5 Brief Analysis of the Continuous Prisoner's Dilemma

In the preceding section we have defined payoff functions (7) for the continuous Prisoner's Dilemma. Figure 1 shows the graphs for the individual payoffs (A: solid boundary; B: dashed boundary) and the joint payoff $p_A(a,b) + p_B(a,b)$ in our typical case (3).

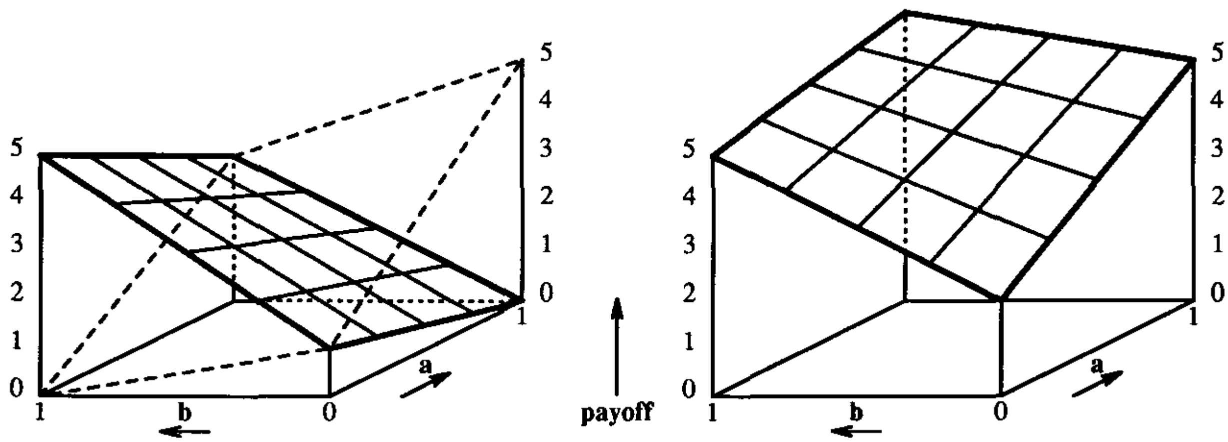

Figure 1: Individual payoff graphs (left) and joint payoff graph (right)

Because the payoff functions were obtained by linear interpolation, the intersection of each graph and a plane perpendicular to either the a-axis or the b-axis consists of a straight line; that is, each graph is a ruled surface. More precisely, the graphs are hyperbolic paraboloids (a type of quadric saddle surface), degenerating to a plane when $R + P = S + T$. In Figure 1, the curvature is not so apparent, but can be inferred by comparing the slopes of opposite boundaries. As a consequence of the ruled nature of the graphs, their global maxima and minima lie on the boundary. In particular, on account of conditions (2) and (5), the joint payoff function attains a global maximum of $2R$ at $(a,b) = (1,1)$.

For which (a,b) do we have $p_A(a,b) = p_B(a,b)$? We calculate

$$\begin{aligned} p_A - p_B &= a\bar{b}(S-T) + b\bar{a}(T-S) \\ &= (b-a)(T-S). \end{aligned}$$

On account of $S < T$ we thus have

$$\begin{aligned} p_A < p_B &\equiv a > b, \\ p_A = p_B &\equiv a = b, \\ p_A > p_B &\equiv a < b. \end{aligned}$$

Taking $a = b$ we get as payoff

$$\begin{aligned} p_A(a,a) &= a^2R + a\bar{a}(S+T) + \bar{a}^2P \\ &= (R+P-2Q)a^2 + 2(Q-P)a + P, \end{aligned}$$

where $Q = (S+T)/2$. From the above observation that the joint payoff function has a global maximum at $(a,b) = (1,1)$ we can conclude that $p_A(a,a)$ where $a \in [0,1]$ has a global maximum at $a = 1$ (regardless of the signs of $R+P-2Q$ and $2(Q-P)$). However, in case $Q < P$ we find a global *minimum* of $p_A(a,a)$—and saddle point of p_A—not at $a = 0$ but at $a = (P-Q)/(R+P-2Q)$: even when choosing the same as one's opponent, one can do worse than P. For example, if $S,P,R,T = 0, 2\frac{3}{4}, 3, 5$ then $p_A(\frac{1}{3},\frac{1}{3}) = 2\frac{2}{3} < P = p_A(0,0)$.

Damped Tit-for-Tat revisited

Let us investigate the continuous strategy $DTFT$ defined in the preceding section. Observe that $DTFT$ is nice ($a_0 = 1$, and $b_k = 1 \Rightarrow a_{k+1} = 1$) and, hence,

$$V(DTFT|DTFT) = R/\bar{w}.$$

Furthermore, when ALL_x (Always-x) plays against $DTFT_r$, the first move is $(x,1)$ and all subsequent moves are $(x, \overline{r\bar{x}})$. Therefore, we find

$$\begin{aligned} V(ALL_x|DTFT_r) &= xR + \bar{x}T + \sum_{k=1}^{\infty}(x\overline{r\bar{x}}R + xr\bar{x}S + \bar{x}\overline{r\bar{x}}T + \bar{x}r\bar{x}P)w^k \\ &= xR + \bar{x}T + (x\overline{r\bar{x}}R + xr\bar{x}S + \bar{x}\overline{r\bar{x}}T + \bar{x}r\bar{x}P)w/\bar{w} \end{aligned}$$

Consider a large population of players employing strategy A and a single player using strategy B. In this situation, each A-player earns $V(A|A)$ per game and the B-player $V(B|A)$. Axelrod says that strategy B can *invade* strategy A when

$$V(B|A) > V(A|A). \tag{9}$$

Consequently, ALL_x can invade $DTFT$ if and only if the above profit exceeds $R/\bar{w}$. In the special case $x = C = 1$, invasion is unconditionally impossible. For $x < 1$ we derive

$$\begin{aligned}
& R/\bar{w} \;<\; xR + \bar{x}T + (x\overline{r\bar{x}}R + xr\bar{x}S + \bar{x}\overline{r\bar{x}}T + \bar{x}r\bar{x}P)w/\bar{w} \\
\equiv \quad & \{\ \bar{w} > 0, \text{ because } w < 1 \text{ assumed }\} \\
& R \;<\; (xR + \bar{x}T)\bar{w} + (x\overline{r\bar{x}}R + xr\bar{x}S + \bar{x}\overline{r\bar{x}}T + \bar{x}r\bar{x}P)w \\
\equiv \quad & \{\ \bar{w} = 1 - w, \text{ collecting terms with } w \text{ on the left and others on the right }\} \\
& (xR + \bar{x}T - x\overline{r\bar{x}}R - xr\bar{x}S - \bar{x}\overline{r\bar{x}}T - \bar{x}r\bar{x}P)w \;<\; xR + \bar{x}T - R \\
\equiv \quad & \{\ \text{combining terms with } R \text{ and } T\ \} \\
& (xr\bar{x}R - xr\bar{x}S + \bar{x}r\bar{x}T - \bar{x}r\bar{x}P)w \;<\; -\bar{x}R + \bar{x}T \\
\equiv \quad & \{\ \bar{x} > 0, \text{ because } x < 1 \text{ assumed; algebra }\} \\
& [x(R-S) + \bar{x}(T-P)]rw \;<\; T - R \\
\equiv \quad & \{\ R - S > 0 \text{ and } T - P > 0, \text{ on account of PD-condition (2) }\} \\
& rw \;<\; \frac{T-R}{x(R-S) + \bar{x}(T-P)}
\end{aligned} \tag{10}$$

Observe that

$$\sup\left\{\frac{T-R}{x(R-S)+\bar{x}(T-P)}\,\middle|\, x\in[0,1)\right\} \;=\; \max\left\{\frac{T-R}{R-S},\frac{T-R}{T-P}\right\}.$$

Consequently, no ALL_x can invade $DTFT_r$ provided $\bar{r}w$ is sufficiently large:

$$\bar{r}w \;\geq\; \max\left\{\frac{T-R}{R-S},\frac{T-R}{T-P}\right\}. \tag{11}$$

For example, in case of the typical payoffs (3), invasion cannot occur when $\bar{r}w \geq \frac{2}{3}$. Thus, when $w > \frac{8}{9}$, invasion cannot occur when $r \leq \frac{1}{4}$. Note that in the typical case, ALL_x is better at invading for *larger* values of x (i.e when more cooperating), since $R-S=3<4=T-P$.

Axelrod calls a strategy *collectively stable* if no strategy can invade it. We now prove that $DTFT_r$ is collectively stable if and only if (11) holds. Condition (11) is obviously necessary, viz. to prevent invasion by ALL_x. To prove that it is sufficient, assume (11) and consider any strategy B. We will show that the best B can do against $DTFT_r$ is always to cooperate. Consider any game of B versus $DTFT_r$. Let B's first and second choice be x and y respectively. The first two moves of the game then are $(x,1)$ followed by $(y,\overline{\bar{r}\bar{x}})$. B's profit V_k on move k satisfies

$$\begin{aligned} V_0 &= xR+\bar{x}T, \\ V_1 &= y\overline{\bar{r}\bar{x}}R + y\bar{r}\bar{x}S + \bar{y}\overline{\bar{r}\bar{x}}T + \bar{y}\bar{r}\bar{x}P \end{aligned}$$

Note that V_k does not depend on x for $k \geq 2$. We investigate B's cumulative profit $p(x)$ when varying B's first choice x. We have

$$p(x) \;=\; xR+\bar{x}T+(y\overline{\bar{r}\bar{x}}R + y\bar{r}\bar{x}S + \bar{y}\overline{\bar{r}\bar{x}}T + \bar{y}\bar{r}\bar{x}P)w + \sum_{k=2}^{\infty} V_k w^k. \tag{12}$$

We now calculate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}p(x) &= R-T+(y\bar{r}R - y\bar{r}S + \bar{y}\bar{r}T - \bar{y}\bar{r}P)w \\ &= [y(R-S)+\bar{y}(T-P)]\bar{r}w - (T-R). \end{aligned}$$

Observe that the derivative does not depend on x. On account of (11) and $y \in [0,1]$, the derivative is at least zero and, hence, $p(x)$ is maximal at $x = C = 1$. However, if B cooperates on the first move, then so does $DTFT$ on the next move and the situation is the same as before. Consequently, B gets a maximal profit by *always cooperating.* We have already seen that the strategy ALL_C (Always-Cooperate) cannot invade $DTFT$ because $DTFT$ always cooperates with itself. Therefore, no strategy can invade $DTFT$. This concludes our stability proof.

Although Tit-for-Tat is a "good" strategy, it has some shortcomings. For example, consider the following strategy.

> *STFT* (Suspicious Tit-for-Tat): Initially defect, then act as *TFT*; that is, $a_0 = 0$ and $a_{k+1} = b_k$ for $k \geq 0$.

When *TFT* plays against *STFT*, they get stuck in out-of-phase alternating cooperate-defect choices. On account of (5) this is worse than mutual cooperation. Such alternation may also appear on account of errors due to noise. A little forgiveness is needed to avoid such locking behavior. The advantage of Damped Tit-for-Tat over Tit-for-Tat is that *DTFT* has the ability to re-converge to total cooperation after errors, because it can forgive defection to a certain extent. For example, consider a game of $DTFT_r$ versus $DTFT_s$, where the initial move (erroneously) was (x, y). The next two moves then are

$$(\overline{r\overline{y}}, \overline{s\overline{x}}) \quad \text{and} \quad (\overline{r\overline{s\overline{x}}}, \overline{s\overline{r\overline{y}}}),$$

because $\overline{\overline{z}} = z$. Thus we have in this game

$$\begin{aligned} a_{2k} &= \overline{t_k\overline{x}}, \\ a_{2k+1} &= \overline{\overline{r}t_k\overline{y}}, \qquad \text{where} \\ t_k &= (\overline{r}\overline{s})^k, \end{aligned}$$

and similarly for b_k. If $r > 0$ or $s > 0$ then $\overline{r}\overline{s} < 1$ and, hence,

$$\lim_{k\to\infty} t_k = 0 \quad \text{and} \quad \lim_{k\to\infty} a_k = 1$$

(the more damping, the faster the convergence). If both $r = 0$ and $s = 0$ (neither damps its response), then the game is locked in an alternation of (x, y) and (y, x) moves. A bit of damping, neither too much (cf. (11)) nor too little ($r > 0$), is advisable.

Axelrod's notion of a collectively stable strategy, involves an environment where almost all players use the same strategy, say A. This sets the "normal" profit of A in that environment at $V(A|A)$. Invasion into this environment by strategy B then requires $V(B|A) > V(A|A)$. However, in a mixed environment containing A, the "normal" profit of A might well differ from $V(A|A)$, say $\widetilde{V}(A)$. For instance, *TFT* does less well when *STFT* is present. Replacement of A by B then requires that the "normal" profit of B exceeds that of A: $\widetilde{V}(B) > \widetilde{V}(A)$. This may be easier for B in the mixed environment than in the homogeneous A-environment when $\widetilde{V}(A) < V(A|A)$. In environments where *DTFT*'s "normal" profit may be lower than $R/\bar{w}$, it is important to employ a smaller damping factor (be less forgiving) than prescribed by (11).

An adaptive variant of Damped Tit-for-Tat

Here is a variant of *DTFT* where the damping factor depends on the preceding choice:

> $ADTFT_r$ (Adaptive $DTFT_r$): Take $a_0 = 1$ and $a_{k+1} = \overline{\overline{r a_k} b_k}$ for $k \geq 0$.

If the opponent persists in total defection ($b_k = 0$), then the response of *ADTFT* will geometrically drop to total defection ($a_k = r^k$ for $k \geq 0$). On the other hand if the opponent cooperates totally ($b_k = 1$), then so does *ADTFT* on the next move ($a_{k+1} = 1$). Thus, *ADTFT* exhibits ***adaptive damping***.

When ALL_D (Always-Defect) plays against *ADTFT* we find

$$\begin{aligned} V(ALL_D|ADTFT_r) &= \sum_{k=0}^{\infty} \left(r^k T + \overline{r^k} P\right) w^k \\ &= (T - P)/\overline{rw} + P/\bar{w}. \end{aligned}$$

Thus, ALL_D can invade $ADTFT_r$ if and only if

$$
\begin{aligned}
& R/\bar{w} < (T-P)/\overline{rw} + P/\bar{w} \\
\equiv \quad & \{\ \bar{w} > 0 \text{ and } \overline{rw} > 0 \text{ because } w < 1 \text{ and } rw < 1\text{; definition of }{}^{-}\ \} \\
& (1-rw)R < (1-w)(T-P) + (1-rw)P \\
\equiv \quad & \{\ \text{algebra}\ \} \\
& w(T - P - rR + rP) < T - R \\
\equiv \quad & \{\ T > rR + \bar{r}P \text{ because } P < R\ \} \\
& w < \frac{T-R}{T-(rR+\bar{r}P)}
\end{aligned} \tag{13}
$$

Note that the right-hand side is less than one unless $r = 1$, because $P < R$. Consequently, ALL_D cannot invade $ADTFT$ provided $r < 1$ and w is sufficiently large. For example, in case of the typical payoffs (3) and $r = \frac{1}{4}$ invasion cannot occur when $w \geq \frac{4}{7}$.

Like *DTFT*, also *ADTFT* recovers from errors when playing against itself, except when incidentally the move $(0,0)$ occurs, which is a fixed point.

New types of strategies

The continuous PD-game allows strategies that are impossible (or impractical) in the discrete version. The following strategy, invented by Renze de Waal, illustrates this.

> SIG_s (Signature-s): Initially play s. If the first choice of the opponent is also s, then cooperate on the second move and continue as Tit-for-Tat. On the other hand, if the opponent's first choice differs from s, then proceed by always defecting.

The parameter s in this strategy can be viewed as a *signature*, by which SIG_s intends to "recognize" others of its kind *in a single move*. SIG_s does well against itself, especially when s is close to one. By taking a "secret" s (in particular, not equal to one), SIG_s will limit the profit of other (in particular, nice) strategies. This makes it hard to invade a large population of SIG_s players.

6 Concluding Remarks and References

We have defined a continuous version of the Prisoner's Dilemma (PD). In the continuous PD, the players choose along a continuum between the usual two options of cooperation and defection. The payoffs vary accordingly. The continuous PD better models some real-life PD-like encounters, such as trade transactions. One interesting feature of the continuous PD is that it allows measured retaliation against defectors.

We have carried out a first analysis of the continuous PD. A "damped" version of the famous Tit-for-Tat strategy, called *DTFT*, turns out to be feasible. We have characterized its resistance to invasion by arbitrary strategies. For appropriate values of the damping factor, *DTFT* cannot be invaded if the future carries enough weight. Damped Tit-for-Tat was also shown to recover from errors due to noise, because it is more forgiving than Tit-for-Tat; that is, unlike Tit-for-Tat it avoids locking into echoing recriminations.

This can be paraphrased as follows in terms of real-world situations. Punishment should at least be so severe, that the other player's payoff will be less than that under mutual cooperation, no matter what the other chooses to do. When punishment is less severe, it does not act as a deterrent. However, punishment need not be maximal, but it should just be sufficiently strong to make defection a less profitable alternative than cooperation for the other. In fact, punishment should be as lenient as possible to maximize the possiblities for reconverging to mutual cooperation. In practice, this is often forgotten and there is even a tendency to punish more severely than the original provocation.

We have also exhibited a new type of strategies using the notion of a signature, which encodes a strategy's identity in a single choice. Such strategies are impossible in the discrete PD.

Further investigation of the continuous PD is still needed to shed more light on the new possibilities it affords. A computer tournament might be a good way to start. Preliminary experiments have shown that *DTFT* and especially its adaptive variant *ADTFT* do well in tournaments.

Acknowledgments

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A Efficient Evaluation of the Payoff Function

In a move of the continuous Prisoner's Dilemma, each of the two players chooses a real number in the closed interval $[0,1]$. Let us call the choices a and b. The payoff for the a-player (i.e. the one choosing a) is $p(a,b)$, defined by

$$p(a,b) \quad = \quad abR + a\bar{b}S + \bar{a}bT + \bar{a}\bar{b}P, \tag{14}$$

where $\bar{x} = 1 - x$ and R, S, T, and P are some constant parameters satisfying

$$S \; < \; P \; < \; R \; < \; T. \tag{15}$$

The payoff for the b-player is $p(b,a)$.

We are interested in "efficient" programs *Eval* that solve

$|[$ **con** P, R, S, T: real $\{ \; S < P < R < T \; \}$;
a, b: real $\{ \; 0 \le a \le 1 \; \wedge \; 0 \le b \le 1 \; \}$;
var pa, pb: real ;
Eval
$\{ \; pa = p(a,b) \; \wedge \; pb = p(b,a) \; \}$
$]|$

We measure efficiency by the number of multiplications.

Evaluation of $p(a,b)$ according to its definition (14) requires eight multiplications. Therefore, *Eval* can be solved with *sixteen* multiplications. Recognizing some common terms this can easily be reduced to *ten* multiplications as shown in the following solution for *Eval*:

$|[$ **var** t, u, v: real ;
$t, u, v := a{*}b{*}R + (1{-}a){*}(1{-}b){*}P,\; a{*}(1{-}b),\; (1{-}a){*}b$;
$pa, pb := t + u{*}S + v{*}T,\; t + v{*}S + u{*}T$
$]|$

However, we can do better than that. First, we calculate

$p(a,b)$
$=$ { definition of p }
$abR + a\bar{b}S + \bar{a}bT + \bar{a}\bar{b}P$
$=$ { definition of $\bar{\ }$ }
$abR + a(1-b)S + (1-a)bT + (1-a)(1-b)P$
$=$ { distribution }
$abR + aS - abS + bT - abT + P - aP - bP + abP$
$=$ { distribution }
$ab(R - S - T + P) + a(S-P) + b(T-P) + P$
$=$ { algebra, defining $Z = R - S - T + P$ }

$$\left[\begin{array}{ll} a(S-P) + b(T-P) + P & \text{if } Z = 0 \\ Z\left(a + \frac{T-P}{Z}\right)\left(b + \frac{S-P}{Z}\right) + \frac{PR-ST}{Z} & \text{if } Z \neq 0 \end{array} \right. \tag{16}$$

Evaluation of $p(a,b)$ according to (16) requires only two multiplications, since for $Z \neq 0$ the constants $(T-P)/Z$, $(S-P)/Z$, and $(P*R-S*T)/Z$ can be precomputed (for example, by the compiler). Thus, *Eval* can be solved with *four* multiplications.

```
|[ var s, t, u, z: real ;
   z := R-S-T+P ;
   if z = 0 →
      pa, pb := a*(S-P) + b*(T-P) + P, a*(T-P) + b*(S-P) + P
   [] z ≠ 0 →
      s, t, u := (S-P)/z, (T-P)/z, (P*R-S*T)/z ;
      pa, pb := z*(a + t)*(b + s) + u, z*(a + s)*(b + t) + u
   fi
]|
```

But there is an even more efficient solution. Observe that

$$\begin{aligned} p(a,b) &= [p(a,b)+p(b,a)]/2 + [p(a,b)-p(b,a)]/2, \\ p(b,a) &= [p(a,b)+p(b,a)]/2 - [p(a,b)-p(b,a)]/2. \end{aligned}$$

We now calculate

$$\begin{aligned} & [p(a,b)+p(b,a)]/2 \\ = & \quad \{ \text{ definition of } p \text{, algebra } \} \\ & abR + (a\bar{b} + \bar{a}b)(S+T)/2 + \bar{a}\bar{b}P \\ = & \quad \{ \text{ definition of } {}^{-} \} \\ & abR + [a(1-b) + (1-a)b](S+T)/2 + (1-a)(1-b)P \\ = & \quad \{ \text{ algebra, defining } Q = (S+T)/2 \} \\ & ab(R-S-T+P) + (a+b)(Q-P) + P \\ = & \quad \{ \text{ algebra, defining } Z = R-S-T+P \} \\ & \left[\begin{array}{ll} (a+b)(Q-P)+P & \text{if } Z = 0 \\ Z\left(a+\frac{Q-P}{Z}\right)\left(b+\frac{Q-P}{Z}\right) + \frac{PR-Q^2}{Z} & \text{if } Z \neq 0 \end{array} \right. \end{aligned}$$

and

$$\begin{aligned} & [p(a,b)-p(b,a)]/2 \\ = & \quad \{ \text{ definition of } p \text{, algebra } \} \\ & (\bar{a}b - a\bar{b})(T-S)/2 \\ = & \quad \{ \text{ definition of } {}^{-} \} \\ & [(1-a)b - a(1-b)](T-S)/2 \\ = & \quad \{ \text{ algebra } \} \\ & (b-a)\tfrac{T-S}{2} \end{aligned}$$

Since for $Z \neq 0$ the constants $(S+T)/2$, $(Q-P)/Z$, $(PR-Q^2)/Z$, and $(T-S)/2$ can be precomputed, this yields a solution for *Eval* with at most *three* multiplications:

```
|[ var q, u, v, z: real ;
  q, z := (S+T)/2, R−S−T+P ;
  if z = 0 →
     u := (a+b)*(q−P) + P
  [] z ≠ 0 →
     u := z*(a + (q−P)/z)*(b + (q−P)/z) + (P*R − q*q)/z
  fi ;
  v := (b−a)*(T−S)/2 ;
  { u = [p(a,b) + p(b,a)]/2 ∧ v = [p(a,b) − p(b,a)]/2 }
  pa, pb := u + v, u − v
]|
```

In the typical case (3)—extensively used by Axelrod in [1]—where

$$S, P, R, T \;=\; 0, 1, 3, 5\,, \tag{17}$$

the coefficient $Z = R-S-T+P$ reduces to -1 and hence *Eval* needs only *two* multiplications:

```
|[ var u, v: real ;
  u, v := 3.25 − (1.5−a)*(1.5+b), 2.5*(b−a) ;
  { u = [p(a,b) + p(b,a)]/2 ∧ v = [p(a,b) − p(b,a)]/2 }
  pa, pb := u + v, u − v
]|
```

When simplifying and reworking our four-multiply solution for this typical case, we obtain another two-multiply solution (this time without auxiliary variables):

```
|[ pa, pb := (4−a)*(b+1) − 3, (4−b)*(a+1) − 3 ]|
```

B Expected Profit in the Iterated PD

In the Iterated Prisoner's Dilemma, a game consists of at least one move, and the probability to meet again after any move is taken to be w with $0 < w < 1$, independent of the game's history. Thus the probability to meet *more than* ℓ times ($\ell \geq 0$) is w^ℓ, and the probability to meet *exactly* ℓ times ($\ell \geq 1$) is $w^{\ell-1}\bar{w}$, where $\bar{w} = 1 - w$. The number of moves in a game has a *geometric distribution.* Let E be the *expected* number of moves in a game and M the *median* number of moves. Concerning the expectation E we then find

$$E \;=\; \sum_{\ell=1}^{\infty} \ell w^{\ell-1}\bar{w} \;=\; 1/\bar{w}, \tag{18}$$

and, hence,

$$w \;=\; (E-1)/E.$$

The median game length M is such that the probability to meet at most M times equals 0.5. Since the probability to meet *at most* ℓ times ($\ell \geq 0$) is $1 - w^\ell$ we find for M:

$$\begin{aligned} w^M &= 0.5, \\ M &= \frac{\ln 0.5}{\ln w}, \\ w &= \sqrt[M]{0.5}. \end{aligned}$$

In computer simulations, one can generate random game lengths with the appropriate distribution as

$$\left\lceil \frac{\ln U}{\ln w} \right\rceil \quad \text{or} \quad \left\lceil \frac{-M \ln U}{\ln 2} \right\rceil, \tag{19}$$

where is U is distributed uniformly in the *open* interval $(0, 1)$ (cf. [5]).

We now derive a formula expressing the expected cumulative profit for a game. First consider strategies A and B that involve no stochastic variables. All their games consist of the same moves, say (a_k, b_k) for $k \geq 0$. Let V_k be A's profit on move k, that is,

$$V_k = p_A(a_k, b_k). \tag{20}$$

For given w, A's expected cumulative profit for a game is then computed as

$$\begin{aligned}
&\sum_{\ell=1}^{\infty} \left[w^{\ell-1} \bar{w} \sum_{k=0}^{\ell-1} V_k \right] \\
= \quad & \{ \text{ swap summation order: } 1 \leq \ell \wedge 0 \leq k \leq \ell - 1 \equiv 0 \leq k \wedge k+1 \leq \ell \ \} \\
&\sum_{k=0}^{\infty} \left[V_k \sum_{\ell=k+1}^{\infty} w^{\ell-1} \bar{w} \right] \\
= \quad & \{ \text{ sum geometric series } \} \\
&\sum_{k=0}^{\infty} V_k w^k
\end{aligned} \tag{21}$$

Since the stopping criterion of a game is independent of the strategies, it can be argued that (21) also holds for stochastic strategies provided that V_k is replaced by A's *expected* profit on move k.

For example, when Random (cooperating with probability q) plays against Tit-for-Tat we have

$$\begin{aligned}
V_0 &= qR + \bar{q}T \\
V_k &= q(qR + \bar{q}T) + \bar{q}(qS + \bar{q}P) \qquad \text{for } k \geq 1.
\end{aligned}$$

Therefore, Random's expected cumulative profit is

$$\begin{aligned}
V(RND_q | TFT) &= V_0 + \sum_{k=1}^{\infty} V_k w^k \\
&= qR + \bar{q}T + \left[q^2 R + q\bar{q}(S+T) + \bar{q}^2 P \right] w/\bar{w}.
\end{aligned}$$

By definition, strategy B can invade strategy A when

$$V(B|A) > V(A|A). \tag{22}$$

For $0 \le w < 1$ and $0 \le q < 1$, we now calculate the condition under which RND_q can invade Tit-for-Tat:

$$\begin{array}{ll}
& V(TFT|TFT) < V(RND_q|TFT) \\
\equiv & \quad \{ \text{ above computations } \} \\
& R/\bar{w} < qR + \bar{q}T + [q^2R + q\bar{q}(S+T) + \bar{q}^2P]\; w/\bar{w} \\
\equiv & \quad \{ \bar{w} > 0 \} \\
& R < (qR + \bar{q}T)\bar{w} + [q^2R + q\bar{q}(S+T) + \bar{q}^2P]\; w \\
\equiv & \quad \{ \bar{w} = 1 - w, \text{ algebra } \} \\
& (qR + \bar{q}T)w - [q^2R + q\bar{q}(S+T) + \bar{q}^2P]\; w < (qR + \bar{q}T) - R \\
\equiv & \quad \{ \text{ algebra, } 1 - q = \bar{q} \} \\
& [q\bar{q}R + \bar{q}T - q\bar{q}(S+T) - \bar{q}^2P]\; w < \bar{q}(T - R) \\
\equiv & \quad \{ \bar{q} > 0 \} \\
& [qR + T - q(S+T) - \bar{q}P]w < T - R \\
\equiv & \quad \{ \text{ algebra, } 1 - q = \bar{q} \} \\
& (q(R-S) + \bar{q}(T-P))w < T - R \\
\equiv & \quad \{ q(R-S) + \bar{q}(T-P) > 0 \text{ on account of 2) } \} \\
& w \; < \; \dfrac{T - R}{q(R-S) + \bar{q}(T-P)}
\end{array} \qquad (23)$$

Compare this result to (10) of ALL_x invading $DTFT$. For the typical payoffs (3) and $q = \frac{1}{2}$ this boils down to $w < \frac{4}{7}$, which corresponds to a median game length of 1.24 moves. RND_1 (Always-Cooperate) cannot invade Tit-for-Tat regardless of w.

When simulating PD-games between two strategies with varying values for w, one might want to normalize the profits so as to ease comparison. Two ways of normalizing the profits from a sample of PD-games come to mind.

1. The first way is to divide the average profit per game by the average game length or, what comes to the same, divide the total profit over all games by the total number of moves in all games. When considering many games this "converges" to the quotient of the expected cumulative profit and the expected game length.

2. The second way is to compute the average (over all games) of the average profit per move per game. This "converges" to the expected average profit per move, that is, to the expectation of the quotient of the cumulative profit and the game length.

We would like to emphasize that, in general, these two ways of normalizing are quite different, because cumulative profit and game length are not necessarily independent stochastic variables. Let us look at an example to illustrate this difference. Consider the games of Always-Defect against Tit-for-Tat, for which we have

$$a_0 = D, \qquad b_0 = C, \qquad \text{and} \qquad a_k = b_k = D \quad \text{for } k \ge 1.$$

For Always-Defect's profit we have

$$V_0 = T \qquad \text{and} \qquad V_k = P \quad \text{for } k \ge 1. \qquad (24)$$

Therefore, given discount parameter w, $0 \leq w < 1$, we have that the expected cumulative profit for Always-Defect equals

$$T + \sum_{k=1}^{\infty} Pw^k \quad = \quad T + wP/\bar{w}.$$

Thus the quotient of the expected cumulative profit and the expected game length (cf. (18)) equals

$$\bar{w}T + wP \quad = \quad P + (T-P)\bar{w}. \tag{25}$$

On the other hand, the expected average profit per move is calculated as

$$\begin{aligned}
& \sum_{\ell=1}^{\infty} \left(w^{\ell-1}\bar{w}\frac{1}{\ell}\sum_{k=0}^{\ell-1} V_k \right) \\
= & \quad \{ \text{ (24) concerning } V_k \text{ } \} \\
& \sum_{\ell=1}^{\infty} \frac{T + (\ell-1)P}{\ell} w^{\ell-1}\bar{w} \\
= & \quad \{ \text{ algebra } \} \\
& P + (T-P)\bar{w}\sum_{\ell=1}^{\infty}\frac{w^{\ell-1}}{\ell} \\
= & \quad \{ \text{ series for the natural logarithm } \} \\
& P + (T-P)\bar{w}\frac{\ln \bar{w}}{-w}
\end{aligned} \tag{26}$$

It is the factor $\frac{\ln \bar{w}}{-w}$ that distinguishes (26) from (25). When taking the limits for $w \downarrow 0$ and $w \uparrow 1$, the distinction disappears (for $w \uparrow 1$, the factor $\bar{w}$ also plays a role). Plugging the typical payoffs (3) and $w = 0.5$ into (25), we find 3 for the quotient of the expectations. Plugging these values into (26), yields approximately 3.8 for the expectation of the quotient, a noticeable difference.

C Alternating Cooperate-Defect Games

Condition (5) on the P, R, S, T-parameters was introduced to exclude optimal profit by out-of-phase alternation of cooperate-defect choices. When the future is discounted (i.e. $w < 1$), this condition still suffices, but it is no longer a necessary condition.

Let us compute the expected cumulative profits for such alternation, that is, for the game with

$$a_{2k} = b_{2k+1} = C \qquad \text{and} \qquad a_{2k+1} = b_{2k} = D \qquad \text{for all } k \geq 0.$$

A's and B's expected cumulative profits are respectively

$$\begin{aligned}
\sum_{k=0}^{\infty} Sw^{2k} + Tw^{2k+1} \quad &= \quad (S + wT)\big/\overline{w^2}, \\
\sum_{k=0}^{\infty} Tw^{2k} + Sw^{2k+1} \quad &= \quad (T + wS)\big/\overline{w^2}.
\end{aligned}$$

The expected cumulative profit of two cooperating players was computed above as $R/\bar{w}$. Alternation is more attractive to *both* players, if and only if

$$\begin{aligned} R/\bar{w} &\leq (S+wT)\big/\overline{w^2} \quad \text{and} \\ R/\bar{w} &\leq (T+wS)\big/\overline{w^2}. \end{aligned}$$

Using $0 \leq w < 1$ and (2) we derive

$$\begin{array}{ll} & R/\bar{w} \leq (S+wT)\big/\overline{w^2} \quad \wedge \quad R/\bar{w} \leq (T+wS)\big/\overline{w^2} \\ \equiv & \{\ \overline{w^2} = (1+w)\bar{w} > 0, \text{ on account of } 0 \leq w < 1\ \} \\ & (1+w)R \leq S+wT \quad \wedge \quad (1+w)R \leq T+wS \\ \equiv & \{\ \text{algebra}\ \} \\ & R-S \leq w(T-R) \quad \wedge \quad w(R-S) \leq T-R \\ \equiv & \{\ T-R > 0 \text{ and } R-S > 0, \text{ on account of (2)}\ \} \\ & \dfrac{R-S}{T-R} \leq w \leq \dfrac{T-R}{R-S} \end{array} \tag{27}$$

On account of (2), the range for w given by (27) is empty if and only if

$$(R-S)^2 \ > \ (T-R)^2,$$

which is equivalent to (5). However, if this range is not empty, alternation is still less attractive than mutual cooperation whenever

$$w \ < \ \frac{R-S}{T-R}. \tag{28}$$

To paraphrase: In sufficiently short alternating games (i.e. with small w), it is not attractive to be the first to cooperate, that is, to assume the role of initial sucker.

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