Solution to Problem 79-15: An identity

When θ is small, M is a slowly-varying function of v, its Fourier harmonics are small, and the series for P_n converges rapidly. We expand M in a power series in θ:

$$\sin^2 \tfrac{1}{2}\theta \cos^2 u = \sin^2 \tfrac{1}{2}\theta - \sin^2 (\tfrac{1}{2}\theta \sin v)$$
$$= \sin[\tfrac{1}{2}\theta(1-\sin v)] \sin [\tfrac{1}{2}\theta(1+\sin v)].$$

Let $v = 90° - 2w$, $p = \theta \sin^2 w$, $q = \theta \cos^2 w$. Then,

$$M^2 = \frac{\sin^2 2w \sin^2 \frac{1}{2}\theta}{\sin p \sin q} = \frac{4pq \sin^2 \frac{1}{2}\theta}{\theta^2 \sin p \sin q}.$$

Now,

$$\frac{\sin p \sin q}{pq} = \left(1 - \frac{\theta^2}{6}\sin^4 w + \cdots\right)\left(1 - \frac{\theta^2}{6}\cos^4 w + \cdots\right)$$
$$= 1 - \frac{\theta^2}{24}(3+\cos 4w) + \cdots,$$

so

$$M = \frac{2}{\theta}\sin\frac{1}{2}\theta\left[1 + \frac{\theta^2}{48}(3-\cos 2v) + O(\theta^4)\right].$$

Through θ^2 there are therefore no terms with $2k > 2$, while the coefficient of J_2 is

$$\frac{4}{\pi}\int_0^{\pi/2} -\frac{\theta^2}{48}\cos^2 2v\, dv = -\frac{\theta^2}{48},$$

and our approximation is

$$P_n(\cos\theta) \cong \frac{2}{\pi}K\left(\sin\frac{1}{2}\theta\right)J_0(x) - \frac{\theta^2}{48}J_2(x).$$

The functions J_0 and $-J_2$ are roughly in phase with each other, and they both have peaks that decrease with increasing x, or n. This proves the conjecture so long as θ is small enough for the approximation to be valid. But calculation shows that, at $\theta = 45°$, the two Bessel terms give about 0.1% error in the first few peaks and in the peaks at $n = 80.5$ and $n = 4000.5$. The hypergeometric formula says successive peaks are $\pi/\theta = 4$ apart in n, so their amplitude ratio is $(n/(n+4))^{1/2}$. For this to be as small as even twice our 0.1%, n must be over 1000 and the "large n" proof is applicable. We conclude the conjecture is true in the range $0 < \theta < 45°$, and therefore for all θ.

REFERENCE

[1] M. ABRAMOWITZ AND I. STEGUN, *Handbook of Mathematical Functions*, U.S. Government Printing Office, National Bureau of Standards, 1965.

An Identity

Problem 79-15, *by* J. D. LOVE (Australian National University, Canberra, Australia).

In an analysis of the electrostatic potential for two charged dielectric spheres, the following identity arises:

$$\operatorname{csch} x = P_n(\cosh x)Q_n(\cosh x) + Q_n(\cosh x)\sum_{m=0}^{n-1} P_m(\cosh x)\, e^{(n-m)x}$$
$$+ P_n(\cosh x)\sum_{m=n+1}^{\infty} Q_m(\cosh x)\, e^{(n-m)x}$$

where $P_n(\cosh x)$ and $Q_n(\cosh x)$ are modified Legendre functions of the first and second kinds, respectively. Prove the identity for real $x > 0$ and nonnegative integers n.

Solution by O. P. LOSSERS (Eindhoven University of Technology, Eindhoven, The Netherlands).

Let the right-hand side of the identity be denoted by S. Then by use of the integral [1, p. 154]

$$P_n(\cosh x)Q_m(\cosh x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)P_m(t)}{\cosh x - t}\,dt, \qquad n \leqq m,$$

S can be reduced to

$$S = \frac{1}{2}e^{nx}\int_{-1}^{1} \frac{P_n(t)}{\cosh x - t}\left(\sum_{m=0}^{\infty} P_m(t)\,e^{-mx}\right)dt.$$

The infinite series can be evaluated by means of the generating function for Legendre polynomials [1, p. 154], viz.,

$$\sum_{m=0}^{\infty} P_m(t)\,e^{-mx} = (1-2\,e^{-x}t + e^{-2x})^{-1/2} = 2^{-1/2}\,e^{1/2x}(\cosh x - t)^{-1/2},$$

thus leading to

$$S = 2^{-3/2}\,e^{(n+1/2)x}\int_{-1}^{1} \frac{P_n(t)}{(\cosh x - t)^{3/2}}\,dt.$$

Because of [2, p. 822, 7.225 (4)],

$$\int_{-1}^{1} \frac{P_n(t)}{(\cosh x - t)^{1/2}}\,dt = \frac{2\sqrt{2}}{2n+1}\,e^{-(2n+1)(x/2)}.$$

It follows by differentiation with respect to x that

$$\int_{-1}^{1} \frac{P_n(t)}{(\cosh x - t)^{3/2}}\,dt = \frac{2\sqrt{2}}{\sinh x}\,e^{-(n+1/2)x},$$

which shows that

$$S = \operatorname{csch} x.$$

Also solved by H. E. FETTIS (Mountain View, California), who used an induction argument *and by* D. K. ROSS (La Trobe University, Australia), who used recurrence relations satisfied by the Legendre functions.

REFERENCES

[1] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1953.
[2] I. S. GRADSTEYN AND I. M. RYZHIK, *Tables of Integrals, etc.*, Academic Press, New York, 1965.

ERRATUM: An Integral Inequality

Problem 77–6 by J. E. WILKINS, JR.

The erratum given for this problem (this Review, 22(1980) p. 102) is in error. It should have read: