

## Solution to Problem 79-15: An identity

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providing details and we will investigate your claim.

When  $\theta$  is small,  $M$  is a slowly-varying function of  $v$ , its Fourier harmonics are small, and the series for  $P_n$  converges rapidly. We expand  $M$  in a power series in  $\theta$ :

$$\begin{aligned} \sin^2 \frac{1}{2}\theta \cos^2 u &= \sin^2 \frac{1}{2}\theta - \sin^2 \left(\frac{1}{2}\theta \sin v\right) \\ &= \sin\left[\frac{1}{2}\theta(1 - \sin v)\right] \sin\left[\frac{1}{2}\theta(1 + \sin v)\right]. \end{aligned}$$

Let  $v = 90^\circ - 2w$ ,  $p = \theta \sin^2 w$ ,  $q = \theta \cos^2 w$ . Then,

$$M^2 = \frac{\sin^2 2w \sin^2 \frac{1}{2}\theta}{\sin p \sin q} = \frac{4pq \sin^2 \frac{1}{2}\theta}{\theta^2 \sin p \sin q}.$$

Now,

$$\begin{aligned} \frac{\sin p \sin q}{pq} &= \left(1 - \frac{\theta^2}{6} \sin^4 w + \dots\right) \left(1 - \frac{\theta^2}{6} \cos^4 w + \dots\right) \\ &= 1 - \frac{\theta^2}{24} (3 + \cos 4w) + \dots, \end{aligned}$$

so

$$M = \frac{2}{\theta} \sin \frac{1}{2} \theta \left[ 1 + \frac{\theta^2}{48} (3 - \cos 2v) + O(\theta^4) \right].$$

Through  $\theta^2$  there are therefore no terms with  $2k > 2$ , while the coefficient of  $J_2$  is

$$\frac{4}{\pi} \int_0^{\pi/2} -\frac{\theta^2}{48} \cos^2 2v \, dv = -\frac{\theta^2}{48},$$

and our approximation is

$$P_n(\cos \theta) \cong \frac{2}{\pi} K\left(\sin \frac{1}{2} \theta\right) J_0(x) - \frac{\theta^2}{48} J_2(x).$$

The functions  $J_0$  and  $-J_2$  are roughly in phase with each other, and they both have peaks that decrease with increasing  $x$ , or  $n$ . This proves the conjecture so long as  $\theta$  is small enough for the approximation to be valid. But calculation shows that, at  $\theta = 45^\circ$ , the two Bessel terms give about 0.1% error in the first few peaks and in the peaks at  $n = 80.5$  and  $n = 4000.5$ . The hypergeometric formula says successive peaks are  $\pi/\theta = 4$  apart in  $n$ , so their amplitude ratio is  $(n/(n+4))^{1/2}$ . For this to be as small as even twice our 0.1%,  $n$  must be over 1000 and the “large  $n$ ” proof is applicable. We conclude the conjecture is true in the range  $0 < \theta < 45^\circ$ , and therefore for all  $\theta$ .

REFERENCE

[1] M. ABRAMOWITZ AND I. STEGUN, *Handbook of Mathematical Functions*, U.S. Government Printing Office, National Bureau of Standards, 1965.

**An Identity**

*Problem 79-15*, by J. D. LOVE (Australian National University, Canberra, Australia).

In an analysis of the electrostatic potential for two charged dielectric spheres, the following identity arises:

$$\begin{aligned} \operatorname{csch} x &= P_n(\cosh x) Q_n(\cosh x) + Q_n(\cosh x) \sum_{m=0}^{n-1} P_m(\cosh x) e^{(n-m)x} \\ &\quad + P_n(\cosh x) \sum_{m=n+1}^{\infty} Q_m(\cosh x) e^{(n-m)x} \end{aligned}$$

where  $P_n(\cosh x)$  and  $Q_n(\cosh x)$  are modified Legendre functions of the first and second kinds, respectively. Prove the identity for real  $x > 0$  and nonnegative integers  $n$ .

*Solution by* O. P. LOSSERS (Eindhoven University of Technology, Eindhoven, The Netherlands).

Let the right-hand side of the identity be denoted by  $S$ . Then by use of the integral [1, p. 154]

$$P_n(\cosh x)Q_m(\cosh x) = \frac{1}{2} \int_{-1}^1 \frac{P_n(t)P_m(t)}{\cosh x - t} dt, \quad n \leq m,$$

$S$  can be reduced to

$$S = \frac{1}{2} e^{nx} \int_{-1}^1 \frac{P_n(t)}{\cosh x - t} \left( \sum_{m=0}^{\infty} P_m(t) e^{-mx} \right) dt.$$

The infinite series can be evaluated by means of the generating function for Legendre polynomials [1, p. 154], viz.,

$$\sum_{m=0}^{\infty} P_m(t) e^{-mx} = (1 - 2e^{-x}t + e^{-2x})^{-1/2} = 2^{-1/2} e^{1/2x} (\cosh x - t)^{-1/2},$$

thus leading to

$$S = 2^{-3/2} e^{(n+1/2)x} \int_{-1}^1 \frac{P_n(t)}{(\cosh x - t)^{3/2}} dt.$$

Because of [2, p. 822, 7.225 (4)],

$$\int_{-1}^1 \frac{P_n(t)}{(\cosh x - t)^{1/2}} dt = \frac{2\sqrt{2}}{2n+1} e^{-(2n+1)(x/2)}.$$

It follows by differentiation with respect to  $x$  that

$$\int_{-1}^1 \frac{P_n(t)}{(\cosh x - t)^{3/2}} dt = \frac{2\sqrt{2}}{\sinh x} e^{-(n+1/2)x},$$

which shows that

$$S = \operatorname{csch} x.$$

*Also solved by* H. E. FETTIS (Mountain View, California), who used an induction argument *and by* D. K. ROSS (La Trobe University, Australia), who used recurrence relations satisfied by the Legendre functions.

REFERENCES

- [1] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1953.
- [2] I. S. GRADSTEYN AND I. M. RYZHIK, *Tables of Integrals, etc.*, Academic Press, New York, 1965.

**ERRATUM: An Integral Inequality**

*Problem 77-6 by* J. E. WILKINS, JR.

The erratum given for this problem (this Review, 22(1980) p. 102) is in error. It should have read: