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On the number of maxima in a discrete sample

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Abstract: Let $M_n = \max(N_1, N_2, ..., N_n)$, where $N_1, N_2, ...$ are i.i.d., positive, integer-valued rv's. We are interested in K_n, the number of values of $j \in \{1, 2, ..., n\}$ for which $N_j = M_n$. It turns out that $K_n \xrightarrow{d} 1$ as $n \to \infty$ in many cases, but not always; the case where N_1 has a geometric distribution is an example of special interest. There is an application of results on K_n to the behaviour of the fractional parts of sample maxima from non-integer populations.

1. Introduction and summary

Lennart Råde (1991) proposes the following problem. Toss n coins, probability p for heads, as follows. First toss all coins, then toss the ones that did not fall heads, again, and so on, until all coins show heads. The, at first rather confusing, question is: What can be said about the behaviour of the number K_n of coins involved in the final toss. A little thought learns that K_n is equal to the number of coins that need the maximum number of tosses to produce heads.

In this paper we consider the following generalization of this problem. Let $N_1, N_2, \ldots$ be i.i.d., positive, integer-valued rv's, and let

$$M_n = \max(N_1, N_2, \ldots, N_n)\,.$$

We shall be interested in the rv K_n defined by

$$(1.1) \qquad K_n = \#\{j \in \{1, 2, \ldots, n\} : N_j = M_n\}\ ,$$

the number of sample elements equal to the sample maximum. We shall use the following notation:

$$(1.2) \qquad p_j = P(N_1 = j),\ \ P_0 = 0,\ \ P_j = \sum_{k=1}^{j} p_k,\ \ \bar{P}_j = 1 - P_{j-1} \quad (j = 1, 2, \ldots)\,.$$

The distribution function of a rv X will be denoted by F_X, its density by f_X. We shall write $\{a\}$ for the fractional part of a, i.e., $\{a\} = a - [a]$ with $[a]$ the largest integer not exceeding a. Our main interest is the behaviour of K_n for large n. In Section 2 we consider the general case, in Section 3 the rather delicate case of the geometric distribution (equivalent to Råde's problem), and in Section 4 we give an application to the behaviour of $\{\max(X_1, \ldots, X_n)\}$, the fractional part of the sample maximum from a non-integer population. Some technical details are collected in an Appendix.

2. A general result

Though the question by Råde, how many coins are in the final toss is at first rather puzzling, the equivalent question, how many of the n coins need the maximal number of tosses, is quite easily answered. In the notation of (1.2) we have the following result.

Lemma 2.1

$$(2.1) \qquad P(K_n = k) = \binom{n}{k} \sum_{j=1}^{\infty} p_j^k P_{j-1}^{n-k} \ \ (k = 1, 2, \ldots, n; n = 1, 2, \ldots)\,.$$

Proof By symmetry and independence we have (using $0^0 = 1$)

$$P(K_n = k) = \binom{n}{k} \sum_{j=1}^{\infty} P(N_1 = ... = N_k = j, N_{k+1} \le j-1, ..., N_n \le j-1)$$

$$= \binom{n}{k} \sum_{j=1}^{\infty} p_j^k P_{j-1}^{n-k} . \qquad \square$$

By a simple calculation we obtain EK_n from (2.1).

Corollary 2.2

$$(2.2) \qquad EK_n = n \sum_{j=1}^{\infty} p_j P_j^{n-1} ,$$

possibly infinite.

The case $k = 1$ of (2.1) is of special interest.

$$(2.3) \qquad P(K_n = 1) = n \sum_{j=1}^{\infty} p_j P_{j-1}^{n-1} .$$

From (2.2) and (2.3) it follows that EK_n is bounded if p_j/p_{j+1} is bounded. Clearly, this is not so if N_1 is bounded, i.e., if $p_m > 0$ for some $m \in I\!N$, and $p_j = 0$ for $j \ge m+1$. In that case we have $P(M_n \to m) = 1$ and $P(K_n \to \infty) = 1$, in agreement with the fact that then

$$\lim_{n\to\infty} P(K_n = k) = \lim_{n\to\infty} \binom{n}{k} \sum_{j=1}^{m} p_j^k P_{j-1}^{n-k} = 0$$

for all $k \in I\!N$.
In what follows we shall assume that N_1 is *unbounded*, i.e., that $P_j < 1$ for all j. Clearly, in this case K_n will take the value 1 infinitely often (i.o.): there will be new records no matter how large the present record is. This is not necessarily true for values larger than 1. As an example we prove

Theorem 2.3 If $p_j = cj^{-a}$ $(j = 1, 2, ...)$ with $1 < a < 2$, then

$$P(K_n = 2 \text{ } i.o.) = 0 .$$

Proof By the Borel-Cantelli lemma it is sufficient to prove that $\sum_{n=2}^{\infty} P(K_n = 2) < \infty$. We have (cf. (2.1), and Section 1 for notation),

$$\sum_{n=2}^{\infty} P(K_n = 2) = \sum_{n=2}^{\infty} \binom{n}{2} \sum_{j=1}^{\infty} p_j^2 P_{j-1}^{n-2} = \sum_{j=1}^{\infty} p_j^2 / \bar{P}_j^3 .$$

Now, since $p_j = cj^{-a}$, we have $\bar{P}_j \sim \frac{c}{a-1} j^{1-a}$ and so $p_j^2/\bar{P}_j^3 \sim$ const. j^{a-3}, which means that the sum above converges if $a - 3 < -1$, i.e., $a < 2$. The condition that $a > 1$ is, of course, necessary for the convergence of Σp_j. □

Similarly it can be proved that $P(K_n = k \ i.o.) = 0$ if $1 < a < k$. It is not hard to see that under the conditions of Theorem 2.3 we have $P(K_n = 1) \to 1$ and even $P(K_n \to 1) = 1$ as $n \to \infty$. We now come to the main result of this section.

Theorem 2.4 If $(p_j)_1^\infty$ is such that

$$(2.4) \qquad \liminf_{j \to \infty} \ (p_j/p_{j-1}) = 1 \ ,$$

then

$$\lim_{n \to \infty} P(K_n = 1) = 1 \ .$$

Proof We have

$$P_j^n - P_{j-1}^n = p_j(P_j^{n-1} + P_{j-1}P_j^{n-2} + \ldots + P_{j-1}^{n-1}) \ ,$$

so

$$(2.5) \qquad np_jP_{j-1}^{n-1} \le P_j^n - P_{j-1}^n \le np_jP_j^{n-1} \ .$$

From (2.2) and (2.5) we obtain, for every $m \in \mathbb{N}$,

$$\begin{aligned} 1 \ge P(K_n = 1) &= n\sum_1^\infty p_jP_{j-1}^{n-1} \ge n \sum_{j=m+1}^\infty p_jP_{j-1}^{n-1} = n \sum_{j=m+1}^\infty \frac{p_j}{p_{j-1}} p_{j-1}P_{j-1}^{n-1} \\ &\ge n(1-\varepsilon(m)) \sum_{j=m}^\infty p_jP_j^{n-1} \ge (1-\varepsilon(m)) \sum_{j=m}^\infty (P_j^n - P_{j-1}^n) \\ &= (1-\varepsilon(m))(1-P_{m-1}^n) \ , \end{aligned}$$

where $1 - \varepsilon(m) = \inf_{j>m} \frac{p_j}{p_{j-1}}$. By (2.4) we know that $\varepsilon(m) < \varepsilon$ for m sufficiently large, whereas $1 - P_{m-1}^n \to 1$ as $n \to \infty$ for each fixed m. □

From (2.5), by writing (2.4) as $\limsup_{j \to \infty} \frac{p_j}{p_{j+1}} = 1$, in a similar way (cf. (2.2)) the following corollary is obtained.

Corollary 2.5 Under the conditions of Theorem 2.4

$$\lim_{n\to\infty} EK_n = 1 \ .$$

Since in a large number of practical cases $p_j/p_{j+1} \to 1$ as $j \to \infty$, in many cases we will have $P(K_n = 1) \to 1$ as $n \to \infty$. We found no examples where K_n has a nondegenerate limit distribution; in some cases $K_n \to \infty$, and in some cases there is no convergence. In the next section we discuss an important example of the latter type.

3. The geometric distribution

Here we have (see Section 1 for notation),

$$(3.1) \qquad p_j = p(1-p)^{j-1}, \ \bar{P}_j = (1-p)^{j-1}, P_{j-1} = 1 - \bar{P}_j \ \ (j = 1, 2, \ldots) \ .$$

Now K_n can be interpreted as the number of coins in the final toss. Intuitively, it makes little difference whether one starts with about 10 coins or with a thousand; after six or seven tosses one is left with about 10 coins again. In a way, this may explain why in this case K_n does not converge in distribution as $n \to \infty$, but converges 'almost', as we will see. We first state and prove a formal theorem.

Theorem 3.1 If p_j and P_{j-1} are given by (3.1), then

$$(3.2) \qquad P(K_n = 1) = p \sum_{l=-\infty}^{\infty} e^{-\lambda(l-\theta_n)} e^{-e^{-\lambda(l-\theta_n)}} + o(1) \ ,$$

as $n \to \infty$, where $\lambda = -\log(1-p)$ and $\theta_n = \{\lambda^{-1} \log \ n\}$ with $\{a\}$ denoting the fractional part of a.

Proof (sketch) Using (3.1), replacing $j-1$ by j and putting $1 - p = e^{-\lambda}$, we obtain (cf. (2.3))

$$P(K_n = 1) = np \sum_{j=0}^{\infty} e^{-\lambda j} (1 - e^{-\lambda j})^{n-1} = p \sum_{j=0}^{\infty} e^{-\lambda j + \log \ n} (1 - \tfrac{1}{n} e^{-\lambda j + \log \ n})^{n-1}$$

$$= p \sum_{l=-m}^{\infty} e^{-\lambda(l-\theta_n)} (1 - \tfrac{1}{n} e^{-\lambda(l-\theta_n)})^{n-1} \ ,$$

where $m = [\log \ n]$.
It is now not very surprising that for $n \to \infty$ we have (3.1); for a more detailed result and proof we refer to Lemma A.2 in the Appendix. □

Since the function in the right-hand side of (3.2) is periodic, and (θ_n) is dense in (0,1) as $n \to \infty$, Theorem 3.1 yields an example where K_n does not converge:

Corollary 3.2 For the geometric distribution, with p_j as in (3.1), the sequence K_n does not converge in distribution, as $n \to \infty$.

Remark 1 Expressions similar to (3.2) can be obtained for $P(K_n = k)$ with $k \geq 2$. One finds (cf. Lemma A.1)

$$P(K_n = k) = \frac{p^k}{k!} \sum_{l=-\infty}^{\infty} e^{-k\lambda(l-\theta_n)} e^{-e^{-\lambda(l-\theta_n)}} + o(1) ,$$

as $n \to \infty$. For EK_n from (2.2), (2.3) and (3.2) we obtain

$$EK_n = \frac{p}{1-p} \sum_{l=-\infty}^{\infty} e^{-\lambda(l-\theta_n)} e^{-e^{-\lambda(l-\theta_n)}} + o(1) .$$

Remark 2 It turns out that the (periodic) functions

$$F_k(\theta; \lambda) := \sum_{l=-\infty}^{\infty} e^{-k\lambda(l-\theta)} e^{-e^{-\lambda(l-\theta)}}$$

are almost constant in θ for moderate values of λ; for $k = \lambda = 1$, i.e., $p = 1 - e^{-1}$ direct computation yields

$$|F_1(\theta; 1) - 1| < 6.5\ 10^{-4} ;$$

see Appendix for more information.
By (3.2) this means that $P(K_n = 1)$ is close to p in this special case, and close to $-p/\log(1-p)$ for more general p. Similarly, one finds $EK_n \approx -p/((1-p)\log(1-p))$, and

$$(3.3) \qquad P(K_n = k) \approx \frac{-p^k}{k\ \log(1-p)} \qquad (k = 1, 2, ...) ,$$

i.e., the distribution of K_n 'almost converges' to the logarithmic distribution. We return to this in the next section.

Remark 3 Things change when p is allowed to depend on n. If we take $p = 1 - \mu/n$, i.e., $\lambda = \log\ (n/\mu)$, then for any fixed $k \in \mathbb{N}$

$$\lim_{n\to\infty} P(K_n = k) = e^{-\mu} \frac{\mu^k}{k!} ,$$

and

$$\lim_{n\to\infty} P(K_n = n) = e^{-\mu} .$$

So K_n has a defective, Poisson limit distribution on $\mathbb{N}$, with mass $e^{-\mu}$ at infinity.

4. Connection with fractional part of maximum

Several papers have been devoted to the study of $\{S_n\}$, where

$$(4.1) \qquad S_n = X_1 + ... + X_n \ ,$$

with $X_1, X_2, ...$ i.i.d. and non-lattice, and $\{a\}$ denoting the fractional part of a (see e.g. Schatte (1983)). As is well known, $\{S_n\} \xrightarrow{d} U$, where U is uniformly distributed on $[0,1)$. In Brands and Wilms (1991) the analogue of (4.1) is considered for maxima, i.e. they consider the behaviour of $\{Z_n\}$, with

$$(4.2) \qquad Z_n = \max(X_1, ..., X_n) \ ,$$

for i.i.d. non-lattice X_j. They show that in many cases $\{Z_n\} \xrightarrow{d} U$. It is known that for exponentially distributed X_j the sequence $(\{Z_n\})_1^\infty$ does not converge (cf. Jagers and Steutel (1990)). This phenomenon is closely connected with the results of Section 3, as we shall see. It is rather difficult to find examples where $\{Z_n\} \xrightarrow{d} V \neq U$. We now use Theorem 2.4 to construct an example of this kind: $\{Z_n\}$ does converge, but not to U.

Theorem 4.1 Let the rv N be such that the $p_j := P(N = j), j = 1, 2, ...$ satisfy the conditions of Theorem 2.4. Further let V be a rv independent of N and such that $P(0 \leq V < 1) = 1$. Finally, let $X_1, X_2, ...$ be i.i.d. and such that $X_1 \overset{d}{=} N + V$. Then for Z_n as defined by (4.2) one has

$$\{Z_n\} \xrightarrow{d} V \quad (n \to \infty) \ .$$

Proof We have

$$Z_n = \max(X_1, ..., X_n) \overset{d}{=} \max(N_1, ..., N_n) + \max(V_1, ..., V_{K_n}) \ ,$$

where $N_j = [X_j]$ and K_n is independent of $V_1, V_2, ...$, which are independent copies of V. It follows that

$$P(\{Z_n\} \leq x) = \sum_{k=1}^{n} P(K_n = k) F_V^k(x) = P_{K_n}(F_V(x)) \ ,$$

where F_V denotes the distribution function of V and P_{K_n} the probability generating function of K_n. So we may write

$$(4.3) \qquad F_{\{Z_n\}}(x) = P_{K_n}(F_V(x)) \ .$$

Now if $K_n \xrightarrow{d} K$, then $F_{\{Z_n\}}(x) \to P_K(F_V(x))$, and in the special case that $K_n \xrightarrow{d} 1$ we have

$$F_{\{Z_n\}}(x) \to F_V(x) ,$$

i.e., $\{Z_n\} \xrightarrow{d} V$. □

This shows that any distribution on $[0,1)$ can occur as a limit distribution of $\{Z_n\}$.
We now return to the geometric distribution. We shall need the following lemma (see Kopocinsky (1988) or Steutel and Thiemann (1989)).

Lemma 4.2 Let Y be exponentially distributed with $EY = \lambda^{-1}$, and let $X = Y + 1$. Then

$$X = N + V ,$$

with N and V independent,

$$P(N = j) = p(1-p)^{j-1} \qquad (j = 1, 2, \ldots) ,$$

with $p = 1 - e^{-\lambda}$, and

$$\text{(4.4)} \qquad F_V(v) = \frac{1 - e^{-\lambda v}}{1 - e^{-\lambda}} \qquad (0 \le v < 1) .$$

From (4.3) and the fact, established in Corollary 3.2, that for this N the sequence K_n does not converge in distribution, it follows that $\{Z_n\}$ does not converge in distribution. On the other hand, (4.3) can be used to obtain information about K_n. Combining (4.3) and (4.4) we get

$$P_{K_n}(z) = F_{\{Z_n\}}\left(\frac{\log(1-pz)}{\log(1-p)}\right) .$$

Now, although $\{Z_n\}$ does not converge in distribution, it is not very far from being uniform for large n. Since

$$\text{(4.5)} \qquad F_{\{Z_n\}}(x) = \sum_{j=0}^{\infty}((1 - e^{-\lambda(j+x)})^n - (1 - e^{-\lambda j})^n) ,$$

Corollary A.4 yields the following theorem ($p = 1 - e^{-\lambda}$).

Theorem 4.3 For $0 < \lambda < 2\pi^2$, i.e., for $0 < p < 1 - 2.6\ 10^{-9}$, and for $n \geq 7$

$$\left| P_{K_n}(z) - \frac{\log(1-pz)}{\log(1-p)} \right| \le 70\lambda^{-1/2}e^{-\pi^2/\lambda} + (3+2\lambda)n^{-1} \ .$$

This means that for large n and moderate values of $p = 1 - e^{-\lambda}$, the random variable K_n is close to having a logarithmic distribution; this result agrees with formula (3.3). Though the bound above is fairly small, it is rather crude; compare the pictures of $T_n(x;\lambda) = F_{\{Z_n\}}(x)$ in the Appendix.

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Appendix

Here we give some of the details that were omitted in sections 3 and 4. We shall use the following notation. For $k, n \in \mathbb{N}, x \in [0,1)$ and $\lambda > 0$ we define

$$\text{(A.1)} \qquad F_k(x;\lambda) = \sum_{l=-\infty}^{\infty} e^{-\lambda k(l-x)} \exp\left(-e^{-\lambda(l-x)}\right) \ ,$$

$$\text{(A.2)} \qquad S_{n,k}(x;\lambda) = \binom{n}{k} \sum_{j=0}^{\infty} e^{-\lambda k(j+x)} \left(1 - e^{-\lambda(j+x)}\right)^{n-k} \ .$$

Lemma A.1 For $n \ge 2k$ the following inequalities hold:

(A.3) $$0 \leq (n-k)^{-k} \binom{n}{k} F_k(\theta - x; \lambda) - S_{n,k}(x;\lambda) \leq R(n,k,\lambda) ,$$

where $\theta = \{\lambda^{-1} \log\,(n-k)\}$, and

(A.4) $$R(n,k,\lambda) =$$

$$(n-k)^{-k} \binom{n}{k} \left[(n-k)^{-1}(\lambda^{-1}(k+1)! + (k+2)^{k+2} e^{-k-2}) + (n-k)^k (1+\lambda^{-1}) e^{-n+k}\right] .$$

Proof. Substitution in (A.2) of $j = m + l$ with $m = [\lambda^{-1} \log(n-k)]$ gives

$$S_{n,k}(x;\lambda) = (n-k)^{-k} \binom{n}{k} \sum_{l=-m}^{\infty} e^{-\lambda k(l+x-\theta)} (1 - (n-k)^{-1} e^{-\lambda(l+x-\theta)})^{n-k} .$$

Putting $u_k(t) = e^{-kt} \exp(-e^{-t})$ and using the inequalities

$$0 \leq e^{-ns} - (1-s)^n \leq ns^2 e^{-ns} \quad (0 \leq s \leq 1, n \in I\!N) ,$$

leads to

$$0 \leq (n-k)^{-k} \binom{n}{k} F_k(\theta - x; \lambda) - S_{n,k}(x,\lambda)$$

$$\leq (n-k)^{-k} \binom{n}{k} \left\{ (n-k)^{-1} \sum_{l=-m}^{\infty} u_{k+2}(\lambda(l+x-\theta)) + \sum_{l=-\infty}^{-m-1} u_k(\lambda(l+x-\theta)) \right\} .$$

Now,

$$\sum_{l=-m}^{\infty} u_{k+2}(\lambda(l+x-\theta)) < F_{k+2}(\theta - x;\lambda) \leq \int_{-\infty}^{\infty} u_{k+2}(\lambda t)dt + \max_{t \in \mathbb{R}} u_{k+2}(t)$$

$$= \lambda^{-1}(k+1)! + e^{-k-2}(k+2)^{k+2} .$$

and, for $n \geq 2k$, (use $\int_a^b s^b e^{-s} ds \leq a^{b+1} e^{-a}$ for $2 \leq b+1 \leq a$)

$$\sum_{l=-\infty}^{-m-1} u_k(\lambda(l+x-\theta)) \leq \int_{-\infty}^{-m-1} u_k(\lambda(y+x-\theta))dy + u_k(\lambda(-m-1+x-\theta))$$

$$\leq (1+\lambda^{-1})(n-k)^k e^{-n+k} .$$

Combination of the inequalities above proves Lemma A.1. □

For the important case $k = 1$ we have

Lemma A.2. For $n \geq 2$

$$(A.5) \qquad |S_{n,1}(x,\lambda) - F_1(\theta - x;\lambda)| \leq R(n,1,\lambda)\,,$$

where $\theta = \{\lambda^{-1}\log(n-1)\}$, and

$$(A.6) \qquad R(n,1,\lambda) = n(n-1)^{-2}(2\lambda^{-1} + 27e^{-3} + (1+\lambda^{-1})(n-1)^2 e^{-n+1})\,.$$

Proof. From the general case in Lemma A.1 we have

$$|S_{n,1}(x;\lambda) - F_1(\theta - x;\lambda)| \leq \max\{R(n,1,\lambda), (n-1)^{-1}F_1(\theta - x;\lambda)\}\,,$$

and the result follows from

$$0 < F_1(\theta - x;\lambda) \leq \int_{-\infty}^{\infty} u_1(\lambda t)dt + \max_{t \in \boldsymbol{R}} u_1(t) = 2\lambda^{-1} + 27e^{-3} \leq (n-1)R(n,1,\lambda)\,. \ \Box$$

The functions $F_k(x,\lambda)$ are almost constant in x. For $k = 1$ we have the following result.

Lemma A.3. For $0 < \lambda < 2\pi^2$

$$(A.7) \qquad |F_1(x;\lambda) - \lambda^{-1}| \leq \delta_1(\lambda)$$

where

$$(A.8) \qquad \delta_1(\lambda) = 4\pi(1 - e^{-4\pi^2/\lambda})^{-1/2}(1 + \lambda/\pi^2 + \lambda^2/(2\pi^4))\lambda^{-3/2}e^{-\pi^2/\lambda}\,.$$

Proof For the Fourier coefficients $C_m(k,\lambda)$ of the functions $F_k(\cdot;\lambda)$, which are periodic with period 1, we have

$$C_m(k,\lambda) = \int_0^1 F_k(x;\lambda)e^{-2\pi i m x}dx = \lambda^{-1}\int_{-\infty}^{\infty} u_k(t)e^{2\pi i\lambda^{-1}mt}dt = \lambda^{-1}\Gamma(k - 2\pi i\lambda^{-1}m)\,.$$

From

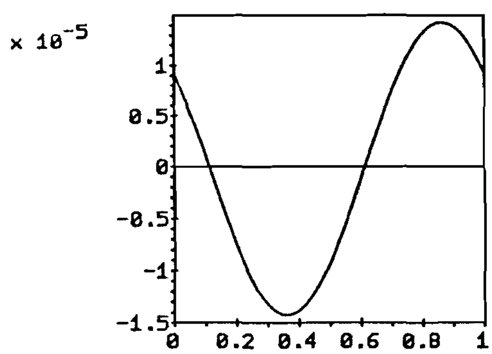
$$|\Gamma(1 \pm iy)|^2 = \Gamma(1+iy)\Gamma(1-iy) = \pi y(\sinh \pi y)^{-1} \text{ and } \Gamma(k+iy) = (1+iy)...(k-1+iy)\Gamma(1+iy)$$

it follows that $|\Gamma(k+iy)| = (\pi y q_k(y)(\sinh \pi y)^{-1})^{1/2}$, where $q_k(y) = \prod_{n=1}^{k-1}(n^2+y^2)$ $(k \geq 2)$, $q_1(y) = 1$. For $k = 1$ we have

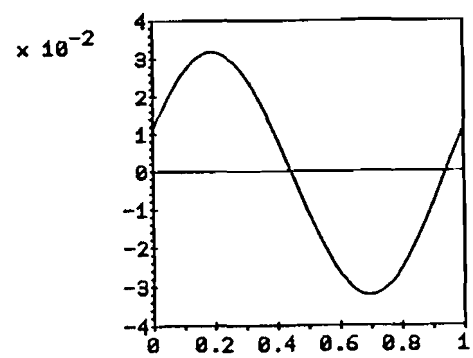
$$|F_1(x;\lambda) - \lambda^{-1}| \leq 2\sum_{m=1}^{\infty} |C_m(1,\lambda)| = \pi(2/\lambda)^{3/2} \sum_{m=1}^{\infty} (m^{1/2}\sinh 2\pi^2\lambda^{-1}m)^{-1/2} .$$

After some fairly straightforward estimations we arrive at the desired result. □

Below pictures of $F_1(x;\lambda) - \lambda^{-1}$ are shown for $\lambda = \log 2$, and $\lambda = 2$ (i.e., $p = \frac{1}{2}$ and $p = 0.865$); the amplitude is increasing in λ.



$F_1(x;\ \log 2)$



$F_1(x;2)$

Remark. From the foregoing it easily follows that

$$F_k(x;\lambda) = (k-1)!\lambda^{-1} + O(\lambda^{-k-1/2}e^{-\pi^2/\lambda}) \quad (\lambda \downarrow 0) .$$

Corollary A.4. Let

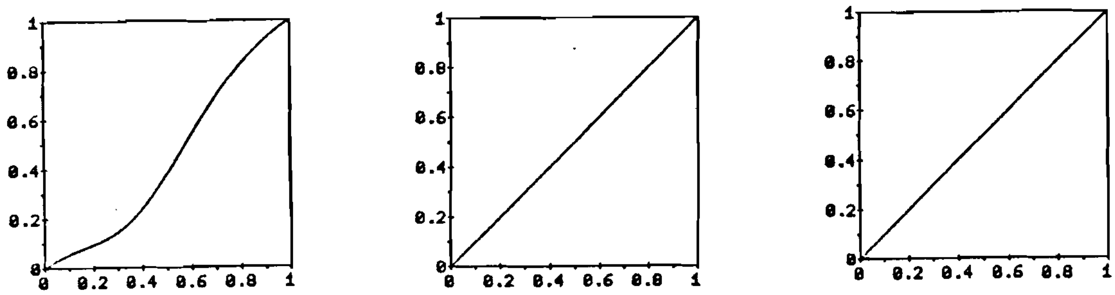
$$T_n(x;\lambda) = \sum_{j=0}^{\infty}((1-e^{-\lambda(j+x)})^n - (1-e^{-\lambda j})^n) .$$

Then for $n \geq 2$ and $0 < \lambda < 2\pi^2$,

$$\text{(A.9)} \qquad |T_n(x;\lambda) - x| \leq 70\lambda^{-1/2}e^{-\pi^2/\lambda} + \frac{n}{(n-1)^2}(2 + 27e^{-3}\lambda + (\lambda+1)(n-1)^2e^{-n+1}) .$$

Proof. Clearly $T_n(x;\lambda) = \lambda\int_0^x S_{n,1}(t;\lambda)dt$. Applying Lemmas A.2 and A.3, and using the fact that $4\pi(1-e^{-4\pi^2/\lambda})^{-1/2}(1+\pi^{-2}\lambda+\frac{1}{2}\pi^{-4}\lambda^2) < 70$ for $0 < \lambda < 2\pi^2$, we obtain (A.9). □

The bound in the right-hand side of (A.9) is rather conservative. For moderate values of λ, and n not very small $T_n(x;\lambda) = F_{\{Z_n\}}(x)$ (cf. (4.5)) can hardly be distinguished from x, as is shown in the pictures below.



$n = 10; \lambda = 4$
($p = 0,98$)

$n = 10; \lambda = 1$
($p = 0,37$)

$n = 25; \lambda = \log 2$
($p = 0,50$)

Pictures of $T_n(x;\lambda) = F_{\{Z_n\}}(x)$

List of COSOR-memoranda - 1992

Number	Month	Author	Title
92-01	January	F.W. Steutel	On the addition of log-convex functions and sequences
92-02	January	P. v.d. Laan	Selection constants for Uniform populations
92-03	February	E.E.M. v. Berkum H.N. Linssen D.A. Overdijk	Data reduction in statistical inference
92-04	February	H.J.C. Huijberts H. Nijmeijer	Strong dynamic input-output decoupling: from linearity to nonlinearity
92-05	March	S.J.L. v. Eijndhoven J.M. Soethoudt	Introduction to a behavioral approach of continuous-time systems
92-06	April	P.J. Zwietering E.H.L. Aarts J. Wessels	The minimal number of layers of a perceptron that sorts
92-07	April	F.P.A. Coolen	Maximum Imprecision Related to Intervals of Measures and Bayesian Inference with Conjugate Imprecise Prior Densities
92-08	May	I.J.B.F. Adan J. Wessels W.H.M. Zijm	A Note on "The effect of varying routing probability in two parallel queues with dynamic routing under a threshold-type scheduling"
92-09	May	I.J.B.F. Adan G.J.J.A.N. v. Houtum J. v.d. Wal	Upper and lower bounds for the waiting time in the symmetric shortest queue system
92-10	May	P. v.d. Laan	Subset Selection: Robustness and Imprecise Selection
92-11	May	R.J.M. Vaessens E.H.L. Aarts J.K. Lenstra	A Local Search Template (Extended Abstract)
92-12	May	F.P.A. Coolen	Elicitation of Expert Knowledge and Assessment of Imprecise Prior Densities for Lifetime Distributions
92-13	May	M.A. Peters A.A. Stoorvogel	Mixed H_2/H_∞ Control in a Stochastic Framework

Number	Month	Author	Title
92-14	June	P.J. Zwietering E.H.L. Aarts J. Wessels	The construction of minimal multi-layered perceptrons: a case study for sorting
92-15	June	P. van der Laan	Experiments: Design, Parametric and Nonparametric Analysis, and Selection
92-16	June	J.J.A.M. Brands F.W. Steutel R.J.G. Wilms	On the number of maxima in a discrete sample