# An asymptotic problem in coding theory

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# AN ASYMPTOTIC PROBLEM IN CODING THEORY

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## ABSTRACT

The asymptotic behaviour of $x \sum_{k=0}^{\infty} (f(kx))^n$ for $n \to \infty$ and of $\sum_{k=0}^{\infty} \prod_{i=0}^{k} f(ix)$ for $x \downarrow 0$ is determined together with explicit bounds for the hidden constants. Here $f$ is a positive nonincreasing function on $[0,\infty)$ with given asymptotic behaviour at zero.

## 1. INTRODUCTION

In [1] the improvement of the Mean Time To Failure for degrading memory systems is studied if a combined test-error-correcting procedure is applied. The authors were confronted with problems which will be formulated explicitly in Section 3 of this paper. These problems can be formulated in a more general setting.

Let $f : [0,\infty) \to [0,\infty)$ be monotonically nonincreasing with the following properties:

There are real numbers $a$, $\alpha$ and $\beta$ with $a > 0$, $0 < \alpha < \beta$ such that $\int_0^\infty f(x)\,dx$ exists and

$$f(x) = 1 - ax^\alpha + O(x^\beta) \quad (x \downarrow 0).$$

It is asked to find lower and upper bounds for

$$S(x,n) := x \sum_{k=0}^{\infty} (f(kx))^n$$

for $x \geq 0$ and $n$ sufficiently large, and for

$$P(x) := \sum_{k=0}^{\infty} \prod_{i=0}^{k} f(ix)$$

for $x > 0$ sufficiently small.

In Section 2 we formulate the results. We apply these results to the original problems from [1] in Section 3. In Section 4 we shall prove a lemma the application of which to the problems stated above is given in Section 5 and 6.

## 2. RESULTS

For $r_1, r_2 \in (0,\infty]$, $p > 0$, $q_1, q_2 \in \mathbb{R}$, $A > 0$, $0 < \mu < \nu$, $y \geq A$, $k \in \mathbb{N}$ and a function $h : [0,\infty) \to [-\infty, 0]$ such that $\int_0^\infty \exp[Ah(s)]\, ds$ exists, all to be specified below, the following expressions are defined:

$$I_k(p) := p^{-(1+k\nu)/\mu}\, \mu^{-1}\, \Gamma((1+k\nu)/\mu)\, y^{-(1+k\nu-k\mu)/\mu}$$

$\tau_1$ and $\tau_2$ are such that $\tau_i \in (0, r_i)$ and $p\tau_i^\mu - q_i \tau_i^\nu > 0$ $(i=1,2)$

$$R_1 := 2p^{-1}\mu^{-1}\tau_1^{1-\mu} y^{-1} \exp[(-p\tau_1^\mu + q_1\tau_1^\nu)y]$$

$$R_2 := \int_0^\infty \exp[Ah(s)]\, ds\, \exp[(-p\tau_2^\mu + q_2\tau_2^\nu)(y-A)]$$

$$A_1 := \begin{cases} \max\{A,\, 2p^{-1}\mu^{-1}\tau_1^{-\mu}(1+\nu-\mu)\} & \text{if } r_1 < \infty \\ A & \text{if } r_1 = \infty \end{cases}$$

$$B_1 := \begin{cases} I_0(p) + q_1 I_1(p) - R_1 & \text{if } r_1 < \infty \\ I_0(p) + q_1 I_1(p) & \text{if } q_1 \leq 0 \text{ and } r_1 = \infty. \end{cases}$$

$$B_2 := \begin{cases} I_0(p) + q_2 I_1(p) + \frac{1}{2} q_2^2 I_2(p - q_2\tau_2^{\nu-\mu}) + R_2 & \text{if } q_2 > 0 \\ I_0(p) + q_2 I_1(p) + \frac{1}{2} q_2^2 I_2(p) + R_2 & \text{if } q_2 \leq 0 \\ I_0(p) + q_2 I_1(p) + \frac{1}{2} q_2^2 I_2(p) & \text{if } q_2 \leq 0 \text{ and } r_2 = \infty. \end{cases}$$

Let $f$ be as in Section 1. Let $\delta_1, \delta_2 \in (0,\infty]$ and $c_1, c_2 \in \mathbb{R}$ be such that

$$\text{(1)} \qquad \begin{aligned} \log f(x) &\geq -a x^\alpha + c_1 x^\gamma \quad (0 \leq x < \delta_1) \\ \log f(x) &\leq -a x^\alpha + c_2 x^\gamma \quad (0 \leq x < \delta_2) \end{aligned}$$

where $\gamma := \min\{2\alpha, \beta\}$.

Then

$$S(x,n) \geq B_1 \qquad (x \geq 0,\, n \geq A_1)$$

$$S(x,n) \leq x + B_2 \quad (x \geq 0,\, n \geq A_1)$$

with

$$p := a,\ \mu := \alpha,\ \nu := \gamma,\ q_1 := c_1,\ q_2 := c_2,\ A = 1,$$

$$r_1 := \delta_1\,,\ r_2 := \delta_2\,,\ h := \log f\,,\ y := n\,,\ \int_0^\infty \exp[A\,h(s)]\,ds = \int_0^\infty f(s)\,ds\,,$$

and

$$P(x) > x^{-1}B_1 - 1 \quad (x \geq A_1^{-1})$$
$$P(x) < x^{-1}B_2 + 1 \quad (x \geq A^{-1})$$

with

$$p := a(\alpha+1)^{-1}\,,\ \mu := \alpha+1\,,\ \nu := \gamma+1\,,\ q_1 := c_1(1+\gamma)^{-1}\,,\ q_2 := c_2(1+\gamma)^{-1}\,,$$

$$A > 0\,,\ r_1 := \delta_1\,,\ r_2 := \delta_2\,,\ h(x) := \int_0^x \log f(s)\,ds\,,\ y := x^{-1}\,,\ \int_0^\infty \exp[A\,h(s)]\,ds \leq A^{-1} + \int_0^\infty f(s)\,ds.$$

## 3. APPLICATION

Now we shall apply the results to the problems in [1].
Let $l,m \in \mathbb{N}$, $1 \leq l < m$. Let

$$g(y) := \sum_{k=0}^{l} \begin{bmatrix} m \\ k \end{bmatrix} y^{m-k}(1-y)^k = 1 - \sum_{k=l+1}^{m} \begin{bmatrix} m \\ k \end{bmatrix} y^{m-k}(1-y)^k \quad (0 \leq y \leq 1).$$

The function $f$ is defined by

$$f(x) := g(e^{-x}) \quad (0 \leq x < \infty).$$

Clearly

$$f(x) \sim \begin{bmatrix} m \\ l \end{bmatrix} e^{-(m-l)x} \quad (x \to \infty)$$

whence $\int_0^\infty f(s)\,ds$ exists.
Furthermore, since

$$(2) \qquad f'(x) = -m \begin{bmatrix} m-1 \\ l \end{bmatrix} e^{-mx}(e^x - 1)^l \quad (x \geq 0)$$

is negative for $x > 0$, $f$ is monotonically decreasing on $[0,\infty)$. Finally

$$f(x) = 1 - \begin{bmatrix} m \\ l+1 \end{bmatrix} x^{l+1} + O(x^{l+2}) \quad (x \downarrow 0).$$

Hence $f$ satisfies all of the required conditions mentioned in Section 1. In order to apply the results of Section 2 we have to determine numbers $\delta_1$, $\delta_2$, $c_1$ and $c_2$ such that (1) holds.

In [1] it has been proved that

$$\log f(x) \geq -\binom{m}{l+1} x^{l+1} \quad (x \geq 0).$$

Hence we can take $c_1 = 0$ and $\delta_1 = \infty$. But the Taylor expansion of $\log f$ about $x = 0$,

$$\log f(x) = -\binom{m}{l+1} x^{l+1} + (m - \tfrac{1}{2} l)(l+1)(l+2)^{-1} \binom{m}{l+1} x^{l+2} \cdots$$

shows that, in principle, a sharper inequality of the kind

$$\log f(x) \geq -\binom{m}{l+1} x^{l+1} + c_1 x^{l+2} \quad (0 \leq x < \delta_1 < \infty)$$

is possible with $c_1 > 0$.

Using (2) we can write

$$\log f(x) \leq -1 + f(x) = -m \binom{m-1}{l} \int_0^x e^{-(m-\frac{1}{2}l)t} (e^{\frac{1}{2}t} - e^{-\frac{1}{2}t})^l \, dt$$

$$\leq -m \binom{m-1}{l} \int_0^x (1 - (m - \tfrac{1}{2} l) t) t^l \, dt$$

$$= -\binom{m}{l+1} x^{l+1} + (m - \tfrac{1}{2} l)(l+1)(l+2)^{-1} \binom{m}{l+1} x^{l+2} \quad (x \geq 0).$$

Hence we can take $c_2 = (m - \frac{1}{2} l)(l+1)(l+2)^{-1} \binom{m}{l+1}$ and $\delta_2 = \infty$.

This choice of $c_2$ is also the best possible since it is precisely the coefficient of $x^{l+2}$ in the Taylor expansion.

Finally, we need an upper bound for $\int_0^\infty f(s)\,ds$

$$\int_0^\infty f(s)\,ds = \sum_{k=0}^{l} \binom{m}{k} \int_0^1 t^{m-k-1} (1-t)^k \, dt = \sum_{k=0}^{l} \frac{1}{m-k}.$$

Substituting in the results of Section 2

$$a := \binom{m}{l+1}, \quad \alpha := l+1, \quad \beta := l+2, \quad c_1 := 0,$$

$$c_2 := (m-\tfrac{1}{2}l)\,(l+1)\,(l+2)^{-1}\begin{bmatrix} m \\ l+1 \end{bmatrix},\ \delta_1 := \infty,\ \delta_2 := \infty,$$

$$\int_0^\infty f(s)\,ds = \sum_{k=0}^{l} \frac{1}{m-k},\ \int_0^\infty \exp[A\int_0^x \log f(s)\,ds]\,dx \le A^{-1} + \sum_{k=0}^{l} \frac{1}{m-k},$$

we can find bounds for $S$ and $P$.

We shall write down the formulas explicitly only in the special case $l = 1$. We find

$$S(x,n) \ge 2^{-1}\pi^{1/2}\begin{bmatrix} m \\ 2 \end{bmatrix}^{-1/2} n^{-1/2} \quad (x \ge 0,\, n \ge 1)$$

$$P(x) \ge 3^{-2/3}\,\Gamma(1/3)\begin{bmatrix} m \\ 2 \end{bmatrix}^{-1/3} x^{-2/3} - 1 \quad (x > 0).$$

For the upper bounds of $S$ and $P$ we have used the expansion, mentioned in the remark at the end of Section 4, with $n = 1$.

$$S(x,n) \le x + 2^{-1}\pi^{1/2}\begin{bmatrix} m \\ 2 \end{bmatrix}^{-1/2} n^{-1/2} + \frac{3}{4}(m-1/2)\begin{bmatrix} m \\ 2 \end{bmatrix}^{-1} n^{-1}$$
$$+ \left[\frac{1}{m} + \frac{1}{m-1}\right] \exp\left[\frac{-2}{27}n\right] \quad (n \ge 1)$$

where we have chosen $\tau_2 = \dfrac{1}{2m-1}$ and where we have used that $8/9 \le m(m-1)\,(m-\tfrac{1}{2})^{-2} < 1 \ (m \ge 2)$.

$$P(x) \le 3^{-2/3}\,\Gamma(1/3)\begin{bmatrix} m \\ 2 \end{bmatrix}^{-1/3} x^{-2/3} + 2^{2/3}\,3^{-1/3}\,\Gamma(5/3)\,(m-\tfrac{1}{2})\begin{bmatrix} m \\ 2 \end{bmatrix}^{-2/3} x^{-1/3}$$
$$+ 1 + \left[1 + \frac{1}{m} + \frac{1}{m-1}\right] x^{-1} \exp[-\frac{2}{27}(m-\tfrac{1}{2})^{-1}(x^{-1}-1)] \quad (0 < x \le 1),$$

where we have chosen $\tau_2 = (m-1/2)^{-1}$ and $A = 1$.

## 4. PROOF

The problems to find bounds for $S$ and $P$ can be reduced to one problem, namely, to find bounds for integrals of the kind $\int_0^\infty \exp[y\,h(s)]\,ds$. Therefore we need the following

### (3) THEOREM

Let $h : [0,\infty) \to [-\infty,0]$ be a monotonically nonincreasing function (where it is understood that if $h(s_1) = -\infty$ for some $s_1 > 0$ then $h(s) = -\infty$ for $s > s_1$) with the following properties:

There are real numbers $p, q_1, q_2, r_1, r_2, \mu, \nu, A$ with $p > 0, r_1 > 0, r_2 > 0$ ($r_1, r_2$ may eventually be $\infty$), $q_1 < q_2, 0 < \mu < \nu, A > 0$ such that

$$h(s) \le -p\, s^{\mu} + q_2\, s^{\nu} \quad (0 \le s < r_2)$$

$$h(s) \ge -p\, s^{\mu} + q_1\, s^{\nu} \quad (0 \le s < r_1)$$

and

$$\int_0^{\infty} \exp[y\, h(s)]\, ds \ \text{ exists for all } y \ge A.$$

Then

$$I := \int_0^{\infty} \exp[y\, h(s)]\, ds$$

satisfies the following inequalities:

$$I \ge B_1 \quad (y \ge A_1),$$

$$I \le B_2 \quad (y \ge A)$$

where $B_1, B_2, A_1$ and all other quantities involved have the same meaning as in Section 2. We mention that in fact $I_k(p) = \int_0^{\infty} \exp[-py\, s^{\mu}]\, y^k\, s^{k\nu}\, ds$.

*Proof of Theorem (3).*
In the sequel we need bounds for the integrals

$$E_k(\tau) := \int_{\tau}^{\infty} \exp[-py\, s^{\mu}]\, y^k\, s^{k\nu}\, ds \quad (k \in I\!N_0).$$

**(4) LEMMA.**
Let $t > 0$. Let $F : [0,\infty) \to (0,\infty)$ be defined by

$$F(z) := \int_z^{\infty} e^{-s}\, s^{t-1}\, ds \quad (z \ge 0).$$

Then

$$F(z) \le 2\, e^{-z}\, z^{t-1} \quad (z \ge \max\{0, 2(t-1)\})$$

$$F(z) \ge \frac{2}{3}\, e^{-z}\, z^{t-1} \quad (z \ge \max\{0, -2(t-1)\}).$$

*Proof of Lemma (4).*
From $e^{-\frac{1}{2}s}\, s^{t-1} \le e^{-\frac{1}{2}z}\, z^{t-1}$ $(s \ge z \ge \max\{0, 2(t-1)\})$ it follows that

$$F(z) \le e^{-\frac{1}{2}z} z^{t-1} \int_z^\infty e^{-\frac{1}{2}s}\, ds = 2\, e^{-z} z^{t-1} \quad (z \ge \max\{0, 2(t-1)\}).$$

From $e^{\frac{1}{2}s} s^{t-1} \ge e^{\frac{1}{2}z} z^{t-1}$ $(s \ge z \ge \max\{0, -2(t-1)\})$ it follows that

$$F(z) \ge e^{\frac{1}{2}z} z^{t-1} \int_z^\infty e^{-\frac{3}{2}s}\, ds = \frac{2}{3} e^{-z} z^{t-1} \quad (z \ge \max\{0, -2(t-1)\}).$$

The substitution $py\, s^\mu = \sigma$ transforms the integrals $E_k(\tau)$ into integrals of the kind $\int_z^\infty e^{-\sigma} \sigma^{t-1}\, d\sigma$. Using Lemma (4) we find

$$E_k(\tau) \le \bar{E}_k(\tau) := 2 p^{-1} \mu^{-1} \tau^{k\nu+1-\mu} y^{k-1} \exp[-py\, \tau^\mu] \quad (py\tau^\mu \ge \max\{0, 2(k\nu+1-\mu)/\mu\})$$

$$E_k(\tau) \ge \underline{E}_k(\tau) := \frac{1}{3} \bar{E}_k(\tau) \quad (py\tau^\mu \ge \max\{0, -2(k\nu+1-\mu)/\mu\}).$$

We will refer to the lower bound $\underline{E}_k(\tau)$ only in the remark at the end of this section.

We remark that

$$\bar{E}_k(\tau) = \bar{E}_0(\tau)\, \tau^{k\nu} y^k.$$

We are now in a position to determine bounds for $I$. To determine a lower bound we consider the two cases $q_1 \le 0$ and $q_1 > 0$.

The case $q_1 \le 0$. Let $\tau_1 \in (0, r_1)$. Then

$$I \ge \int_0^{\tau_1} \exp[-pys^\mu + q_1 y\, s^\nu]\, ds = \int_0^\infty - \int_{\tau_1}^\infty$$

$$\ge \int_0^\infty \exp[-pys^\mu]\, (1 + q_1 y\, s^\nu)\, ds - \exp[q_1 y\, \tau_1^\nu]\, E_0(\tau_1)$$

$$\ge I_0(p) + q_1 I_1(p) - \exp[q_1 y\, \tau_1^\nu]\, \bar{E}_0(\tau_1) \quad (y \ge \max\{A, 2(1-\mu)\mu^{-1} p^{-1} \tau_1^{-\mu}\}).$$

We remark that if $r_1 = \infty$ we can take the limit $\tau_1 \to \infty$. Then we find

$$I \ge I_0(p) + q_1 I(p) \quad (y \ge A).$$

The case $q_1 > 0$. Let $\tau_1 \in (0, r_1)$ be such that $-p\, \tau_1^\mu + q_1 \tau_1^\nu < 0$. Then

$$I \ge \int_0^{\tau_1} \exp[-pys^\mu + q_1 y\, s^\nu]\, ds \ge \int_0^{\tau_1} \exp[-pys^\mu]\, (1 + q_1 y\, s^\nu)\, ds$$

$$= I_0(p) + q_1 I_1(p) - E_0(\tau_1) - q_1 E_1(\tau_1)$$

$$\geq I_0(p) + q_1 I_1(p) - \bar{E}_0(\tau_1)\,(1 + q_1 y \tau_1^\nu)$$

$$\geq I_0(p) + q_1 I_1(p) - \exp[q_1 y \tau_1^\nu]\,\bar{E}_0(\tau_1) \quad (y \geq \max\{A, 2(\nu+1-\mu)\mu^{-1} p^{-1} \tau_1^{-\mu}\}.$$

The last step has been taken only in order to get the same formula for both cases. Otherwise we have to distinguish between the two cases.

In determining an upper bound for $I$ we distinguish between the two cases $q_2 \leq 0$ and $q_2 > 0$.

The case $q_2 \leq 0$. We take $\tau_2 \in (0, r_2)$. Then

$$I \leq \int_0^{\tau_2} \exp[-p y s^\mu + q_2 y s^\nu]\, ds + R(\tau_2)$$

$$\leq \int_0^{\infty} \exp[-p y s^\mu + q_2 y s^\nu]\, ds + R(\tau_2)$$

$$\leq \int_0^{\infty} \exp[-p y s^\mu]\,(1 + q_2 y s^\nu + \tfrac{1}{2} q_2^2 y^2 s^{2\nu})\, ds + R(\tau_2)$$

$$= I_0(p) + q_2 I_1(p) + \tfrac{1}{2} q_2^2 I_2(p) + R_2(\tau_2) \quad (y \geq A),$$

where $R(\tau)$ and $R_2(\tau)$ are defined for $0 \leq \tau < r_2$, $y \geq A$ as follows:

$$R(\tau) := \int_\tau^\infty \exp[y\, h(s)]\, ds \leq \exp[(y-A)\, h(\tau)] \int_0^\infty \exp[A\, h(s)]\, ds$$

$$\leq \exp[(y-A)\,(-p\tau^\mu + q_2 \tau^\nu)] \int_0^\infty \exp[A\, h(s)]\, ds =: R_2(\tau).$$

If $r_2 = \infty$ then, letting $\tau_2 \to \infty$, we find

$$I \leq I_0(p) + q_2 I_1(p) + \tfrac{1}{2} q_2^2 I_2(p) \quad (y \geq A).$$

The case $q_2 > 0$. Let $\tau_2 \in (0, r_2)$ be such that $-p\tau_2^\mu + q_2 \tau_2^\nu < 0$. Then

$$I \leq \int_0^{\tau_2} \exp[-p y s^\mu + q_2 y s^\nu]\, ds + R(\tau_2)$$

$$\leq \int_0^{\tau_2} \exp[-p\,y\,s^{\mu}]\,(1+q_2\,y\,s^{\nu}+\tfrac{1}{2}\,q_2^2\,y^2\,s^{2\nu}\exp[q_2\,y\,s^{\nu}])\,ds + R_2(\tau_2)$$

$$\leq I_0(p)+q_2\,I_1(p)+\tfrac{1}{2}\,q_2^2 \int_0^{\tau_2} y^2\,s^{2\nu}\exp[(-p+q_2\,\tau_2^{\nu-\mu})\,y\,s^{\mu}]\,ds + R_2(\tau_2)$$

$$\leq I_0(p)+q_2\,I_1(p)+\tfrac{1}{2}\,q_2^2\,I_2(p-q_2\,\tau_2^{\nu-\mu})+R(\tau_2) \quad (y\geq A).$$

**REMARK**

Keeping in mind that

$$\sum_{k=0}^{n-1}\frac{x^k}{k!}\leq e^x\leq \sum_{k=0}^{n-1}\frac{x^k}{k!}+\frac{x^n}{n!}\,e^x \quad (n\in \mathbb{N}, x\geq 0)$$

$$\sum_{k=0}^{n-1}\frac{x^k}{k!}\leq e^x\leq \sum_{k=0}^{n}\frac{x^k}{k!} \quad (n\in 2\,\mathbb{N}, x\leq 0)$$

we could have been chosen more (or less) terms in the Taylor expansion of $\exp[q\,y\,s^{\nu}]$. For instance, in the case $q_2>0$ we could have derived that for $n\in \mathbb{N}$

$$I\leq \sum_{k=0}^{n-1}\frac{1}{k!}\,q_2^k\,I_k(p)+\frac{1}{n!}\,q_2^n\,I_n(p-q_2\,\tau_2^{\nu-\mu})+R_2(\tau_2) \quad (y\geq A).$$

To get simple formulas we have chosen the shortest possible expansions such that at least the linear contributions with respect to $q_1$ and $q_2$ are given exactly.

Furthermore, in striving after simple formulas we have neglected several exponential small terms. We give an example. Deriving an upper bound in the case $q_2>0$ we have neglected $\underline{E}_0(\tau_2)+q_2\,\underline{E}_1(\tau_2)$ in

$$\int_0^{\tau_2}\exp[-p\,y\,s^{\mu}]\,(1+q_2\,y\,s^{\nu})\,ds\leq \int_0^{\infty}\exp[-p\,y\,s^{\mu}]\,(1+q_2\,y\,s^{\nu})\,ds \;-\underline{E}_0(\tau_2)-q_2\,\underline{E}_1(\tau_2).$$

## 5. BOUNDS OF $S$

From the monotonicity of $f$ it follows that

$$I\leq S(x,n)\leq x+I \quad (x>0,\,n\in \mathbb{N}),$$

where

$$I:=x\int_0^{\infty}(f(x\,t))^n\,dt=\int_0^{\infty}(f(s))^n\,ds.$$

It follows that $\lim_{x\downarrow 0} S(x,n)=I$.

The asymptotic behaviour of $f$ implies that

$$\log f(s) = -a\, s^{\alpha} + O(s^{\gamma}) \quad (s \downarrow 0),$$

where $\gamma = \min\{\beta, 2\alpha\}$. Clearly there are numbers $\delta_1, \delta_2 \in (0,\infty]$ and constants $c_1$ and $c_2$ such that

$$\text{(5)} \qquad \begin{aligned} &\log f(s) \geq -a\, s^{\alpha} + c_1\, s^{\gamma} \quad (0 \leq s < \delta_1) \\ &\log f(s) \leq -a\, s^{\alpha} + c_2\, s^{\gamma} \quad (0 \leq s < \delta_2). \end{aligned}$$

It follows that $h(s) := \log f(s)$ satisfies the requirements of Theorem (3). Hence we can take the result of Theorem (3) with

$$p := a\,,\ \mu := \alpha,\ \nu = \gamma,\ q_1 = c_1\,,\ q_2 := c_2\,,\ r_1 := \delta_1\,,\ r_2 := \delta_2 \ \text{ and } \ A = 1.$$

## 6. BOUNDS OF $P$

We define

$$L(x,k) := \sum_{i=0}^{k} \log f(ix) \quad (x > 0),$$

where we take $L(x,k) = -\infty$ if the value $f(ix) = 0$ occurs in the sum.
Since $\log f$ is monotonically decreasing and $\log f(0) = 0$ we get

$$\int_1^{k+1} \log f(xs)\, ds \leq L(x,k) \leq \int_0^{k} \log f(xs)\, ds\ ,$$

or, equivalently

$$x^{-1} J((k+1)x) - x^{-1} J(x) \leq L(x,k) \leq x^{-1} J(kx)\ ,$$

where

$$J(x) := \int_0^{x} \log f(s)\, ds \quad (x \geq 0).$$

Using these bounds for $L(x,k)$ in

$$\int_0^{\infty} \exp L(x,k)\, dk \leq P(x) = \sum_{k=0}^{\infty} \exp L(x,k) \leq 1 + \int_0^{\infty} \exp L(x,k)\, dk$$

we find (using that $J$ is negative)

$$P(x) \leq 1 + x^{-1} \int_0^{\infty} \exp[x^{-1} J(s)]\, ds\ ,$$

$$P(x) \geq x^{-1} \exp[-x^{-1} J(x)] \left( \int_0^\infty \exp[x^{-1} J(s)]\, ds - \int_0^x \exp[x^{-1} J(s)]\, ds \right)$$

$$\geq -1 + x^{-1} \int_0^\infty \exp[x^{-1} J(s)]\, ds.$$

Integrating both sides of (5) we get

$$J(s) \leq -a(\alpha+1)^{-1} s^{\alpha+1} + c_2(\gamma+1)^{-1} s^{\gamma+1} \quad (0 \leq s < \delta_2)$$

$$J(s) \geq -a(\alpha+1)^{-1} s^{\alpha+1} + c_1(\gamma+1)^{-1} s^{\gamma+1} \quad (0 \leq s < \delta_1).$$

Clearly $J$ satisfies the conditions of Theorem (3) with $p := a(\alpha+1)^{-1}$, $\mu = \alpha + 1$, $\nu := \gamma + 1$, $q_1 = (\gamma+1)^{-1} c_1$, $q_2 := c_2(\gamma+1)^{-1}$, $r_1 := \delta_1$, $r_2 := \delta_2$, and $A > 0$ at choice.

That $\int_0^\infty \exp[y\, J(s)]\, ds$ exists if $y > 0$ follows from the fact that $J'(x) = \log f(x)$ tends to $-\infty$ whence $J(x) < -x$ if $x$ is sufficiently large.

The factor $\int_0^\infty \exp[A\, J(x)]\, dx$ occuring in the term $B_2$ in the upper bound of $P$ is bounded above by $A^{-1} + \int_0^\infty f(x)\, ds$. For, since $\log f(x) \leq -1 + f(x)$ $(x \geq 0)$, we can write

$$\int_0^\infty \exp[A\, J(x)]\, dx \leq \int_0^\infty \exp[-Ax + A \int_0^x f(s)\, ds]\, dx$$

$$= A^{-1} + A^{-1} \int_0^\infty \exp[-Ax + A \int_0^x f(s)\, ds]\, A\, f(x)\, dx \leq A^{-1} + \int_0^\infty f(x)\, dx.$$

## REFERENCES

[1] H. Vinck, K. Post, On the influence of coding on the Mean Time To Failure for degrading memories with defects. To appear in IEEE Transactions on Information Theory (July 1990).