

# Modeling and analysis of a long thin stripline in a stripline environment

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# Modeling and Analysis of a Long Thin Stripline in a Stripline Environment

Dave Bekers

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## Preface

The research described in this report has been inspired by a model problem in the course “Blok B” of the post graduate programme Mathematics for Industry at the Eindhoven University of Technology, and by a final project of the same programme, as described in [1]. I would like to thank dr. ir. A.A.F. van de Ven for initiating the research, and Daniel Chandra for the pleasant cooperation during the aforementioned model problem of “Blok B”. Furthermore, I would like to thank dr. ir. S.J.L. van Eijndhoven for reading carefully all preliminary versions of this report. His comments have led to many improvements.

Dave Bekers  
May 2002

On September 12th and 13th, 2002, this work has been presented at the Universidad Mayor de San Andrés (Facultad de Ciencias Puras y Naturales, Carrera de Matemática) in La Paz (Bolivia), which led to improvements of the first version of the report. I would like to thank Mgr. Porfirio Suñaga Salgado, director de Carrera de Matemática, for giving me the opportunity to present this work, and the audience for their attention and useful comments.

Dave Bekers  
September 2002

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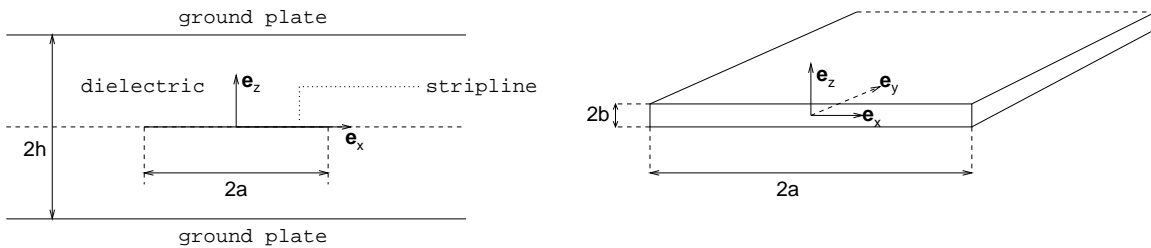
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## CHAPTER 1

## Introduction

An electromagnetic transmission line is used for the excitation of antennas. A stripline is a planar type of transmission line, which can be produced by photolithography. In this report, we consider a stripline in a so-called stripline environment, the intersection of which is shown in Figure 1.1 (left). It consists of a long, thin, conducting strip of width  $2a$ , called the stripline,



**Figure 1.1** Left: Cross-section of a stripline in a stripline environment, Right: Strip in a dielectric.

centered between two wide, conducting ground plates. Strip and ground plates are parallel. The ground plates are separated by a distance  $2h$  and the space between the ground plates and the strip is filled with a dielectric. The ratio  $\zeta = h/a$  is mostly of order 1. We introduce a coordinate system the origin of which is in the center of the strip (see Figure 1.1 (left)). The  $x$ -axis is transverse to the strip, the  $y$ -axis is in its length-direction, and the  $z$ -axis is normal to it.

In high frequency applications on L-band ( $f = 1 - 2\text{GHz}$ , where  $f$  is the frequency and  $\omega = 2\pi f$  the radian frequency), the height  $2h$  and the width  $2a$  are of the same order (both a few millimeters up to about a centimeter). Furthermore, the wavelength is much larger than the height and the width. Note that in case the dielectric is vacuum, the wavelength in the dielectric is about 15-30cm for L-band frequencies. Thickness of the stripline (about  $40\mu\text{m}$ ) is much smaller than its width. Width and length of the ground plates are much larger than the width and the length of the stripline. Since a stripline consists usually of copper, its conductivity  $\sigma$  is high ( $\sigma = 5.8824 \cdot 10^7 \text{ohm}^{-1}\text{m}^{-1}$ ).

We assume that the length of the stripline is much larger than the wavelength, such that it can support a propagating wave in  $y$ -direction. We assume that the stripline carries a current, which is a planar wave in  $y$ -direction. In this report, we show how such a current can be generated. The current induces an electromagnetic field in the dielectric and the strip, which is also a planar wave in  $y$ -direction. Assuming that this wave is not reflected at the

ends of the stripline and neglecting edge effects, we can consider the stripline infinitely long, and therewith, the ground plates also.

The aim of this report is to calculate the electromagnetic fields generated in the stripline environment and the characteristic impedance, or shortly impedance, of the stripline. To model the stripline, we consider first an infinitely long, thin, conducting strip of thickness  $2b$  and width  $2a$  in an infinitely large dielectric, as given in Figure 1.1 (right). We assume that the strip carries a current, which is a planar wave in  $y$ -direction. We show that for the given numerical values, the strip can be considered infinitely thin and perfectly conducting ( $\sigma \rightarrow \infty$ ), and we deduce boundary conditions for the strip. Then, the stripline in the stripline environment is modeled as infinitely thin and perfectly conducting. After that, the electromagnetic fields in the stripline environment are calculated, whereby we consider only waves of transverse electromagnetic (TEM) type. Mostly, such calculations are carried out by application of conformal mapping theory (see for example [3, Chapter 4] and [9]). However, we solve the boundary value problem for the potential of the electric field directly. Two different approaches are given. The resulting integral equations are solved asymptotically as well as numerically. We compare our results for the impedance with formulae given by Pozar in [7, p. 156], and by Howe in [6, p. 34-35].

The analysis of an infinitely long thin strip in the dielectric is given in Chapter 2. In Chapter 3, we consider the stripline environment with the infinitely thin, perfectly conducting stripline. Numerical results are given in Chapter 4, and a survey and conclusions in Chapter 5.

## CHAPTER 2

# An Infinitely Long Thin Strip in an Infinitely Large Dielectric

In this chapter, we consider an infinitely long thin (conducting) strip in an infinitely large dielectric. The dielectric is homogeneous and isotropic. Geometry and numerical values for radian frequency  $\omega$ , thickness  $2b$ , width  $2a$ , and conductivity  $\sigma$  are given in the introduction of Chapter 1.

In Section 2.1, we present the governing equations for the electromagnetic fields in the dielectric and the strip, and we deduce the transition conditions between these two media. Subsequently, we derive the governing equations for a planar wave in  $y$ -direction in Section 2.2, we analyse the case  $b/a \ll 1$  in Section 2.3, and we analyse the case  $b/\sqrt{\sigma\omega\mu_0} \gg 1$  in Section 2.4. This leads to boundary conditions for an infinitely thin, perfectly conducting strip. Finally, we recapitulate the results.

## 2.1 Balance Equations and Transition Conditions

The electromagnetic fields in the strip and the dielectric are governed by Maxwell's equations. We assume that the two media, strip and dielectric, are linear, homogeneous, and isotropic. Maxwell's equations for such a medium are given by

$$\begin{aligned} \mu \frac{\partial \mathcal{H}}{\partial t} &= -\text{rot } \mathcal{E}, & \text{div } \mathcal{H} &= 0, \\ \varepsilon \frac{\partial \mathcal{E}}{\partial t} + \mathcal{J} &= \text{rot } \mathcal{H}, & \text{div } \mathcal{E} &= \frac{\rho}{\varepsilon}, \end{aligned} \tag{2.1}$$

for  $\mathbf{x} \in \Omega$ . Here,  $\mathcal{E}$  is the electric field,  $\mathcal{H}$  the magnetic field,  $\mathcal{J}$  the current density,  $\rho$  the charge density,  $\varepsilon$  the permittivity,  $\mu$  the permeability, and  $\Omega \subset \mathbb{R}^3$ . We formulate these equations for the strip and the dielectric separately. We denote the quantities in dielectric and strip by superscripts  $(d)$  and  $(s)$  and the material parameters by subscripts  $(d)$  and  $(s)$ . The permeability of both media equals the permeability of vacuum,  $\mu_{(s)} = \mu_{(d)} = \mu_0$ .



In the dielectric, i.e.  $\mathcal{J}^{(d)} = \mathbf{0}$  and  $\varrho^{(d)} = 0$ , Maxwell's equations are

$$\begin{aligned} \mu_0 \frac{\partial \mathcal{H}^{(d)}}{\partial t} &= -\text{rot } \mathcal{E}^{(d)}, & \text{div } \mathcal{H}^{(d)} &= 0, \\ \varepsilon_{(d)} \frac{\partial \mathcal{E}^{(d)}}{\partial t} &= \text{rot } \mathcal{H}^{(d)}, & \text{div } \mathcal{E}^{(d)} &= 0. \end{aligned} \quad (2.2)$$

From these equations, we obtain wave equations for  $\mathcal{H}^{(d)}$  and  $\mathcal{E}^{(d)}$  by differentiating (2.2)<sup>1,3</sup> with respect to the time  $t$  and applying (2.2)<sup>2,4</sup>,

$$\varepsilon_{(d)} \frac{\partial^2 \mathcal{E}^{(d)}}{\partial t^2} = \text{rot } \frac{\partial \mathcal{H}^{(d)}}{\partial t} = -\frac{1}{\mu_0} \text{rot rot } \mathcal{E}^{(d)} = \frac{1}{\mu_0} \Delta \mathcal{E}^{(d)}, \quad (2.3)$$

$$\mu_0 \frac{\partial^2 \mathcal{H}^{(d)}}{\partial t^2} = -\text{rot } \frac{\partial \mathcal{E}^{(d)}}{\partial t} = -\frac{1}{\varepsilon_{(d)}} \text{rot rot } \mathcal{H}^{(d)} = \frac{1}{\varepsilon_{(d)}} \Delta \mathcal{H}^{(d)}, \quad (2.4)$$

or,

$$\frac{\partial^2 \mathcal{E}^{(d)}}{\partial t^2} - c_{(d)}^2 \Delta \mathcal{E}^{(d)} = \mathbf{0}, \quad \frac{\partial^2 \mathcal{H}^{(d)}}{\partial t^2} - c_{(d)}^2 \Delta \mathcal{H}^{(d)} = \mathbf{0}, \quad (2.5)$$

where  $c_{(d)}^2 = 1/(\mu_0 \varepsilon_{(d)})$  is the squared velocity of the electromagnetic waves in the dielectric. If either  $\mathcal{E}^{(d)}$ , with  $\text{div } \mathcal{E}^{(d)} = 0$ , or  $\mathcal{H}^{(d)}$ , with  $\text{div } \mathcal{H}^{(d)} = 0$ , is known, the other quantity can be obtained from (2.2)<sup>1</sup> or (2.2)<sup>3</sup>, respectively. Then, this quantity is solenoidal, i.e. it satisfies (2.2)<sup>2</sup> or (2.2)<sup>4</sup>.

In the strip, free charge distribution cannot exist. Furthermore, the strip carries a current with distribution  $\mathcal{J}^{(s)}$ . The relation between  $\mathcal{J}^{(s)}$  and  $\mathcal{E}^{(s)}$  is given by Ohm's law,

$$\mathcal{E}^{(s)} = \frac{1}{\sigma} \mathcal{J}^{(s)}, \quad (2.6)$$

where  $\sigma$  is the conductivity. Maxwell's equations for the strip are then given by

$$\begin{aligned} \mu_0 \frac{\partial \mathcal{H}^{(s)}}{\partial t} &= -\text{rot } \mathcal{E}^{(s)}, & \text{div } \mathcal{H}^{(s)} &= 0, \\ \varepsilon_{(s)} \frac{\partial \mathcal{E}^{(s)}}{\partial t} + \mathcal{J}^{(s)} &= \text{rot } \mathcal{H}^{(s)}, & \text{div } \mathcal{E}^{(s)} &= 0, & \mathcal{E}^{(s)} &= \frac{1}{\sigma} \mathcal{J}^{(s)}. \end{aligned} \quad (2.7)$$

We obtain a damped wave equation for  $\mathcal{J}^{(s)}$  by differentiating (2.7)<sup>3</sup> with respect to the time  $t$  and using (2.7)<sup>1,5</sup>,

$$\frac{\partial^2 \mathcal{J}^{(s)}}{\partial t^2} + \frac{\sigma}{\varepsilon_{(s)}} \frac{\partial \mathcal{J}^{(s)}}{\partial t} = c_{(s)}^2 \Delta \mathcal{J}^{(s)}, \quad (2.8)$$

where  $c_{(s)}^2 = 1/(\mu_0 \varepsilon_{(s)})$ . If  $\mathcal{J}^{(s)}$  is known with  $\text{div } \mathcal{J}^{(s)} = 0$ , the quantities  $\mathcal{E}^{(s)}$  and  $\mathcal{H}^{(s)}$  follow from (2.7)<sup>1,3,5</sup>. Then, these quantities are solenoidal, i.e. they satisfy (2.7)<sup>2,4</sup>.

The system of equations for strip and dielectric should be supplemented by a set of transition conditions between the two media. We consider only the conditions at  $z = \pm b$ . The

conditions at  $x = \pm a$  are omitted for reasons we explain in Section 2.3. The normal component of the dielectric displacement is discontinuous across the interfaces  $z = \pm b$ , if there exist surface charges at these interfaces. Let  $\varrho_S^\pm$  be the (unknown) surface charge density at  $z = \pm b$ . Because of symmetry  $\varrho_S^+ = \varrho_S^- =: \varrho_S$ . Then,

$$\begin{aligned} \varrho_S &= \varepsilon_{(d)}(\boldsymbol{\mathcal{E}}^{(d)}, \mathbf{n}) - \varepsilon_{(s)}(\boldsymbol{\mathcal{E}}^{(s)}, \mathbf{n}) \Big|_{z=\pm b} = \\ &= \pm \varepsilon_{(d)}(\boldsymbol{\mathcal{E}}^{(d)}, \mathbf{e}_z) \mp \frac{\varepsilon_{(s)}}{\sigma}(\boldsymbol{\mathcal{J}}^{(s)}, \mathbf{e}_z) \Big|_{z=\pm b} = \pm \varepsilon_{(d)} \mathcal{E}_z^{(d)} \Big|_{z=\pm b}, \end{aligned} \quad (2.9)$$

where we used that  $\mathcal{J}_z^{(s)} = 0$  on  $z = \pm b$  with normal  $\mathbf{n} = \pm \mathbf{e}_z$  (the + and - sign are chosen in accordance with the sign of  $z$ ). The normal component of the magnetic induction is continuous across  $z = \pm b$ ,

$$0 = \mu_0 \left( (\boldsymbol{\mathcal{H}}^{(d)}, \mathbf{n}) - (\boldsymbol{\mathcal{H}}^{(s)}, \mathbf{n}) \right) \Big|_{z=\pm b} = \mu_0 \left( \pm \mathcal{H}_z^{(d)} \mp \mathcal{H}_z^{(s)} \right) \Big|_{z=\pm b}. \quad (2.10)$$

The tangential component of the electric field is also continuous across  $z = \pm b$ ,

$$\begin{aligned} \mathbf{0} &= (\boldsymbol{\mathcal{E}}^{(d)} \times \mathbf{n}) - (\boldsymbol{\mathcal{E}}^{(s)} \times \mathbf{n}) \Big|_{z=\pm b} = \\ &= \pm \left( \mathcal{E}_y^{(d)} - \frac{1}{\sigma} \mathcal{J}_y^{(s)} \right) \mathbf{e}_x \mp \left( \mathcal{E}_x^{(d)} - \frac{1}{\sigma} \mathcal{J}_x^{(s)} \right) \mathbf{e}_y \Big|_{z=\pm b}. \end{aligned} \quad (2.11)$$

The tangential component of the magnetic field is discontinuous across an interface, if a surface current exists. But, since the (volume) current  $\mathcal{J}^{(s)}$  is finite at the boundary, no surface current exists. Therefore, the tangential component of the magnetic field is continuous,

$$\mathbf{0} = (\boldsymbol{\mathcal{H}}^{(d)} \times \mathbf{n}) - (\boldsymbol{\mathcal{H}}^{(s)} \times \mathbf{n}) \Big|_{z=\pm b} = \pm \left( \mathcal{H}_y^{(d)} - \mathcal{H}_y^{(s)} \right) \mathbf{e}_x \mp \left( \mathcal{H}_x^{(d)} - \mathcal{H}_x^{(s)} \right) \mathbf{e}_y \Big|_{z=\pm b}. \quad (2.12)$$

## 2.2 A Planar Wave in $y$ -Direction

We assume that the current in the strip is a planar wave in  $y$ -direction,

$$\boldsymbol{\mathcal{J}}^{(s)}(\mathbf{x}, t) = \mathbf{J}^{(s)}(x, z) e^{i(\beta y - \omega t)}. \quad (2.13)$$

where  $\beta$  is the wave number. Then, from Ohm's law and Maxwell's equations for the strip, it follows that the electric and magnetic field and the charge distribution in the strip have the same nature. From the transition conditions, it follows that the electric and magnetic field in the dielectric, and the surface charge have also the same nature. So,

$$\boldsymbol{\mathcal{E}}^{(m)}(\mathbf{x}, t) = \mathbf{E}^{(m)}(x, z) e^{i(\beta y - \omega t)}, \quad \boldsymbol{\mathcal{H}}^{(m)}(\mathbf{x}, t) = \mathbf{H}^{(m)}(x, z) e^{i(\beta y - \omega t)}, \quad (2.14)$$

where  $m = s, d$ , and

$$\varrho_S(x, y, t) = \rho_S(x) e^{i(\beta y - \omega t)}. \quad (2.15)$$

The assumption that the current is a planar wave in  $y$ -direction can be regarded as a boundary condition on the infinitely long strip.

We reformulate the system of equations as given in the previous section to a system in which only the electric field of the dielectric and the current in the strip appear as unknowns. The magnetic field in the dielectric, the magnetic and electric field in the strip, and the surface charge can then be calculated from (2.2)<sup>1</sup>, (2.7)<sup>1,5</sup>, and (2.9), respectively. First, we express the magnetic field of the strip in terms of the current in the strip by using Maxwell's equations (2.7)<sup>1,5</sup> and the planar wave expressions for the fields,

$$\mathbf{H}^{(s)} e^{i\beta y} = \frac{1}{i\sigma\omega\mu_0} \text{rot} \left( \mathbf{J}^{(s)} e^{i\beta y} \right). \quad (2.16)$$

Secondly, we express the magnetic field of the dielectric into the electric field by Maxwell's equation (2.2)<sup>1</sup> and the planar wave expression for the fields,

$$\mathbf{H}^{(d)} e^{i\beta y} = \frac{1}{i\omega\mu_0} \text{rot} \left( \mathbf{E}^{(d)} e^{i\beta y} \right). \quad (2.17)$$

Using (2.16) and (2.17), we rewrite boundary condition (2.10) as

$$\left. \frac{\partial E_y^{(d)}}{\partial x} - i\beta E_x^{(d)} - \frac{1}{\sigma} \left( \frac{\partial J_y^{(s)}}{\partial x} - i\beta J_x^{(s)} \right) \right|_{z=\pm b} = 0. \quad (2.18)$$

This equation is satisfied by (2.11), given by

$$\left. E_y^{(d)} - \frac{1}{\sigma} J_y^{(s)} \right|_{z=\pm b} = 0, \quad \left. E_x^{(d)} - \frac{1}{\sigma} J_x^{(s)} \right|_{z=\pm b} = 0. \quad (2.19)$$

Using (2.16) and (2.17), we rewrite boundary condition (2.12) as

$$\left. -\frac{\partial E_z^{(d)}}{\partial x} + \frac{\partial E_x^{(d)}}{\partial z} + \frac{1}{\sigma} \frac{\partial J_x^{(s)}}{\partial z} \right|_{z=\pm b} = 0, \quad \left. i\beta E_z^{(d)} - \frac{\partial E_y^{(d)}}{\partial z} + \frac{1}{\sigma} \frac{\partial J_y^{(s)}}{\partial z} \right|_{z=\pm b} = 0. \quad (2.20)$$

Then, from (2.5)<sup>1</sup>, (2.2)<sup>1</sup>, (2.8), (2.7)<sup>4,5</sup>, and (2.18)-(2.20) follow the equations for  $\mathbf{E}^{(d)}$  and  $\mathbf{J}^{(s)}$ , and the transition conditions between these quantities,

$$\begin{aligned} \frac{\partial^2 \mathbf{E}^{(d)}}{\partial x^2} - (\beta^2 - k_{(d)}^2) \mathbf{E}^{(d)} + \frac{\partial^2 \mathbf{E}^{(d)}}{\partial z^2} &= \mathbf{0}, \\ \frac{\partial E_x^{(d)}}{\partial x} + i\beta E_y^{(d)} + \frac{\partial E_z^{(d)}}{\partial z} &= 0, \\ \frac{\partial^2 \mathbf{J}^{(s)}}{\partial x^2} - (\beta^2 - k_{(s)}^2 - i\sigma\omega\mu_0) \mathbf{J}^{(s)} + \frac{\partial^2 \mathbf{J}^{(s)}}{\partial z^2} &= \mathbf{0}, \\ \frac{\partial J_x^{(s)}}{\partial x} + i\beta J_y^{(s)} + \frac{\partial J_z^{(s)}}{\partial z} &= 0, \end{aligned} \quad (2.21)$$

where  $k_{(d)} = \omega/c_{(d)}$  and  $k_{(s)} = \omega/c_{(s)}$  are the wave numbers in dielectric and strip, and

$$\begin{aligned} \left. E_y^{(d)} - \frac{1}{\sigma} J_y^{(s)} \right|_{z=\pm b} &= 0, & \left. E_x^{(d)} - \frac{1}{\sigma} J_x^{(s)} \right|_{z=\pm b} &= 0, \\ \left. -\frac{\partial E_z^{(d)}}{\partial x} + \frac{\partial E_x^{(d)}}{\partial z} + \frac{1}{\sigma} \frac{\partial J_x^{(s)}}{\partial z} \right|_{z=\pm b} &= 0, & \left. i\beta E_z^{(d)} - \frac{\partial E_y^{(d)}}{\partial z} + \frac{1}{\sigma} \frac{\partial J_y^{(s)}}{\partial z} \right|_{z=\pm b} &= 0. \end{aligned} \quad (2.22)$$

The surface charge density follows from (2.9), i.e.

$$\rho_S = \pm \varepsilon_{(d)} E_z^{(d)} \Big|_{z=\pm b}. \quad (2.23)$$

### 2.3 The Case $b \ll a$

In this section, we consider the case  $b \ll a$ . Introduce the dimensionless coordinates  $\hat{x} = x/a$  and  $\hat{z} = z/b$ , and the parameters

$$\epsilon = \frac{b}{a}, \quad \xi^2 = b^2(\beta^2 - k_{(d)}^2), \quad \frac{1}{\delta^2} = b^2(\beta^2 - k_{(s)}^2 - i\sigma\omega\mu_0) = A^2 \left( \frac{v^2}{A^2} - i \right), \quad (2.24)$$

where the dimensionless numbers  $A$  and  $v^2$  are defined by

$$A = b\sqrt{\sigma\omega\mu_0}, \quad v^2 = b^2(\beta^2 - k_{(s)}^2). \quad (2.25)$$

Then, the equations (2.21) turn into (we omit the superscripts)

$$\begin{aligned} \epsilon^2 \frac{\partial^2 \mathbf{E}}{\partial \hat{x}^2} - \xi^2 \mathbf{E} + \frac{\partial^2 \mathbf{E}}{\partial \hat{z}^2} &= \mathbf{0}, & \frac{\partial E_x}{\partial \hat{x}} + i\beta b E_y + \frac{\partial E_z}{\partial \hat{z}} &= 0, \\ \epsilon^2 \frac{\partial^2 \mathbf{J}}{\partial \hat{x}^2} - \frac{1}{\delta^2} \mathbf{J} + \frac{\partial^2 \mathbf{J}}{\partial \hat{z}^2} &= \mathbf{0}, & \frac{\partial J_x}{\partial \hat{x}} + i\beta b J_y + \frac{\partial J_z}{\partial \hat{z}} &= 0, \end{aligned} \quad (2.26)$$

and the transition conditions (2.22) and (2.23) into

$$\begin{aligned} E_y - \frac{1}{\sigma} J_y \Big|_{\hat{z}=\pm 1} &= 0, & E_x - \frac{1}{\sigma} J_x \Big|_{\hat{z}=\pm 1} &= 0, \\ -\epsilon \frac{\partial E_z}{\partial \hat{x}} + \frac{\partial E_x}{\partial \hat{z}} + \frac{1}{\sigma} \frac{\partial J_x}{\partial \hat{z}} \Big|_{\hat{z}=\pm 1} &= 0, & i\beta b E_z - \frac{\partial E_y}{\partial \hat{z}} + \frac{1}{\sigma} \frac{\partial J_y}{\partial \hat{z}} \Big|_{\hat{z}=\pm 1} &= 0, \end{aligned} \quad (2.27)$$

$$\rho_S = \pm \varepsilon_{(d)} E_z \Big|_{\hat{z}=\pm 1}.$$

Since the problem of the strip in the infinitely large dielectric serves as model problem for the stripline in the stripline environment, we are interested only in the fields in and in the neighbourhood of the strip. So, we are interested in  $\mathbf{E}$  in the neighbourhood of the strip ( $|\hat{x}|, |\hat{z}| > 1$ , but  $\hat{x}, \hat{z} = O(1)$ ) and  $\mathbf{J}$  in the strip ( $-1 < \hat{x}, \hat{z} < 1$ ). For  $-1 < \hat{x} < 1$  with  $1 - |\hat{x}| \neq O(\epsilon)$ , and for  $\hat{z} = O(1)$ , the field quantities are only weakly dependent on  $\hat{x}$ , because  $\epsilon \ll 1$  (for the numerical values given in Chapter 1,  $\epsilon = O(10^{-2})$ ). Then, we can describe the fields  $\mathbf{J}$  and  $\mathbf{E}$  by the boundary conditions at  $\hat{z} = \pm 1$  and the differential equations with  $\epsilon = 0$ , whereby the fields are independent of  $\hat{x}$ . It follows that  $J_x = 0$  for  $-1 \leq \hat{z} \leq 1$ , because  $J_x = 0$  at  $\hat{x} = 0$  due to symmetry. Then, according to (2.27)<sup>2</sup>,  $E_x = 0$  at  $\hat{z} = \pm 1$ . Because the differential equation (2.26)<sup>1</sup> for  $E_x$  is homogeneous, we obtain  $E_x = 0$  for  $|\hat{z}| \geq 1$ , if we require  $E_x \rightarrow 0$  as  $|\hat{z}| \rightarrow \infty$ . For the same kind of reasons, i.e.  $J_z = 0$  at  $\hat{z} = \pm 1$  and the differential equation (2.26)<sup>3</sup> for  $J_z$  is homogeneous, we obtain also  $J_z = 0$  for  $-1 \leq \hat{z} \leq 1$ .

Since  $J_x = J_z = 0$ , the local continuity equation, i.e. (2.26)<sup>4</sup>, yields  $\beta b J_y = 0$ , which can only be satisfied by  $J_y = 0$ . However, this would yield the trivial solution  $\mathbf{J} = \mathbf{0}$ . So, the local continuity equation cannot be satisfied. This is due to the infinitely propagating planar wave we have assumed. Neglecting boundary effects at  $\hat{x} = \pm 1$ , the integral form of the continuity

equation is satisfied on a cylindrical volume with ends  $y = y_1$  and  $y = y_1 + 2\pi/\beta$ , which encloses the strip. This is due to the periodicity of charge and current in  $y$ -direction. Based on the aforementioned observations, we assume that  $J_y \neq 0$  and we do not take the local form of the continuity equation into account. We will show that  $E_z(\pm 1) \neq 0$ , which implies that there exists a surface charge with density  $\rho_S$ . This shows also that the continuity equation, i.e. the conservation of charge, cannot be satisfied in local sense.

By the above, the differential equations (2.26) and the transition conditions (2.27) for the fields  $\mathbf{E}$  and  $\mathbf{J}$  turn into

$$\begin{aligned} \frac{\partial^2 E_y}{\partial \hat{z}^2} - \xi^2 E_y &= 0, & \frac{\partial^2 E_z}{\partial \hat{z}^2} - \xi^2 E_z &= 0, \\ i\beta b E_y + \frac{\partial E_z}{\partial \hat{z}} &= 0, & \frac{\partial^2 J_y}{\partial \hat{z}^2} - \frac{1}{\delta^2} J_y &= 0, \end{aligned} \quad (2.28)$$

and

$$E_y - \frac{1}{\sigma} J_y \Big|_{\hat{z}=\pm 1} = 0, \quad i\beta b E_z - \frac{\partial E_y}{\partial \hat{z}} + \frac{1}{\sigma} \frac{\partial J_y}{\partial \hat{z}} \Big|_{\hat{z}=\pm 1} = 0, \quad \rho_S = \pm \varepsilon_{(d)} E_z \Big|_{\hat{z}=\pm 1} \quad (2.29)$$

The other field components equal zero. Because of symmetry,  $J_y$  is even in  $\hat{z}$ . Then, it follows from (2.29)<sup>1</sup> that  $E_y(-1) = E_y(1)$ . This implies that  $E_y$  is even in  $\hat{z}$ , because  $E_y$  is described by the same homogeneous differential equation above and below the strip and  $E_y \rightarrow 0$  for  $|\hat{z}| \rightarrow \infty$ . Then, it follows from (2.28)<sup>3</sup> that  $E_z$  is odd.

We consider the case  $\xi^2 = b^2(\beta^2 - k_{(d)}^2) \notin \{z \in \mathcal{C} \mid \operatorname{Re} z \leq 0, \operatorname{Im} z = 0\}$  only, and we choose  $\operatorname{Re} \xi > 0$ . Note that  $b$  and  $k_{(d)}$  are real and positive. Furthermore, if  $\beta$  has a nonzero imaginary part, the propagating wave is attenuated in either the positive or the negative  $y$ -direction, and its amplitude tends to infinity in the opposite direction. First, we calculate  $J_y$ . Since  $J_y(\hat{z})$  is even in  $\hat{z}$ ,

$$J_y(\hat{z}) = \frac{J_y(1) \cosh(\hat{z}/\delta)}{\cosh(1/\delta)}, \quad |\hat{z}| \leq 1. \quad (2.30)$$

For the numerical values given in Chapter 1,  $A \gg 1$  and  $|\delta| \ll 1$ . For instance, for  $f = 1\text{GHz}$ ,  $\sigma = 5.8824 \cdot 10^7 \text{ mho/m}$ ,  $b = 40 \mu\text{m}$ ,  $\mu_0 = 4\pi \cdot 10^{-7} \text{ Vs/Am}$ , the constant  $A$  equals approximately 27. Furthermore, assuming that the difference between  $\beta^2$  and  $k_{(s)}^2$  is of the order of  $k^2$  in vacuum ( $\approx 438.6 \text{m}^{-2}$ ), we obtain  $v^2 = O(10^{-6})$ . Then,  $v^2/A^2 \ll 1$ , and therefore, we approximate  $1/\delta$  by

$$\frac{1}{\delta} \approx A e^{-\pi i/4}. \quad (2.31)$$

By assuming  $b\beta \ll 1$  and  $bk_{(s)} \ll 1$ , we find the same approximation.

The electric field components  $E_y$  and  $E_z$  are obtained in the same way as the  $y$ -component  $J_y$  of the current. Because of symmetry, we consider the case  $\hat{z} \geq 1$  only. Using the boundary conditions (2.29)<sup>1,3</sup> and requiring that  $E_y(\hat{z}), E_z(\hat{z}) \rightarrow 0$  as  $\hat{z} \rightarrow \infty$ , we obtain

$$E_y(\hat{z}) = \frac{1}{\sigma} J_y(1) e^{-\xi(\hat{z}-1)}, \quad E_z(\hat{z}) = \frac{\rho_S}{\varepsilon_{(d)}} e^{-\xi(\hat{z}-1)}, \quad \hat{z} \geq 1. \quad (2.32)$$

Because  $E_y$  is even in  $\hat{z}$  and  $E_z$  is odd in  $\hat{z}$ , we arrive at

$$E_y(\hat{z}) = \frac{1}{\sigma} J_y(1) e^{-\xi(|\hat{z}|-1)}, \quad E_z(\hat{z}) = \operatorname{sign}(\hat{z}) \frac{\rho_S}{\varepsilon_{(d)}} e^{-\xi(|\hat{z}|-1)} \quad |\hat{z}| \geq 1. \quad (2.33)$$

for  $|\hat{z}| > 1$ .

The only unknowns left are  $J_y(1)$  and  $\rho_S$ . By (2.29)<sup>2</sup>, we obtain

$$\frac{i\beta b \rho_S}{\varepsilon_{(d)}} + \frac{\xi}{\sigma} J_y(1) + \frac{J_y(1)}{\sigma \delta} \tanh(1/\delta) = 0. \quad (2.34)$$

Multiplying this equation by  $1/ib\omega\mu_0$  and using the approximation (2.31), we obtain

$$\frac{\beta \rho_S c_{(d)}^2}{\omega} - \delta b J_y(1) (\delta \xi + \tanh(1/\delta)) = 0, \quad (2.35)$$

or,

$$J_y(1) = \frac{\beta \rho_S c_{(d)}^2}{\delta b \omega} \frac{\coth(1/\delta)}{1 + \delta \xi \coth(1/\delta)}. \quad (2.36)$$

Now, the only unknown left is  $\rho_S$ . The only equation that needs to be satisfied is (2.28)<sup>3</sup>, but  $\rho_S$  cannot be determined from this equation. It puts restrictions on  $\beta$  only as we show later on. Therefore, we need another equation to determine  $\rho_S$ . Let  $\mathcal{I}(y, t)$  be the total current passing at time  $t$  through the cross section  $\Sigma(y)$  of the strip, i.e.

$$\Sigma(y) = \{(x, y, z) \in \mathbb{R}^3 \mid -a \leq x \leq a, -b \leq z \leq b\}. \quad (2.37)$$

Neglecting boundary effects near  $\hat{x} = \pm 1$  and using the obtained solution for  $J_y$  on the square  $-1 < \hat{x}, \hat{z} < 1$ , we deduce

$$\mathcal{I}(y, t) = \iint_{\Sigma(y)} \mathcal{J}_y dS = 2ab \int_{-1}^1 J_y(\hat{z}) d\hat{z} e^{i(\beta y - \omega t)} = I e^{i(\beta y - \omega t)}, \quad (2.38)$$

with

$$I = \frac{4a\beta \rho_S c_{(d)}^2}{\omega} \frac{1}{1 + \delta \xi \coth(1/\delta)}. \quad (2.39)$$

So, the surface charge density  $\rho_S$  can be expressed in terms of  $I$  by

$$\rho_S = \frac{\omega I}{4a\beta c_{(d)}^2} (1 + \delta \xi \coth(1/\delta)). \quad (2.40)$$

The total current  $I$  does not depend on  $\delta$ . In fact, (2.39) is the law of Ampère. To show this, we start from the law of Ampère-Maxwell, which for a linear medium is given by

$$\int_{\partial\tilde{\Sigma}} (\mathcal{H} \cdot \mathbf{t}) ds = \int_{\tilde{\Sigma}} \left( \mathcal{J} + \varepsilon \frac{\partial \mathcal{E}}{\partial t} \right) \cdot \mathbf{n} d\tilde{\Sigma}, \quad (2.41)$$

where  $\Sigma$  is a surface with boundary  $\partial\Sigma$ ,  $\mathbf{n}$  is the normal on  $\Sigma$  and  $\mathbf{t}$  is the tangential vector to  $\partial\Sigma$ . Let  $\Sigma = \Sigma(y)$  and  $\partial\Sigma = \partial\Sigma(y)$  its boundary. Then, all the fields in (2.41) are fields in the strip. Since the tangential component of the magnetic field is continuous at the boundary,  $\mathcal{H}^{(s)}$  can be replaced by  $\mathcal{H}^{(d)}$ . Using Ohm's law and the planar wave form of the fields, we write (2.41) as

$$\int_{\partial\Sigma(y)} (\mathbf{H}^{(d)} \cdot \mathbf{t}) ds = \int_{\Sigma(y)} \left( 1 + \frac{\varepsilon(s)}{\sigma} \right) \mathbf{J}^{(s)} \cdot \mathbf{n} d\Sigma(y), \quad (2.42)$$

Because  $\varepsilon/\sigma \ll 1$ , (2.42) can be approximated by Ampère's law,

$$\int_{\partial\Sigma(y)} (\mathbf{H} \cdot \mathbf{t}) ds = \int_{\Sigma(y)} (\mathbf{J} \cdot \mathbf{n}) d\Sigma(y), \quad (2.43)$$

where  $\mathbf{H} = \mathbf{H}^{(d)}$  and  $\mathbf{J} = \mathbf{J}^{(s)}$ . The right-hand side of this equation equals  $I$  as defined in (2.39). To calculate the left-hand side, we calculate first the field  $\mathbf{H}$  from Maxwell's equation (2.2)<sup>1</sup>. Because the  $x$ -component of the electric field equals zero and the electric field does not depend on  $x$ ,  $H_y = H_z = 0$ . Using the planar wave form of the fields, we write the  $x$ -component of (2.2)<sup>1</sup> as

$$H_x = \frac{1}{ib\omega\mu_0} \left( i\beta b E_z - \frac{\partial E_y}{\partial \hat{z}} \right). \quad (2.44)$$

Since  $E_z$  is odd and  $E_y$  is even,  $H_x$  is odd. It follows from (2.33), (2.36), and (2.31) that

$$H_x(\hat{z}) = \text{sign}(\hat{z}) \frac{\beta \rho_S c_{(d)}^2}{\omega} \frac{1}{1 + \delta \xi \coth(1/\delta)} e^{-\xi(|\hat{z}|-1)}, \quad |\hat{z}| \geq 1. \quad (2.45)$$

Neglecting boundary effects near  $\hat{x} = \pm 1$ , we deduce by Ampere's law

$$\begin{aligned} I &= \int_{\partial\Sigma(y)} (\mathbf{H}, \mathbf{t}) ds = a \int_{-1}^1 H_x(\hat{z} = 1) d\hat{x} - a \int_{-1}^1 H_x(\hat{z} = -1) d\hat{x} = \\ &= 2a \int_{-1}^1 H_x(\hat{z} = 1) d\hat{x} = \frac{4a\beta \rho_S c_{(d)}^2}{\omega} \frac{1}{1 + \delta \xi \coth(1/\delta)}. \end{aligned} \quad (2.46)$$

Note that the tangential component of  $\mathbf{H}$  equals zero on the boundaries  $\hat{x} = \pm 1$ , because  $H_y = H_z = 0$ . The result (2.46) is the same as (2.39).

For the analysis of the stripline in the stripline environment, we introduce the quantity  $I^{(b)} = I/2a$ , which expresses the total free current through a line segment  $-b \leq z \leq b$  of the cross section  $\Sigma(y)$  of the strip. Then, it follows from (2.36) and (2.40) that

$$\rho_S = \frac{\omega I^{(b)}}{2\beta c_{(d)}^2} (1 + \delta \xi \coth(1/\delta)), \quad J_y(1) = \frac{I^{(b)}}{2\delta b} \coth(1/\delta). \quad (2.47)$$

Substituting these expressions into the expressions for  $J_y$ ,  $E_y$ ,  $E_z$ , and  $H_x$ , given by (2.33), (2.30), and (2.45), respectively, we arrive at

$$\begin{aligned} J_y(\hat{z}) &= \frac{I^{(b)} \cosh(\hat{z}/\delta)}{2b\delta \sinh(1/\delta)}, \\ E_y(\hat{z}) &= -\frac{i\delta b\omega\mu_0 I^{(b)}}{2} \coth(1/\delta) e^{-\xi(|\hat{z}|-1)}, \\ E_z(\hat{z}) &= \text{sign}(\hat{z}) \frac{\omega\mu_0 I^{(b)}}{2\beta} (1 + \delta \xi \coth(1/\delta)) e^{-\xi(|\hat{z}|-1)}, \\ H_x(\hat{z}) &= \text{sign}(\hat{z}) \frac{I^{(b)}}{2} e^{-\xi(|\hat{z}|-1)}. \end{aligned} \quad (2.48)$$

The only equation that still needs to be satisfied, is (2.28)<sup>3</sup>. Substituting  $E_y$  and  $E_z$  into this equation yields

$$\delta\beta^2 b^2 \coth(1/\delta) - \xi(1 + \delta\xi \coth(1/\delta)) = 0, \quad (2.49)$$

or, using (2.24)<sup>2</sup>,

$$\beta^2 = k_{(d)}^2(1 + \delta^2 b^2 k_{(d)}^2 \coth^2(1/\delta)) \approx k_{(d)}^2, \quad (2.50)$$

because we have assumed  $b^2 k_{(d)}^2 \ll 1$  and  $|\delta| \ll 1$ . For the given numerical values, we obtain  $\text{Re } \delta^2 b^2 k_{(d)}^2 \coth^2(1/\delta) = O(10^{-8})$  and  $\text{Im } \delta^2 b^2 k_{(d)}^2 \coth^2(1/\delta) = O(10^{-5})$  with positive imaginary part. Hence, the wave is attenuated in the positive  $y$ -direction at very large distances from a fixed origin and its amplitude tends “slowly” to infinity in the opposite direction. Approximating  $\beta$  by  $k_{(d)}$ , we obtain a purely propagating wave in  $y$ -direction.

## 2.4 The Case $b\sqrt{\sigma\omega\mu_0} \gg 1$ : Perfect Conduction

As aforementioned, the dimensionless constant  $A = b\sqrt{\sigma\omega\mu_0} \gg 1$ , and therefore,  $|\delta| \ll 1$ . We have given a numerical example for which  $A \approx 27$  and  $|\delta| \approx 0.037$ . Furthermore,  $|\xi| \ll 1$ , because we have assumed that  $b\beta \ll 1$  and  $b k_{(d)} \ll 1$  (see below (2.31)). Then, the ratio of  $E_y$  and  $E_z$  can be estimated by

$$\left| \frac{E_y(\hat{z})}{E_z(\hat{z})} \right| \approx \delta\beta b \ll 1, \quad |\hat{z}| > 1. \quad (2.51)$$

This shows that  $E_y$  can be neglected with respect to  $E_z$ . Furthermore,  $E_z$  can be approximated by

$$E_z(\hat{z}) = \text{sign}(\hat{z}) \frac{\omega\mu_0 I^{(b)}}{2\beta} e^{-\xi(|\hat{z}|-1)}, \quad |\hat{z}| \geq 1. \quad (2.52)$$

So, the electric field is approximated by its first order term for  $|\delta| \downarrow 0$ . We see that  $H_x$  does not depend on  $\delta$ , and is therefore also approximated by its first order term for  $|\delta| \downarrow 0$ . From (2.47)<sup>1</sup> and (2.50), it follows that

$$\rho_S = \frac{\omega I^{(b)}}{2\beta c_{(d)}^2}, \quad \beta^2 = k_{(d)}^2, \quad (2.53)$$

for  $|\delta| \downarrow 0$ .

Now the only quantity to approximate left is the current given by (2.48)<sup>3</sup>. The shape of the current is determined by the factor  $\cosh(\hat{z}/\delta)/\sinh(1/\delta)$ , while its magnitude is determined by the factor  $I^{(b)}/2b\delta$ . Since  $|\delta| \ll 1$ , the absolute value of the factor  $\cosh(\hat{z}/\delta)/\sinh(1/\delta)$  is about 1 for  $\hat{z} = \pm 1$ , while it is much lower than 1 on the larger part of the interval  $-1 < \hat{z} < 1$ . So, the current is mainly located on the boundaries of the strip. We deduce a measure for the thickness of the layers in which the current is concentrated. Because  $|\delta| \ll 1$ , the dominating term in  $\cosh(\hat{z}/\delta)/\sinh(1/\delta)$  is

$$e^{(|\hat{z}|-1)/\delta}. \quad (2.54)$$



Using (2.31), we approximate it by

$$e^{(|\hat{z}|-1)A(1+i)/\sqrt{2}}, \quad (2.55)$$

with amplitude

$$e^{(|\hat{z}|-1)A/\sqrt{2}}. \quad (2.56)$$

The amplitude determines the shape of the current and therewith, the layer thickness. We define the relative layer thickness (in comparison to the thickness of the strip) as that value of  $1 - |\hat{z}|$  for which the power of the amplitude equals  $-1$ . The actual layer thickness is then given by  $b(1 - |\hat{z}|)$ . Since the power equals  $-1$  if  $|\hat{z}| = 1 - \sqrt{2}/A$ , the relative layer thickness is  $\sqrt{2}/A = \sqrt{2}/(b\sqrt{\sigma\omega\mu_0})$  and the actual layer thickness is

$$\delta_{skin} = \sqrt{\frac{2}{\sigma\omega\mu_0}}, \quad (2.57)$$

which is called the skin depth. Since  $A \gg 1$ , the relative layer thickness is much smaller than 1, which implies that the skin depth is much smaller than the thickness of the strip. For the copper strip as introduced in Chapter 1 with frequency 1GHz, the layer thickness or skin depth is about  $2.1\mu\text{m}$ , which is about 5% of the thickness of the strip.

Taking also for  $J_y$  the limit  $|\delta| \downarrow 0$ , we need to calculate

$$\lim_{|\delta| \downarrow 0} J_y(\hat{z}; \delta) \quad (2.58)$$

in distributional sense, where we add the parameter  $\delta$  to show the dependence of  $J_y$  on  $\delta$  explicitly. It can be shown that

$$\lim_{|\delta| \downarrow 0} J_y(\hat{z}; \delta) = \Lambda (\delta_f(\hat{z} - 1) + \delta_f(\hat{z} + 1)), \quad (2.59)$$

where  $\delta_f$  is the delta function and  $\Lambda$  is a constant. This means that

$$\lim_{|\delta| \downarrow 0} \int_{-\infty}^{\infty} J_y(\hat{z}; \delta) \varphi(\hat{z}) d\hat{z} = \Lambda (\varphi(1) + \varphi(-1)) \quad (2.60)$$

for every function  $\varphi \in C_c^\infty(\mathbb{R})$ . Here,  $\varphi \in C_c^\infty(\mathbb{R})$  is the space of all infinitely many times continuously differentiable functions on  $\mathbb{R}$  with compact support. To calculate  $\Lambda$ , we take such a function  $\varphi$ , which has the property  $\varphi(\hat{z}) = 1$  for  $-1 \leq \hat{z} \leq 1$ . Note that  $J_z(\hat{z}) = 0$  for  $|\hat{z}| > 1$ . Using (2.48)<sup>3</sup>, we deduce

$$\Lambda = \frac{1}{2} \lim_{|\delta| \downarrow 0} \int_{-\infty}^{\infty} J_y(\hat{z}; \delta) \varphi(\hat{z}) d\hat{z} = \frac{1}{2} \lim_{|\delta| \downarrow 0} \int_{-1}^1 J_y(\hat{z}; \delta) d\hat{z} = \frac{I^{(b)}}{2b}. \quad (2.61)$$

Summarizing, we have deduced the following approximation of the field distribution in the neighbourhood of and in the strip:

$$\begin{aligned} E_z(\hat{z}) &= \text{sign}(\hat{z}) \frac{\omega\mu_0 I^{(b)}}{2\beta} e^{-\xi(|\hat{z}|-1)}, & E_y(\hat{z}) &= 0, \\ J_y(\hat{z}) &= \frac{I^{(b)}}{2b} (\delta_f(\hat{z} - 1) + \delta_f(\hat{z} + 1)), & H_x(\hat{z}) &= \text{sign}(\hat{z}) \frac{I^{(b)}}{2} e^{-\xi(|\hat{z}|-1)}, \\ \rho_S &= \frac{\omega I^{(b)}}{2\beta c_{(d)}^2}, & \beta^2 &= k_{(d)}^2, \end{aligned} \quad (2.62)$$

The other field components are zero as we have shown above. Note that  $\mathbf{E}$  and  $\mathbf{H}$  are the electric and magnetic field in the dielectric and  $\mathbf{J}$  is the current in the strip.

The approximation of the fields by taking  $|\delta| \downarrow 0$ , or  $A = b\sqrt{\sigma\omega\mu_0} \rightarrow \infty$ , can be interpreted as assuming that the strip is a perfect conductor, i.e.  $\sigma \rightarrow \infty$ . In correspondence with results in literature, the tangential electric field vanishes at the surface of this perfectly conducting strip. Furthermore, the current is located only at the (upper and lower) surface of the strip, and consequently, the electric field inside the strip is zero. Neglecting boundary effects at  $x = \pm a$ , the strip can be assumed infinitely thin. Then, the current is a surface current in an infinitely thin sheet. The boundary conditions for the electric and magnetic field at this sheet follow from (2.62) by putting  $\hat{z} = \pm 1$ . The constant  $I^{(b)}$  is the total free current through a line segment  $-b \leq z \leq b$  of the strip. For the infinitely thin strip,  $I^{(b)}$  should be interpreted as the surface current  $J$  on the strip, which depends on  $x$ . Then, the electric and magnetic field and the charge density  $\rho_S$  at the surface of the perfectly conducting, infinitely thin strip are given by

$$\begin{aligned} E_z &= \pm \frac{\omega\mu_0 J(x)}{2\beta}, & H_x &= \pm \frac{J(x)}{2}, \\ E_x = E_y &= 0, & H_y = H_z &= 0, \\ \rho_S(x) &= \frac{\omega J(x)}{2\beta c_{(d)}^2}, & \beta^2 &= k_{(d)}^2. \end{aligned} \tag{2.63}$$

whereby  $c_{(d)} = 1/\sqrt{\mu_0\epsilon_{(d)}}$  is the velocity of light in the dielectric. Note that the signs in  $E_z$  and  $H_x$  should be chosen in correspondence with the sign of the normal  $\mathbf{n} = \pm\mathbf{e}_z$  on the strip.

## 2.5 Recapitulation

In the previous sections, we have considered an infinitely long, thin strip in an infinitely large dielectric (see Figure 1.1 in Chapter 1). Both strip and dielectric are homogeneous, isotropic, and linear. The width  $2a$  of the strip is much larger than the thickness  $2b$  of the strip, and both sizes are much smaller than the wavelength. Furthermore,  $A = b\sqrt{\sigma\omega\mu_0} \gg 1$ , where  $\sigma$  is the conductivity and  $\omega$  is the radian frequency. The current in the strip is a planar wave in  $y$ -direction with wave number  $\beta$ . It is assumed that the thickness  $b$  is much smaller than the wavelength of this wave, i.e.  $b\beta \ll 1$ , and much smaller than the wavelengths in the dielectric and the strip, i.e.  $bk_{(s)} \ll 1$  and  $bk_{(d)} \ll 1$ , whereby  $k_{(s)}$  and  $k_{(d)}$  are the wave numbers in strip and dielectric.

It is shown that the fields in the strip and the dielectric can be described by a set of differential equations and transition conditions for the electric field in the dielectric and the current in the strip. Because  $b \ll a$ , the current in the strip depends only weakly on the  $x$ -coordinate (width coordinate), except near the sides  $x = \pm a$ . The electric field near the strip (i.e. at distances of the order  $b$ ) depend also weakly on  $x$ . Neglecting the  $x$ -dependency of the fields, the current in the strip and the electric field near the strip are calculated. It is shown that the  $x$ -component of the electric field and the  $x$  and  $z$ -component of the current vanish. The other field components are calculated, whereby the conditions  $b\beta \ll 1$  and  $bk_{(s)} \ll 1$  are used. Furthermore, it is shown that the local continuity equation, i.e. conservation of charge, cannot be satisfied due to the assumed infinitely propagating planar wave. A surface charge is generated at the surfaces  $z = \pm b$  of the strip. After calculation of the electric field

in the dielectric, the magnetic field in the dielectric is calculated from Maxwell's equations. It is shown that the  $y$  and  $z$ -component vanish. A relation between the surface charge and the total current is deduced, which comes down to Ampère's law. By this relation, all field components can be expressed into the total current. Finally, it is shown that the wave number  $\beta^2$  equals approximately  $k_{(d)}^2$ .

Then, the case  $A \gg 1$  is considered. It is shown that the electric and magnetic field in the dielectric may be approximated by their first order terms for  $A \rightarrow \infty$ , and that the first order term of  $\beta^2$  equals  $k_{(d)}^2$ . Furthermore, it is shown that the current in the strip is mainly located on its (upper and lower) boundaries. A measure for the thickness of the layers in which the current is concentrated, is deduced. This measure is the so-called skin depth, defined by

$$\delta_{skin} = \frac{\sqrt{2}b}{A} = \sqrt{\frac{2}{\sigma\omega\mu_0}}, \quad (2.64)$$

Since  $A \gg 1$ , the layer thickness is much smaller than the thickness ( $2b$ ) of the strip. Approximation of the current by taking the limit  $A \rightarrow \infty$  yields two delta peaks at the upper and lower boundary of the strip.

The approximation of the fields by taking  $A = b\sqrt{\sigma\omega\mu_0} \rightarrow \infty$  can be interpreted as assuming that the strip is a perfect conductor, i.e.  $\sigma \rightarrow \infty$ . In correspondence with results in literature, the tangential electric field vanishes at the surface of this perfectly conducting strip. Neglecting boundary effects at  $x = \pm a$ , the strip can be assumed infinitely thin. Then, the current is a surface current in an infinitely thin sheet. The boundary conditions for the electric and magnetic field at this sheet follow from the analysis of the finitely thin strip, and are given by

$$\begin{aligned} E_z &= \pm \frac{\omega\mu_0 J(x)}{2\beta}, & H_x &= \pm \frac{J(x)}{2}, \\ E_x = E_y &= 0, & H_y = H_z &= 0, \\ \rho_S(x) &= \frac{\omega J(x)}{2\beta c_{(d)}^2}, & \beta^2 &= k_{(d)}^2. \end{aligned} \quad (2.65)$$

whereby  $J$  is the surface current in the strip and  $c_{(d)} = 1/\sqrt{\mu_0\varepsilon_{(d)}}$  is the velocity of light in the dielectric. Note that the sign in  $E_z$  and  $H_x$  should be chosen in correspondence with the sign of the normal  $\mathbf{n} = \pm\mathbf{e}_z$  on the strip.

In the analysis, it is shown that the copper strip as introduced in Chapter 1 can be modeled as infinitely thin and perfectly conducting. For the numerical values as given in Chapter 1, we have:

- The ratio of the width of the strip to the wavelength (in vacuum) is  $\lesssim 0.067$ . The ratio of the thickness of the strip to the wavelength (in vacuum) is of the order  $10^{-3}$ .
- The ratio of thickness to width of the strip is of the order  $10^{-3}$ .
- The skin depth is about 5% of the thickness of the strip and  $A = b\sqrt{\sigma\omega\mu_0} \approx 27.26$ .

For these values, the strip can be modeled as infinitely thin and perfectly conducting, whereby the boundary conditions at the strip are given by (2.65).

## CHAPTER 3

# An Infinitely Thin, Perfectly Conducting Stripline

In this chapter, we consider the stripline in the stripline environment introduced in Chapter 1. The aim of this chapter is to calculate the electromagnetic field in the dielectric, the current in the stripline, and the (characteristic) impedance of the stripline. Using the results of the previous chapter, in particular (2.65), we model the stripline as an infinitely thin and perfectly conducting strip.

In Section 3.1, we present the balance equations and boundary conditions for the electromagnetic field in the dielectric. In Section 3.2, we deduce a boundary value problem for the potential of the electric field assuming a transverse electromagnetic (TEM) wave in the stripline. From this boundary value problem, we calculate the potential of the electric field and the current in the stripline in Section 3.3. Two different approaches are presented. In Section 3.4, we present asymptotic approximations for the potential of the electric field, the current, and the impedance. Finally, we present in Section 3.5 an algorithm to calculate potential, current, and impedance numerically.

## 3.1 Balance Equations and Boundary Conditions

The balance equations for the (homogeneous, isotropic, and linear) dielectric are given by (2.2), i.e.

$$\begin{aligned} \mu_0 \frac{\partial \mathcal{H}}{\partial t} &= -\text{rot } \mathcal{E}, \quad \text{div } \mathcal{H} = 0, \\ \varepsilon \frac{\partial \mathcal{E}}{\partial t} &= \text{rot } \mathcal{H}, \quad \text{div } \mathcal{E} = 0, \end{aligned} \tag{3.1}$$

where we omitted the superscripts <sup>(d)</sup> and subscripts <sub>(d)</sub>. Because of symmetry with respect to the plane  $z = 0$ , we take  $\mathbf{x} \in G^+ = \{\mathbf{x} \in \mathbb{R}^3 \mid x, y \in \mathbb{R}, 0 < z < h\}$ . As in the previous chapter, we assume that the (surface) current on the stripline is a planar wave in  $y$ -direction,

$$\mathcal{J}(\mathbf{x}, t) = \mathbf{J}(x) e^{i(\beta y - \omega t)} = J_y(x) e^{i(\beta y - \omega t)} \mathbf{e}_y \tag{3.2}$$

for  $\mathbf{x} \in S = \{\mathbf{x} \in \mathbb{R}^3 \mid -a < x < a, y \in \mathbb{R}, z = 0\}$ . This assumption can be regarded as a boundary condition on the infinitely long stripline. Although we have shown in the previous chapter that  $\beta = k$  (see (2.65)<sup>6</sup>), where  $k = \omega/c$  and  $c = 1/\sqrt{\varepsilon\mu_0}$  are the wave number and

the velocity of light in the dielectric, we put  $\beta$  as the (unknown) wave number. We will show that the stripline can support a transverse electromagnetic wave only if  $\beta = k$ .

The electric and magnetic field in the dielectric are also planar waves in  $y$ -direction,

$$\mathcal{E}(\mathbf{x}, t) = \mathbf{E}(x, z) e^{i(\beta y - \omega t)}, \quad \mathcal{H}(\mathbf{x}, t) = \mathbf{H}(x, z) e^{i(\beta y - \omega t)}, \quad (3.3)$$

as is the (surface) charge density (per unit length) on the stripline,

$$\varrho_S(x, y, t) = \rho_S(x) e^{i(\beta y - \omega t)}. \quad (3.4)$$

In this way, each quantity is written as the multiplication of its wave behaviour, which depends only on time and length direction, and its ‘stationary’ part (with respect to the propagation of the wave), which depends only on the coordinates in the intersection of the stripline environment. Then, (3.1)<sup>1</sup>, (3.1)<sup>3</sup>, and (3.1)<sup>2,4</sup> turn into

$$\begin{aligned} -i\omega\mu_0 H_x &= -i\beta E_z + \frac{\partial E_y}{\partial z}, \\ -i\omega\mu_0 H_y &= \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}, \\ -i\omega\mu_0 H_z &= -\frac{\partial E_y}{\partial x} + i\beta E_x, \end{aligned} \quad (3.5)$$

$$\begin{aligned} -i\omega\varepsilon E_x &= i\beta H_z - \frac{\partial H_y}{\partial z}, \\ -i\omega\varepsilon E_y &= -\frac{\partial H_z}{\partial x} + \frac{\partial H_x}{\partial z}, \\ -i\omega\varepsilon E_z &= \frac{\partial H_y}{\partial x} - i\beta H_x, \end{aligned} \quad (3.6)$$

$$\frac{\partial E_x}{\partial x} + i\beta E_y + \frac{\partial E_z}{\partial z} = 0, \quad \frac{\partial H_x}{\partial x} + i\beta H_y + \frac{\partial H_z}{\partial z} = 0, \quad (3.7)$$

respectively. Note that (3.7) is trivially satisfied, if (3.5) and (3.6) are satisfied.

The boundary conditions are formulated as follows. At  $z = h$ , the tangential component of the electric field and the normal component of the magnetic field are continuous across the interface. If we assume perfectly conducting ground plates, the electric and magnetic fields inside the ground plates ( $z > h$ ) vanish (as we have shown for the strip in the previous chapter). Then,

$$\mathbf{E} \times \mathbf{e}_z = \mathbf{0}, \quad \mathbf{H} \cdot \mathbf{e}_z = 0, \quad z = h, \quad (3.8)$$

or

$$E_x = E_y = 0, \quad H_z = 0, \quad z = h. \quad (3.9)$$

Furthermore, the jump in the normal component of the electric field is proportional to the surface charge density, and the jump in the tangential component of the magnetic field is proportional to the surface current density at  $z = h$ . However, the surface current density and

the surface charge density are not known. So, both jump conditions do not yield additional information.

At  $z = 0$ , we need to distinguish  $|x| < a$  and  $|x| > a$ . First, we consider the boundary conditions at  $z = 0$ ,  $|x| > a$ , which follow from symmetry. Since  $E_z$  is odd in  $z$ ,  $E_z(x, 0) = 0$ . Considering (3.6)<sup>3</sup>, we see that  $E_z$  is odd in  $z$ , if  $H_x$  and  $H_y$  are both odd in  $z$ . Therefore, we assume that  $H_x$  and  $H_y$  are both odd in  $z$ . Considering (3.6)<sup>1</sup>, we see that  $H_y$  is odd in  $z$ , if  $E_x$  and  $H_z$  are both even in  $z$ . Therefore, we assume that  $E_x$  and  $H_z$  are both even in  $z$ . Since  $H_x$  is assumed odd in  $z$  and  $H_z$  is assumed even in  $z$ ,  $E_y$  is even in  $z$  by (3.6)<sup>2</sup>. Summarizing, we see that  $H_x$ ,  $H_y$ , and  $E_z$  are odd in  $z$ , and  $E_x$ ,  $E_y$ , and  $H_z$  are even in  $z$ . These properties are in correspondence with (3.5). Therefore, all assumptions are justified. Consequently, the boundary conditions are given by

$$E_z = 0, \quad H_x = H_y = 0, \quad z = 0, |x| > a, \quad (3.10)$$

and

$$\frac{\partial E_x}{\partial z} = \frac{\partial E_y}{\partial z} = 0, \quad \frac{\partial H_z}{\partial z} = 0, \quad z = 0, |x| > a, \quad (3.11)$$

The boundary conditions at  $z = 0^+$ ,  $|x| < a$  follow from the results obtained in the previous chapter (see (2.65)),

$$E_x = E_y = 0, \quad H_y = H_z = 0, \quad z = 0^+, |x| < a, \quad (3.12)$$

and

$$E_z = \frac{\omega\mu_0}{2\beta} J_y(x) = \frac{\rho_S(x)}{2\varepsilon}, \quad H_x = \frac{1}{2} J_y(x), \quad z = 0^+, |x| < a. \quad (3.13)$$

### 3.2 A TEM Wave: Derivation of a Boundary Value Problem for the Electric Potential

A transverse electromagnetic (TEM) wave is a wave of which the phasor field quantities are functions of the distance along a single axis only. Furthermore, the electric and magnetic field are perpendicular to each other and transverse to the direction of propagation of the wave. So, the wave we consider, is a TEM wave, if we require  $E_y = 0$  and  $H_y = 0$ . Then, it follows from (3.5)<sup>3</sup> and (3.6)<sup>1</sup> that

$$H_z = -\frac{\beta}{\omega\mu_0} E_x, \quad E_x = -\frac{\beta}{\omega\varepsilon} H_z, \quad (3.14)$$

which implies that

$$\beta^2 = \omega^2 \varepsilon \mu_0 = k^2. \quad (3.15)$$

From (3.5)<sup>1</sup> and (3.6)<sup>3</sup>, i.e.

$$H_x = \frac{\beta}{\omega\mu_0} E_z, \quad E_z = \frac{\beta}{\omega\varepsilon} H_x, \quad (3.16)$$

we obtain the same result. This means that the stripline can support a TEM wave only if  $\beta = k$ . This is in correspondence with the result (2.65)<sup>6</sup> of the previous chapter. It follows from (3.14) and (3.16) that

$$\begin{pmatrix} H_x \\ H_z \end{pmatrix} = \kappa \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_Q \begin{pmatrix} E_x \\ E_z \end{pmatrix}, \quad (3.17)$$

where

$$\kappa = \frac{k}{\omega\mu_0} = \sqrt{\varepsilon/\mu_0}. \quad (3.18)$$

This relation shows that the electric field is perpendicular to the magnetic field. The matrix  $Q$  describes a rotation over an angle of 90 degrees. The factor  $\kappa$  is called the intrinsic admittance of the dielectric and  $1/\kappa$  is the intrinsic impedance. For free space,  $1/\kappa = 120\pi$  ohm.

From (3.5)<sup>2</sup>, (3.6)<sup>2</sup>, and (3.7) it follows that

$$\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = 0, \quad \frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} = 0, \quad (3.19)$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0, \quad \frac{\partial H_x}{\partial x} + \frac{\partial H_z}{\partial z} = 0. \quad (3.20)$$

Because of (3.19),  $\mathbf{E}$  and  $\mathbf{H}$  are irrotational. Then, because the region  $G^+$  is simply connected,  $\mathbf{E}$  and  $\mathbf{H}$  are both conservative. This implies that  $\mathbf{E}$  and  $\mathbf{H}$  are gradients of scalar functions  $\phi(x, z)$  and  $\psi(x, z)$ ,

$$\mathbf{E} = -\text{grad } \phi, \quad \mathbf{H} = -\text{grad } \psi. \quad (3.21)$$

The functions  $\phi(x, z)$  and  $\psi(x, z)$  are called the electric and magnetic potential, respectively. They are harmonic,

$$\Delta\phi = 0, \quad \Delta\psi = 0, \quad (3.22)$$

by (3.20), and they satisfy the Cauchy-Riemann relations

$$-\frac{\partial\psi}{\partial z} = \frac{k}{\omega\mu_0} \frac{\partial\phi}{\partial x}, \quad \frac{\partial\psi}{\partial x} = \frac{k}{\omega\mu_0} \frac{\partial\phi}{\partial z}, \quad (3.23)$$

by (3.16). Because of (3.23), we do not need to solve the boundary value problems for both  $\phi$  and  $\psi$ . Therefore, we deduce and solve the boundary value problem for  $\phi$  only.

We rewrite the boundary conditions for the electric field in terms of  $\phi$ . From (3.9)<sup>1</sup>, we obtain

$$\frac{\partial\phi}{\partial x} = 0, \quad z = h, \quad (3.24)$$

and from (3.10)<sup>1</sup> and (3.11)<sup>1</sup>

$$\frac{\partial\phi}{\partial z} = 0, \quad \frac{\partial}{\partial z} \left( \frac{\partial\phi}{\partial x} \right) = 0, \quad z = 0, |x| > a. \quad (3.25)$$

Note that (3.25)<sup>1</sup> implies (3.25)<sup>2</sup>. From (3.12) and (3.13), we obtain

$$\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial \phi}{\partial z} = -\frac{\omega \mu_0}{2k} J_y(x), \quad z = 0^+, |x| < a. \quad (3.26)$$

Summarizing, we arrive at the following boundary value problem for  $\phi$ :

$$\begin{aligned} \Delta \phi &= 0, & |x| < \infty, 0 < z < h, \\ \frac{\partial \phi}{\partial x} &= 0, & z = h, \\ \frac{\partial \phi}{\partial z} &= 0, & z = 0^+, |x| > a, \\ \frac{\partial \phi}{\partial z} &= -\frac{\omega \mu_0}{2k} J_y(x), & z = 0^+, |x| < a, \\ \frac{\partial \phi}{\partial x} &= 0, & z = 0^+, |x| < a. \end{aligned} \quad (3.27)$$

The solution to this problem is not unique. Since the tangential derivatives of  $\phi$  at  $z = h$  and at  $z = 0, |x| < a$  are zero, we may prescribe a constant potential at these boundaries. Without loss of generality, we put  $\phi = 0$  at  $z = h$  and  $\phi = \phi_0$  at  $z = 0, |x| < a$ , thus arriving at a mixed (Dirichlet-Neumann) boundary value problem for  $\phi$ ,

$$\begin{aligned} \Delta \phi &= 0, & |x| < \infty, 0 < z < h, \\ \phi &= 0, & z = h, \\ \frac{\partial \phi}{\partial z} &= 0, & z = 0^+, |x| > a, \\ \frac{\partial \phi}{\partial z} &= -\frac{\omega \mu_0}{2k} J_y(x), & z = 0^+, |x| < a, \\ \phi &= \phi_0, & z = 0^+, |x| < a. \end{aligned} \quad (3.28)$$

The potential  $\phi_0$  can be interpreted as the generator of the current on the stripline. To normalize the problem for  $\phi$ , we scale  $x$  with the length  $a$ ,  $z$  with the distance  $h$ , and  $\phi$  in accordance with the right-hand side of (3.28)<sup>5</sup>,

$$x = a\hat{x}, \quad z = h\hat{z}, \quad \phi = \phi_0\hat{\phi}. \quad (3.29)$$

We scale the current  $J_y$  in accordance with the right-hand side of (3.28)<sup>4</sup>,

$$J_y = j_y \hat{J}_y, \quad j_y = \frac{2k\phi_0}{\omega \mu_0 h}. \quad (3.30)$$



Then, the boundary value problem for  $\hat{\phi}$  is given by

$$\begin{aligned}
\frac{\partial^2 \hat{\phi}}{\partial \hat{x}^2} + \frac{1}{\zeta^2} \frac{\partial^2 \hat{\phi}}{\partial \hat{z}^2} &= 0, & |\hat{x}| < \infty, 0 < \hat{z} < 1, \\
\hat{\phi} &= 0, & \hat{z} = 1, \\
\frac{\partial \hat{\phi}}{\partial \hat{z}} &= 0, & \hat{z} = 0^+, |\hat{x}| > 1, \\
\frac{\partial \hat{\phi}}{\partial \hat{z}} &= -\hat{J}(\hat{x}), & \hat{z} = 0^+, |\hat{x}| < 1, \\
\hat{\phi} &= 1, & \hat{z} = 0^+, |\hat{x}| < 1,
\end{aligned} \tag{3.31}$$

where  $\zeta = h/a$  and  $\hat{J}_y(\hat{x}) = \hat{J}(\hat{x})$ . In the analysis to follow, we omit the hats. From (3.31)<sup>1</sup>, we conclude that  $\phi$  depends on  $\zeta$ , and therefore,  $J$  depends also on  $\zeta$ . To show these dependencies explicitly, we write  $\phi(x, z; \zeta)$  and  $J(x; \zeta)$ .

### 3.3 Calculation of Potential and Current

To solve the boundary value problem (3.31) for  $\phi$ , we apply the Fourier transform with respect to  $x$ ,

$$\Phi(s, z; \zeta) = \mathcal{F}\{\phi(x, z; \zeta); x \rightarrow s\} = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} \phi(x, z; \zeta) e^{isx} dx, \tag{3.32}$$

and its inverse transform

$$\phi(x, z; \zeta) = \mathcal{F}^{-1}\{\Phi(s, z; \zeta); s \rightarrow x\} = \frac{1}{\sqrt{2\pi}} \int_{s=-\infty}^{\infty} \Phi(s, z; \zeta) e^{-isx} ds. \tag{3.33}$$

Assuming that  $\phi$  and its second derivative with respect to  $x$  are square integrable, we obtain

$$\mathcal{F}\left\{\frac{\partial^2 \phi}{\partial z^2}\right\} = \frac{\partial^2 \Phi}{\partial z^2}, \quad \mathcal{F}\left\{\frac{\partial^2 \phi}{\partial x^2}\right\} = -s^2 \Phi(s, z), \tag{3.34}$$

respectively. Then, the differential equation for  $\phi$  is transformed to

$$\frac{\partial^2 \Phi}{\partial z^2} - (\zeta s)^2 \Phi(s, z) = 0, \tag{3.35}$$

and the boundary condition at  $z = 1$ , i.e. (3.31)<sup>2</sup>, to  $\Phi = 0$  at  $z = 1$ . The solution for  $\Phi$  is given by

$$\Phi(s, z; \zeta) = C(s; \zeta) \sinh(\zeta s(1 - z)), \tag{3.36}$$

where  $C$  is an unknown function. Applying the inverse Fourier transform, we obtain the following integral expression for  $\phi$ :

$$\phi(x, z; \zeta) = \frac{1}{\sqrt{2\pi}} \int_{s=-\infty}^{\infty} C(s; \zeta) \sinh(\zeta s(1 - z)) e^{-isx} ds, \quad 0 < z < 1, |x| < \infty. \tag{3.37}$$

We distinguish two approaches to calculate  $C$  and  $J$ . In the first approach, we choose the condition (3.31)<sup>4</sup> to calculate the potential  $\phi$ , i.e. to express  $\phi$  in terms of the current. The current is calculated from the condition for  $\phi$  on the plate, i.e. (3.31)<sup>5</sup>. In the second approach we choose the condition (3.31)<sup>5</sup> to calculate the potential  $\phi$ . The current is calculated from the condition (3.31)<sup>4</sup>. First, we consider both approaches and then, we address the question whether they yield the same result.

### 3.3.1 First Approach

The boundary conditions (3.31)<sup>3,4</sup>, i.e.

$$\begin{aligned} \frac{\partial \phi}{\partial z} &= -J(x; \zeta), \quad z = 0^+, \quad |x| < 1, \\ \frac{\partial \phi}{\partial z} &= 0, \quad z = 0^+, \quad |x| > 1, \end{aligned} \tag{3.38}$$

yield two integral equations for  $C(s; \zeta)$ ,

$$\begin{aligned} \left. \frac{\partial \phi}{\partial z} \right|_{z=0^+} &= -\frac{\zeta}{\sqrt{2\pi}} \int_{s=-\infty}^{\infty} s C(s; \zeta) e^{-isx} \cosh(\zeta s) ds = -J(x; \zeta), \quad |x| < 1, \\ \left. \frac{\partial \phi}{\partial z} \right|_{z=0^+} &= -\frac{\zeta}{\sqrt{2\pi}} \int_{s=-\infty}^{\infty} s C(s; \zeta) e^{-isx} \cosh(\zeta s) ds = 0, \quad |x| > 1. \end{aligned} \tag{3.39}$$

Consequently,

$$\frac{1}{\sqrt{2\pi}} \int_{s=-\infty}^{\infty} \tilde{C}(s; \zeta) e^{-isx} ds = \begin{cases} J(x; \zeta), & |x| < 1, \\ 0, & |x| > 1, \end{cases} \tag{3.40}$$

where

$$\tilde{C}(s; \zeta) = \zeta s C(s; \zeta) \cosh(\zeta s). \tag{3.41}$$

The right-hand side of (3.40) represents the inverse Fourier transform of  $\tilde{C}(s; \zeta)$ . By transformation, we obtain

$$\tilde{C}(s; \zeta) = \frac{1}{\sqrt{2\pi}} \int_{u=-1}^1 J(u; \zeta) e^{isu} du. \tag{3.42}$$

Then,  $C(s; \zeta)$  is given by

$$C(s; \zeta) = \frac{1}{\sqrt{2\pi} \zeta s \cosh(\zeta s)} \int_{u=-1}^1 J(u; \zeta) e^{isu} du. \tag{3.43}$$

Substitution of this expression into (3.37) yields

$$\begin{aligned} \phi(x, z; \zeta) &= \frac{1}{2\pi\zeta} \int_{s=-\infty}^{\infty} \frac{e^{-isx} \sinh(\zeta s(1-z))}{s \cosh(\zeta s)} \int_{u=-1}^1 J(u; \zeta) e^{isu} du ds = \\ &= \int_{u=-1}^1 J(u; \zeta) F(x, z; u, \zeta) du. \end{aligned} \tag{3.44}$$

where the function  $F(x, z, u; \zeta)$  is defined by

$$\begin{aligned} F(x, z, u; \zeta) &= \frac{1}{2\pi\zeta} \int_{s=-\infty}^{\infty} \frac{e^{-is(x-u)} \sinh(\zeta s(1-z))}{s \cosh(\zeta s)} ds = \\ &= \frac{1}{\pi\zeta} \int_{s=0}^{\infty} \frac{\cos(s(x-u)) \sinh(\zeta s(1-z))}{s \cosh(\zeta s)} ds \end{aligned}$$

Since

$$k(\alpha, z; \zeta) = \frac{1}{\pi\zeta} \int_{s=0}^{\infty} \frac{\cos(\alpha s) \sinh(\zeta s(1-z))}{s \cosh(\zeta s)} ds = \frac{1}{2\pi\zeta} \log \left[ \frac{\cosh\left(\frac{\alpha\pi}{2\zeta}\right) + \sin\left(\frac{(1-z)\pi}{2}\right)}{\cosh\left(\frac{\alpha\pi}{2\zeta}\right) - \sin\left(\frac{(1-z)\pi}{2}\right)} \right] \quad (3.45)$$

(see Erdélyi et al. [4], table 1.9 (34), p. 33), the solution for  $\phi$  can be written as

$$\phi(x, z; \zeta) = \int_{u=-1}^1 J(u; \zeta) k(x-u, z; \zeta) du. \quad (3.46)$$

The current  $J$  can be calculated from the condition (3.31)<sup>5</sup>, i.e.  $\phi = 1$  for  $z = 0^+$ ,  $|x| < 1$ , and satisfies a Fredholm equation of the first kind,

$$\int_{u=-1}^1 J(u; \zeta) k(x-u, 0; \zeta) du = 1, \quad |x| < 1. \quad (3.47)$$

Using the doubling formulas for the cosine and sine hyperbolic, we obtain

$$k(\alpha, 0; \zeta) = \frac{1}{2\pi\zeta} \log \left[ \frac{\cosh\left(\frac{\alpha\pi}{2\zeta}\right) + 1}{\cosh\left(\frac{\alpha\pi}{2\zeta}\right) - 1} \right] = \frac{1}{\pi\zeta} \log \left| \coth \frac{\alpha\pi}{4\zeta} \right| =: k(\alpha; \zeta). \quad (3.48)$$

Then, the integral equation for  $J$  turns into

$$\mathcal{K}_\zeta J = 1, \quad (3.49)$$

where  $\mathcal{K}_\zeta$  is defined by

$$(\mathcal{K}_\zeta J)(x) = \int_{u=-1}^1 J(u; \zeta) k(x-u; \zeta) du, \quad |x| < 1. \quad (3.50)$$

We write  $k = k_1 + k_2$  with  $k_1(\alpha) = \log |\alpha|$ . Then,  $k_2 \in H_{2,1}[-2, 2]$ , where  $H_{2,1}[a, b]$  is the space of all absolutely continuous functions  $f$ , which have a generalized derivative  $f'$  in  $L_2[a, b]$ , i.e.

$$f(x) = A + \int_0^x f'(\xi) d\xi. \quad (3.51)$$

We can prove that the operator  $\mathcal{K}_\zeta$  maps a function in  $L_2[-1, 1]$  onto a function in  $H_{2,1}[-1, 1]$ . For that, we use a result in [8, p. 20] and the general result that  $\mathcal{K}_\zeta$  with kernel  $k \in H_{2,1}[-1, 1]$

maps a function in  $L_2[-1, 1]$  onto a function in  $H_{2,1}[-1, 1]$ . The latter is shown in [2, Appendix D].

The aforementioned observations do not provide information about the existence and uniqueness of the solution of (3.49) in  $L_2[-1, 1]$  for the  $H_{2,1}[-1, 1]$  function in the right-hand side. It can be shown that the solution of the boundary value problem (3.31) is unique, as well as the solution of (3.49). However, the solution of the latter is not in  $L_2[-1, 1]$ , because the current has a square root singularity at the edges  $x = \pm 1$  as is shown in Appendix B. We do not go into further details. In Section 3.4 and Section 3.5, we show how the solution of (3.49) can be approximated asymptotically and numerically.

Using the uniqueness of (3.49), we show that the solution  $J(u; \zeta)$  is even in  $u$ . Let for any function  $f$  the function  $f^\vee$  be defined by  $f^\vee(x) = f(-x)$ . For a function  $J(u)$ , we derive

$$\begin{aligned} (\mathcal{K}_\zeta J^\vee)(x) &= \int_{u=-1}^1 J(-u) k(x-u; \zeta) du = \int_{u=-1}^1 J(u) k(x+u; \zeta) du = \\ &= (\mathcal{K}_\zeta J)(-x) = (\mathcal{K}_\zeta J)^\vee(x). \end{aligned} \quad (3.52)$$

So,

$$\mathcal{K}_\zeta J^\vee = (\mathcal{K}_\zeta J)^\vee. \quad (3.53)$$

Let  $J(u; \zeta)$  be a solution of the integral equation (3.49). Then,  $J^\vee(u; \zeta) := J(-u; \zeta)$  is by (3.53) also a solution of the integral equation. Since the solution of the integral equation is unique, we have  $J = J^\vee$ , which means that  $J(u; \zeta)$  is even in  $u$ .

### 3.3.2 Second Approach

The boundary conditions (3.31)<sup>3,5</sup>, i.e.

$$\begin{aligned} \phi = 1, \quad z = 0^+, \quad |x| < 1, \\ \frac{\partial \phi}{\partial z} = 0, \quad z = 0^+, \quad |x| > 1, \end{aligned} \quad (3.54)$$

yield a system of two integral equations for  $C(s; \zeta)$ ,

$$\begin{aligned} \phi(x, 0; \zeta) &= \frac{1}{\sqrt{2\pi}} \int_{s=-\infty}^{\infty} C(s; \zeta) e^{-isx} \sinh(\zeta s) ds = 1, \quad |x| < 1, \\ \frac{\partial \phi}{\partial z} \Big|_{z=0^+} &= -\frac{\zeta}{\sqrt{2\pi}} \int_{s=-\infty}^{\infty} s C(s; \zeta) e^{-isx} \cosh(\zeta s) ds = 0, \quad |x| > 1. \end{aligned} \quad (3.55)$$

Let  $C(s; \zeta)$  be a solution to this system. Then,  $-C(-s; \zeta)$  is also a solution of the system. This can be shown by replacing  $x$  by  $-x$  and  $s$  by  $-s$ . So, assuming that the solution to the system is unique, we conclude that  $C(s; \zeta)$  is odd in  $s$ . To solve the system, we write

$$s C(s; \zeta) \cosh(\zeta s) = \int_{t=-1}^1 g(t; \zeta) B(t, s) dt, \quad (3.56)$$

where  $B(t, s)$  is a given function, which is even in  $s$ . We substitute the expression for  $C(s; \zeta)$  into the integral of (3.55)<sup>2</sup> and deduce

$$\begin{aligned} \int_{s=-\infty}^{\infty} s C(s; \zeta) e^{-isx} \cosh(\zeta s) ds &= \int_{s=-\infty}^{\infty} e^{-isx} \int_{t=-1}^1 g(t; \zeta) B(t, s) dt ds = \\ &= \int_{t=-1}^1 g(t; \zeta) \int_{s=-\infty}^{\infty} e^{-isx} B(t, s) ds dt = \sqrt{2\pi} \int_{t=-1}^1 g(t; \zeta) \mathcal{F}^{-1}\{B(t, s); s \rightarrow x\} dt. \end{aligned} \quad (3.57)$$

To satisfy (3.55)<sup>2</sup>, we assume that

$$\sqrt{2\pi} \mathcal{F}^{-1}\{B(t, s); s \rightarrow x\} = \frac{H(|t| - |x|)}{D(t, x)}, \quad (3.58)$$

where  $H(\cdot)$  is the Heaviside function and  $D(t, x)$  is an unknown function. Substituting (3.58) into (3.57), we obtain

$$\begin{aligned} \int_{s=-\infty}^{\infty} s C(s; \zeta) e^{-isx} \cosh(\zeta s) ds &= \int_{t=-1}^1 g(t; \zeta) \frac{H(|t| - |x|)}{D(t, x)} dt = \\ &= \begin{cases} 0, & |x| > 1, \\ \left( \int_{t=|x|}^1 + \int_{t=-1}^{-|x|} \right) \frac{g(t; \zeta)}{D(t, x)} dt, & |x| < 1. \end{cases} \end{aligned} \quad (3.59)$$

The result for  $|x| > 1$  implies that the second integral equation (3.55)<sup>2</sup> is satisfied. We have to determine  $g(t; \zeta)$  such that the first integral equation (3.55)<sup>1</sup> is satisfied also. Substituting (3.56) into (3.37), we obtain

$$\begin{aligned} \phi(x, z; \zeta) &= \frac{1}{\sqrt{2\pi}} \int_{s=-\infty}^{\infty} \frac{e^{-isx} \sinh(\zeta s(1-z))}{s \cosh(\zeta s)} \int_{t=-1}^1 g(t; \zeta) B(t, s) dt ds = \\ &= \int_{t=-1}^1 g(t; \zeta) G(x, z, t; \zeta) dt, \end{aligned} \quad (3.60)$$

where

$$\begin{aligned} G(x, z, t; \zeta) &= \frac{1}{\sqrt{2\pi}} \int_{s=-\infty}^{\infty} \frac{e^{-isx} \sinh(\zeta s(1-z))}{s \cosh(\zeta s)} B(t, s) ds = \\ &= \sqrt{\frac{2}{\pi}} \int_{s=0}^{\infty} \frac{\cos(sx) \sinh(\zeta s(1-z))}{s \cosh(\zeta s)} B(t, s) ds, \end{aligned} \quad (3.61)$$

because  $B(t, s)$  is even in  $s$ . Then, the first integral equation (3.55)<sup>1</sup> turns into

$$\int_{t=-1}^1 g(t; \zeta) G(x, 0, t; \zeta) dt = 1, \quad |x| < 1, \quad (3.62)$$

which is a Fredholm equation of the first kind. After calculation of  $g(t; \zeta)$  from (3.62), the potential  $\phi$  is calculated from (3.60). The current  $J$  is calculated from the boundary condition (3.31)<sup>4</sup>, i.e.

$$\frac{\partial \phi}{\partial z} = -J(x; \zeta), \quad z = 0^+, \quad |x| < 1. \quad (3.63)$$

Because  $\phi$  is even in  $x$ ,  $J$  is also even in  $x$ .

The inverse Fourier transform in the left-hand side of (3.58) is even in  $x$ , because  $B(t, s)$  is even in  $s$ . Therefore,  $D(t, x)$  has to be even in  $x$ . Because  $B(t, s)$  is even in  $s$ , we can replace (3.58) by

$$\int_{s=0}^{\infty} \cos(sx) B(t, s) ds = \frac{H(|t| - |x|)}{2D(t, x)}, \quad (3.64)$$

i.e.  $\sqrt{\pi/2}$  times the Fourier cosine transform of  $B(t, s)$ . Up to now, we have not specified  $B(t, s)$ . It should be chosen such that the integral (3.61) and the inverse Fourier transform (3.58) exist. If we take  $B(t, s)$  even in  $t$ ,  $G(x, z, t; \zeta)$  is also even in  $t$  and therefore,  $g(t; \zeta)$  is even in  $t$ . By Fourier transformation, we obtain from (3.58)

$$B(t, s) = \frac{1}{2\pi} \int_{x=-|t|}^{|t|} \frac{e^{isx}}{D(t, x)} dx = \frac{1}{\pi} \int_{x=0}^{|t|} \frac{\cos(sx)}{D(t, x)} dx, \quad (3.65)$$

because  $D(t, x)$  is even in  $x$ . Choices for  $D(t, x)$  are for example  $D(t, x) = 1$  with

$$B(t, s) = \frac{1}{\pi} \frac{\sin(s|t|)}{s}, \quad (3.66)$$

and  $D(t, x) = \frac{1}{2}\sqrt{t^2 - x^2}$  with

$$B(t, s) = \frac{2}{\pi} \int_{x=0}^{|t|} \frac{\cos(sx)}{\sqrt{t^2 - x^2}} dx = \mathcal{J}_0(ts). \quad (3.67)$$

For both choices, (3.58) and (3.61) exist. We consider the choice  $D(t, x) = 1$  first. Substituting (3.66) into the expression for  $G$ , i.e. (3.61), we obtain

$$\begin{aligned} G(x, z, t; \zeta) &= \frac{1}{2\pi} \sqrt{\frac{2}{\pi}} \int_{s=0}^{\infty} \frac{\sinh(\zeta s(1-z))}{s^2 \cosh(\zeta s)} [\sin s(|t| + x) + \sin s(|t| - x)] ds = \\ &= \frac{\zeta}{\sqrt{2\pi}} [K(|t| + x, z; \zeta) + K(|t| - x, z; \zeta)], \end{aligned} \quad (3.68)$$

where  $K$  is defined by

$$K(\alpha, z; \zeta) = \frac{1}{\pi\zeta} \int_{s=0}^{\infty} \frac{\sin(\alpha s) \sinh(\zeta s(1-z))}{s^2 \cosh(\zeta s)} ds. \quad (3.69)$$

A series expansion for  $K$  is deduced in Appendix A. Note that  $\partial K/\partial \alpha = k$ . Substituting (3.68) into the integral equation for  $g$ , i.e. (3.62), we obtain

$$\zeta \sqrt{\frac{2}{\pi}} \int_{t=-1}^1 g(t; \zeta) [K(|t| + x, 0; \zeta) + K(|t| - x, 0; \zeta)] dt = 1, \quad |x| < 1. \quad (3.70)$$

Let us now consider the choice  $D(t, x) = \frac{1}{2}\sqrt{t^2 - x^2}$ . Substituting the integral expression (3.67) for  $B(t, s)$  into the expression (3.61) for  $G$  and reversing integrals, we obtain

$$\begin{aligned} G(x, z, t; \zeta) &= \\ &= \frac{1}{\pi} \sqrt{\frac{2}{\pi}} \int_{r=0}^{|t|} \frac{1}{\sqrt{t^2 - r^2}} \int_{s=0}^{\infty} \frac{\sinh(\zeta s(1-z))}{s \cosh(\zeta s)} [\cos(s(x+r)) + \cos(s(x-r))] ds. \end{aligned} \quad (3.71)$$

By definition (3.45) of  $k$ , this expression can be written as

$$G(x, z, t; \zeta) = \zeta \sqrt{\frac{2}{\pi}} \int_{r=-|t|}^{|t|} \frac{1}{\sqrt{t^2 - r^2}} k(x - r, z; \zeta) dr. \quad (3.72)$$

Substituting (3.72) into the integral equation for  $g$ , i.e. (3.62), and using the definition (3.48) of  $k$ , we obtain

$$\zeta \sqrt{\frac{2}{\pi}} \int_{t=-1}^1 g(t; \zeta) \int_{r=-|t|}^{|t|} \frac{1}{\sqrt{t^2 - r^2}} k(x - r; \zeta) dr = 1, \quad |x| < 1. \quad (3.73)$$

In the next subsection, we relate the functions  $J$  and  $g$ , and we show that the integral equations (3.70) and (3.73) for  $g$  can be rewritten to the integral equation (3.49) for  $J$ .

### 3.3.3 Comparison of the Two Approaches

The difference between the two approaches is that in the first approach, we determine  $J$  first, while in the second one, we determine  $C$  first. The two approaches yield the same result for  $C$  and, consequently, for  $J$ , if the result for  $C$  of the first approach satisfies the set of integral equations of the second approach. Note that we assume that the solutions of the integral equation (3.40) of the first approach and the set of integral equations (3.55) of the second approach are unique and that  $C$  is determined uniquely in both approaches. The expression for  $C$  of the first approach is given by (3.43), i.e.

$$C(s; \zeta) = \frac{1}{\sqrt{2\pi}} \frac{1}{\zeta s \cosh(\zeta s)} \int_{u=-1}^1 J(u; \zeta) e^{isu} du. \quad (3.74)$$

This function satisfies the second integral equation of the second approach, i.e. (3.55)<sup>2</sup>, because the second integral equation of the first approach, i.e. (3.39)<sup>2</sup>, is the same. Substitution of (3.74) into the first integral equation of the second approach, i.e. (3.55)<sup>1</sup>, yields

$$\frac{1}{2\pi\zeta} \int_{s=-\infty}^{\infty} \frac{e^{-isx} \sinh(\zeta s)}{s \cosh(\zeta s)} ds \int_{u=-1}^1 J(u; \zeta) e^{isu} du = 1, \quad |x| < 1. \quad (3.75)$$

This is exactly the equation  $\phi(x, 0; \zeta) = 1$  in the first approach (see (3.44)). So, we obtain the same integral equation for  $J$  as in the first approach. This shows that the two approaches yield the same result for  $J$  and  $C$ .

In the first approach, we solve an integral equation for  $J$ , while in the second approach, we solve an integral equation for  $g$ . A relation between  $J$  and  $g$  can be deduced as follows. We substitute the expression (3.56) for  $C(s; \zeta)$  of the second approach into the integral equation of the first approach, i.e. (3.40),

$$\frac{\zeta}{\sqrt{2\pi}} \int_{s=-\infty}^{\infty} e^{-isx} \int_{t=-1}^1 g(t; \zeta) B(t, s) dt ds = \begin{cases} J(x; \zeta), & |x| < 1, \\ 0, & |x| > 1, \end{cases} \quad (3.76)$$

As in (3.59), we obtain

$$J(x; \zeta) = \frac{\zeta}{\sqrt{2\pi}} \left( \int_{t=|x|}^1 + \int_{t=-1}^{-|x|} \right) \frac{g(t; \zeta)}{D(t, x)} dt, \quad |x| < 1, \quad (3.77)$$

or, assuming that  $D(t, x)$  is even in  $t$  (and so,  $g(t; \zeta)$  is even in  $t$ ),

$$J(x; \zeta) = \zeta \sqrt{\frac{2}{\pi}} \int_{t=|x|}^1 \frac{g(t; \zeta)}{D(t, x)} dt, \quad |x| < 1. \quad (3.78)$$

Substituting (3.77) with  $D(t, x) = 1$  into the integral equation (3.49) for  $J$  and reversing the integrals with respect to  $t$  and  $u$ , we arrive at the integral equation (3.70) for  $g$ . In a similar way, we arrive at the integral equation (3.49) for  $J$  by reversing the integrals with respect to  $t$  and  $r$  in the integral equation (3.73) for  $g$  and by using the relation (3.77) between  $g$  and  $J$ . So, the integral equations of both choices of  $D(t, x)$  can be transformed into the integral equation (3.49) for  $J$ . To show this generally, we substitute (3.65) into (3.61), reverse the integrals, and use the doubling formulas of the cosine,

$$\begin{aligned} G(x, 0, t; \zeta) &= \\ &= \frac{1}{4\pi} \sqrt{\frac{2}{\pi}} \int_{u=-|t|}^{|t|} \frac{1}{D(t, u)} \int_{s=0}^{\infty} \frac{[\cos s(x+u) + \cos s(x-u)] \sinh(\zeta s(1-z))}{s \cosh(\zeta s)} ds du. \end{aligned} \quad (3.79)$$

Because  $D(t, u)$  is even in  $u$ , we can write

$$\begin{aligned} G(x, 0, t; \zeta) &= \frac{1}{4\pi} \sqrt{\frac{2}{\pi}} \int_{u=-|t|}^{|t|} \frac{1}{D(t, u)} \int_{s=0}^{\infty} \frac{\cos s(x-u) \sinh(\zeta s(1-z))}{s \cosh(\zeta s)} ds du = \\ &= \frac{\zeta}{\sqrt{2\pi}} \int_{u=-|t|}^{|t|} \frac{1}{D(t, u)} k(x-u, z; \zeta) du. \end{aligned} \quad (3.80)$$

Substituting this expression into (3.60) and reversing the integrals with respect to  $u$  and  $t$ , we obtain the integral equation (3.49) for  $J$  by the relation (3.77) between  $g$  and  $J$ . As above, this shows that the two approaches yield the same result for  $J$ .

**Remark:** As aforementioned, we do not only want to calculate the electromagnetic field in the dielectric and the current in the stripline, but also the impedance of the stripline. As shown in Appendix C, the impedance is a constant divided by the integral of  $J(x; \zeta)$  from 0 to 1. Assuming that  $D(t, x)$  is even in  $t$ , we transform this integral to an integral of  $g(t; \zeta)$  by (3.78),

$$\int_{x=0}^1 J(x; \zeta) dx = \zeta \sqrt{\frac{2}{\pi}} \int_{t=0}^1 g(t; \zeta) r(t) dt, \quad (3.81)$$

where

$$r(t) = \int_{x=0}^t \frac{1}{D(t, x)} dx, \quad (3.82)$$

because  $D(t, x)$  is even in  $x$ . For  $D(t, x) = 1$ , we obtain  $r(t) = t$  and for  $D(t, x) = \sqrt{t^2 - x^2}$ ,  $r(t) = \pi/2$ .

### 3.4 Asymptotic Approximations of Current, Potential, and Impedance

In this section, we deduce asymptotic approximations for  $\phi$ ,  $J$ , and  $Z$  for the cases  $\zeta \gg 1$  and  $\zeta \ll 1$ .



### 3.4.1 Asymptotic Approximations for $\zeta \ll 1$

We consider the case  $\zeta = h/a \ll 1$ , i.e. the width  $2a$  of the stripline is much larger than the distance  $2h$  between the ground plates. To deduce asymptotic approximations for the potential, current, and impedance, we use a combination of both approaches. We start from the expression (3.60) for the potential obtained in the second approach. The function  $g$  in this expression is determined by the equation (3.62). We approximate  $G(x, 0, t; \zeta)$  in this equation by

$$G(x, 0, t; \zeta) = \zeta \sqrt{\frac{2}{\pi}} \int_{s=0}^{\infty} \cos(sx) B(t, s) ds = \zeta \sqrt{\frac{2}{\pi}} \frac{H(|t| - |x|)}{2D(t, x)}. \quad (3.83)$$

The second equality follows from (3.64). Assuming that  $D(t, x)$  is even in  $t$ , we write the equation for  $g(t; \zeta)$ , i.e. (3.62), as

$$\zeta \sqrt{\frac{2}{\pi}} \int_{t=|x|}^1 \frac{g(t; \zeta)}{D(t, x)} dt = 1, \quad |x| < 1. \quad (3.84)$$

From the relation (3.78) between  $J$  and  $g$ , it follows that

$$J(x; \zeta) = 1, \quad |x| < 1. \quad (3.85)$$

So, the current on the stripline is almost uniform.

To calculate the potential  $\phi$ , we can either use the expression (3.46) of the first approach or the expression (3.60) of the second approach. Since the current is constant,  $\partial K / \partial \alpha = k$  (see (3.69) and (3.45)), and a series expansion of  $K$  is known (see Appendix A), we use the expression of the first approach. The potential  $\phi$  is given by

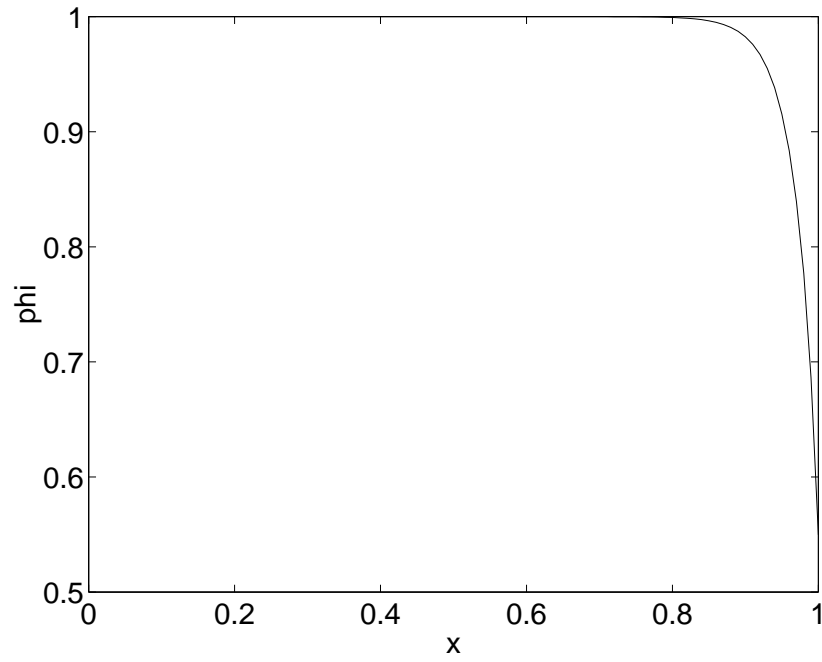
$$\phi(x, z; \zeta) = \int_{u=-1}^1 k(x-u, z; \zeta) du = K(1+x, z; \zeta) + K(1-x, z; \zeta), \quad (3.86)$$

(see (3.46)). It can be shown that the approximation (3.86) satisfies the differential equation for  $\phi$ , the boundary condition at  $z = 1$ , and the boundary condition  $\partial \phi / \partial z = -J(x; \zeta) = -1$  at  $z = 0^+$ ,  $|x| < 1$ . The boundary condition  $\phi = 1$  at  $z = 0^+$ ,  $|x| < 1$  is not satisfied. This condition serves as a measure for precision of the approximation. Figure 3.1 shows  $\phi(x, z; \zeta)$  at  $z = 0^+$ ,  $|x| < 1$  for  $\zeta = 0.05$ . Note that because the series expansion of  $K(\alpha, z; \zeta)$  converges rapidly (except near  $\alpha = 0$ ), we need only the first term to calculate  $\phi$ . We see that the potential is almost uniform at the stripline. Figure 3.2 shows the equipotential lines for the case  $\zeta = 0.05$ . The impedance of the stripline can be calculated with (C.6). With  $J(x; \zeta) = 1$ , we obtain

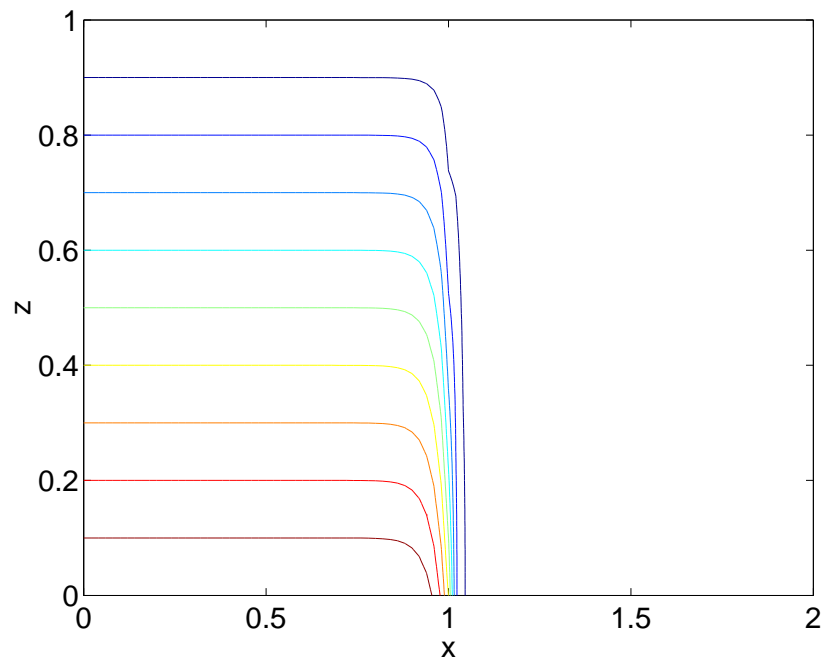
$$Z = \frac{\zeta}{4\sqrt{\epsilon_r}} \sqrt{\frac{\mu_0}{\epsilon}} \approx \frac{30\pi}{\sqrt{\epsilon_r}} \zeta, \quad (3.87)$$

where we used  $\sqrt{\mu_0/\epsilon} \approx 120\pi$ . We compare this with the approximations of Pozar [7, p. 156] and Howe [6, p. 34-35]. For  $\zeta < 1/0.35$ , the impedance approximation of Pozar is given by

$$Z = \frac{30\pi}{\sqrt{\epsilon_r}} \frac{1}{\frac{1}{\zeta} + 0.441}. \quad (3.88)$$



**Figure 3.1** The first order approximation ( $\zeta \ll 1$ ) of the potential  $\phi$  as a function of  $x$  at  $z = 0^+$  with  $\zeta = 0.05$ .



**Figure 3.2** The equipotential lines for  $\zeta = 0.05$ , calculated with the first order asymptotic approximation.

This equals almost our approximation (3.87) for  $\zeta \ll 1$ . For  $\zeta < 1/0.35$ , Howe's impedance approximation is given by

$$Z = \frac{1}{\sqrt{\varepsilon_r}} \frac{94.15}{\frac{1}{\zeta} + \frac{2}{\pi} \log 2}, \quad (3.89)$$

which is almost equal to Pozar's approximation, and therefore, it equals almost our approximation for  $\zeta \ll 1$ .

### 3.4.2 Asymptotic Approximations for $\zeta \gg 1$

We consider the case  $\zeta = h/a \gg 1$ , i.e. the width  $2a$  of the plate is much smaller than the distance  $2h$  between the ground plates. First, we discuss an approximation of the impedance by the second approach and then, we discuss an approximation of current and impedance by the first approach.

We take  $D(t, x) = 1$  in the second approach, which yields  $B(t, s) = \frac{2}{\pi} \frac{\sin s|t|}{s}$ . Then, the expression for  $G(x, 0, t; \zeta)$  is given by

$$G(x, 0, t; \zeta) = \sqrt{\frac{2}{\pi}} \int_{s=0}^{\infty} \frac{\cos(sx) \sinh(\zeta s)}{s^2 \cosh(\zeta s)} \sin(s|t|) ds. \quad (3.90)$$

We rewrite  $G(x, 0, t; \zeta)$  as follows:

$$\begin{aligned} G(x, 0, t; \zeta) &= \zeta \sqrt{\frac{2}{\pi}} \int_{s=0}^{\infty} \frac{\cos \frac{sx}{\zeta} \sinh s}{s^2 \cosh s} \sin \frac{s|t|}{\zeta} ds = \\ &= |t| \underbrace{\sqrt{\frac{2}{\pi}} \int_{s=0}^{\infty} \frac{\cos \frac{sx}{\zeta} \sinh s}{s \cosh s} ds}_{(1)} + \zeta \underbrace{\sqrt{\frac{2}{\pi}} \int_{s=0}^{\infty} \frac{\cos \frac{sx}{\zeta} \sinh s}{s^2 \cosh s} \left( \sin \frac{s|t|}{\zeta} - \frac{s|t|}{\zeta} \right) ds}_{(2)}. \end{aligned} \quad (3.91)$$

In a rough approximation, we neglect the second integral and deduce

$$\begin{aligned} G(x, 0, t; \zeta) &= |t| \sqrt{\frac{2}{\pi}} \pi \zeta k(x/\zeta, 0; 1) = \frac{1}{2} |t| \sqrt{\frac{2}{\pi}} \log \left[ \frac{\cosh \left( \frac{x\pi}{2\zeta} \right) + 1}{\cosh \left( \frac{x\pi}{2\zeta} \right) - 1} \right] \approx \\ &\approx |t| \sqrt{\frac{2}{\pi}} \log \left( \frac{4\zeta}{|x|\pi} \right). \end{aligned} \quad (3.92)$$

Substituting this expression into the integral equation (3.62) for  $g(t; \zeta)$ , we obtain

$$\sqrt{\frac{2}{\pi}} \log \left( \frac{4\zeta}{|x|\pi} \right) \int_{t=0}^1 t g(t; \zeta) dt = 1, \quad 0 < |x| < 1. \quad (3.93)$$

Note that we used that  $D(t, x)$  is even in  $t$  and consequently,  $g(t; \zeta)$  is also even in  $t$ . This equation can only be satisfied for one value of  $x$  in  $(-1, 1) \setminus \{0\}$ . However, for large values of  $\zeta$ , the equation will be satisfied 'approximately'. Take  $x = x_0$  and require

$$\sqrt{\frac{2}{\pi}} \log \left( \frac{4\zeta}{x_0\pi} \right) \int_{t=0}^1 t g(t; \zeta) dt = 1. \quad (3.94)$$

Then,

$$\int_{t=0}^1 t g(t; \zeta) dt = \sqrt{\frac{\pi}{2}} \left( \log \left( \frac{4\zeta}{x_0\pi} \right) \right)^{-1}. \quad (3.95)$$

By (3.81) with  $r(t) = t$ , this equation turns into

$$\int_{x=0}^1 J(x; \zeta) dx = \frac{\zeta\pi}{2} \left( \log \left( \frac{4\zeta}{x_0\pi} \right) \right)^{-1}. \quad (3.96)$$

Consequently, the impedance is given by (see C.6)

$$Z = \frac{60}{\sqrt{\varepsilon_r}} \log \left( \frac{4\zeta}{x_0\pi} \right). \quad (3.97)$$

The choice of  $x_0$  is a difficulty. The choice  $x_0 = 0.5$  yields the formula of Howe [6, p. 34-35] for the impedance when  $\zeta > 1/0.35$ . Further research is necessary to obtain a less arbitrary approximation.

To obtain an approximation of current and impedance by the first approach, we consider the integral equation (3.49) for  $J$ . For  $\zeta \gg 1$ , we approximate the kernel  $k$  by

$$k(|\alpha|; \zeta) = \frac{1}{\pi\zeta} \log \frac{4\zeta}{|\alpha|\pi}. \quad (3.98)$$

Then, the integral equation for  $J$  turns into

$$\frac{1}{\pi\zeta} \int_{u=-1}^1 J(u; \zeta) \log \frac{4\zeta}{|x-u|\pi} du = 1, \quad |x| < 1. \quad (3.99)$$

In Appendix B, it is shown that the current has a square root singularity near the edges of the stripline. Extracting this singularity from the function  $J$ , we define  $\tilde{J}$  by

$$\tilde{J}(x; \zeta) = \sqrt{1-x^2} J(x; \zeta), \quad |x| \leq 1. \quad (3.100)$$

Then, with  $x = \cos \theta$  and  $u = \cos \theta'$ , (3.99) turns into

$$\frac{1}{\pi\zeta} \int_{\theta'=0}^{\pi} \tilde{J}(\cos \theta'; \zeta) \log \frac{4\zeta}{|\cos \theta - \cos \theta'|\pi} d\theta' = 1, \quad 0 \leq \theta \leq \pi. \quad (3.101)$$

Using that

$$\log |\cos \theta - \cos \theta'| = -\log 2 - 2 \sum_{l=1}^{\infty} \frac{1}{l} \cos l\theta \cos l\theta', \quad (3.102)$$

we write (3.101) as

$$\frac{1}{\pi\zeta} \log \frac{8\zeta}{\pi} \int_{\theta'=0}^{\pi} \tilde{J}(\cos \theta'; \zeta) d\theta' + \frac{2}{\pi\zeta} \sum_{l=1}^{\infty} \frac{1}{l} \cos l\theta \int_{\theta'=0}^{\pi} \tilde{J}(\cos \theta'; \zeta) \cos l\theta' d\theta' = 1, \quad 0 \leq \theta \leq \pi. \quad (3.103)$$

We can expand  $\tilde{J}$  into a Fourier series, because this function is square integrable. Since  $\tilde{J}(\cos \theta; \zeta)$  is even in  $\theta$  and  $\tilde{J}$  is even in  $x$ , the Fourier series expansion of  $\tilde{J}(\cos \theta; \zeta)$  (in  $\theta$ ) is given by

$$\tilde{J}(\cos \theta; \zeta) = \sum_{n=0}^{\infty} \alpha_n(\zeta) \cos 2n\theta. \quad (3.104)$$

Because  $x = \cos \theta$  and  $T_n(\cos \theta) = \cos n\theta$ , where  $T_n$  is the Chebyshev polynomial of order  $n$ , we see that  $\tilde{J}(x; \zeta)$  is expressed into a series of Chebyshev polynomials of even order. Substituting (3.104) into (3.103), we obtain

$$\frac{\alpha_0(\zeta)}{\zeta} \log \frac{8\zeta}{\pi} + \frac{1}{2\zeta} \sum_{n=1}^{\infty} \frac{\alpha_n(\zeta)}{n} \cos 2n\theta = 1, \quad 0 \leq \theta \leq \pi. \quad (3.105)$$

Identifying corresponding Fourier coefficients, we arrive at

$$\alpha_0(\zeta) = \frac{\zeta}{\log \frac{8\zeta}{\pi}}, \quad \alpha_n(\zeta) = 0, \quad n \geq 1. \quad (3.106)$$

Then, by (3.100) and (3.104),

$$J(x; \zeta) = \frac{\zeta}{\log \left( \frac{8\zeta}{\pi} \right) \sqrt{1-x^2}}. \quad (3.107)$$

This is the exact solution of the approximated integral equation (3.99). The accuracy of the solution can be measured by substituting it into the original integral equation (3.49). However, since we have not found an analytical solution for the resulting integral, we can check the integral equation only numerically. By Appendix C, (C.6), the impedance is given by

$$Z = \frac{60}{\sqrt{\varepsilon_r}} \log \frac{8\zeta}{\pi}. \quad (3.108)$$

This is the formula of Howe for the impedance in case  $\zeta > 1/0.35$ .

### 3.5 Numerical Approximation of Current, Potential, and Impedance

In this section, we present a method to approximate the current, potential, and impedance numerically. We use the integral equation (3.49) for the current, i.e.

$$\int_{u=-1}^1 J(u; \zeta) k(x-u; \zeta) du = 1, \quad |x| < 1. \quad (3.109)$$

This equation can be written as

$$\int_{u=0}^1 J(u; \zeta) [k(x-u; \zeta) + k(x+u; \zeta)] du = 1, \quad |x| < 1, \quad (3.110)$$

because  $J(u; \zeta)$  is even in  $u$ . Since the kernel is symmetric with respect to  $x$ , we need only to consider the interval  $0 \leq x < 1$ . As in Subsection 3.4.2, we extract the square root singularity of  $J$  and define  $\tilde{J}$  by

$$\tilde{J}(x; \zeta) = \sqrt{1-x^2} J(x; \zeta), \quad |x| \leq 1. \quad (3.111)$$

Then, with  $x = \cos \theta$  and  $u = \cos \theta'$ , (3.110) turns into

$$\int_{\theta'=0}^{\pi/2} \tilde{J}(\cos \theta'; \zeta) [k(\cos \theta - \cos \theta'; \zeta) + k(\cos \theta + \cos \theta'; \zeta)] d\theta' = 1, \quad 0 \leq \theta \leq \pi/2. \quad (3.112)$$

As in Subsection 3.4.2, we approximate  $\tilde{J}(\cos \theta; \zeta)$  by the Fourier series (3.104). Then, the current  $J(x; \zeta)$  is given by

$$J(x; \zeta) = \frac{1}{\sqrt{1-x^2}} \sum_{n=1}^N \alpha_n(\zeta) T_{2n-2}(x), \quad |x| < 1, \quad (3.113)$$

where  $T_{2n-2}(x)$  is the Chebyshev polynomial of order  $2n-2$ . Substituting the expansion for  $\tilde{J}(\cos \theta; \zeta)$  into the integral equation (3.112), we obtain

$$\sum_{n=1}^N \alpha_n(\zeta) \int_{\theta'=0}^{\pi/2} \cos((2n-2)\theta') [k(\cos \theta - \cos \theta'; \zeta) + k(\cos \theta + \cos \theta'; \zeta)] d\theta' = 1, \quad 0 \leq \theta \leq \pi/2. \quad (3.114)$$

To calculate the coefficients  $\alpha_n(\zeta)$ , we use the method of collocation. We choose points  $0 < \theta_m < 1$ ,  $m = 1, \dots, N$ , and we require

$$\sum_{n=1}^N \alpha_n(\zeta) \int_{\theta'=0}^{\pi/2} \cos((2n-2)\theta') [k(\cos \theta_m - \cos \theta'; \zeta) + k(\cos \theta_m + \cos \theta'; \zeta)] d\theta' = 1, \quad m = 1, \dots, N. \quad (3.115)$$

In matrix notation, this can be written as

$$\mathcal{Z}\alpha = \mathcal{V}, \quad (3.116)$$

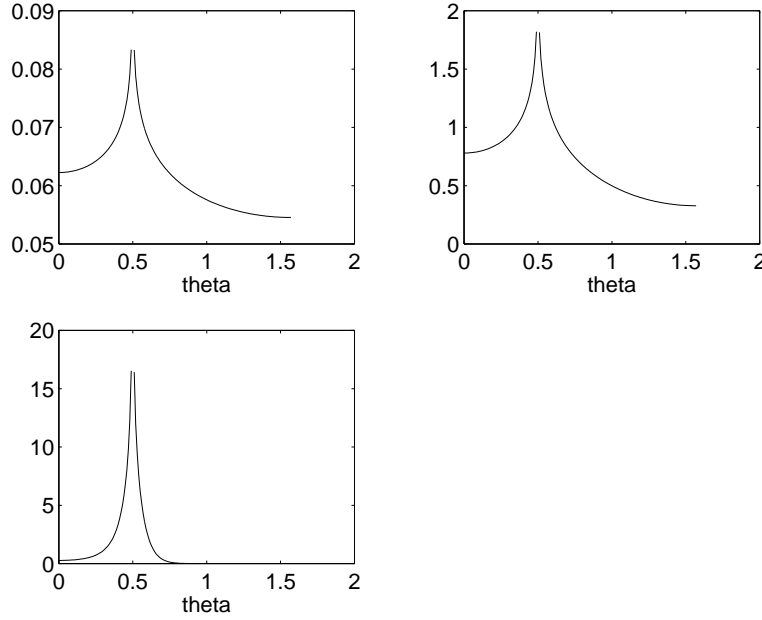
where  $\alpha = (\alpha_1, \dots, \alpha_N)^T$ ,  $\mathcal{V} = (1, \dots, 1)^T$  and

$$\mathcal{Z}(m, n) = \int_{\theta'=0}^{\pi/2} \cos((2n-2)\theta') [k(\cos \theta_m - \cos \theta'; \zeta) + k(\cos \theta_m + \cos \theta'; \zeta)] d\theta'. \quad (3.117)$$

The computation of  $\mathcal{Z}$  is carried out numerically. The integrand of the components  $\mathcal{Z}(m, n)$  is singular in  $\theta' = \theta_m$ . Figure 3.3 shows the integrand as a function of  $\theta$  for  $n = 1$ ,  $\theta_m = 0.5$  and  $\zeta = 50, 1, 0.05$ . We see that for  $\zeta = 50$ , the slope of the curve near the singularity at  $\theta' = \theta_m = 0.5$  is less steep than for  $\zeta = 0.05$ . This can also be seen from the expression of the kernel  $k$  (see (3.48)). To account for the behaviour of the integrand in the numerical integration, we use the following algorithm to compute  $\mathcal{Z}(m, n)$ . The interval  $[0, \pi/2]$  is divided into two intervals:  $int_1 = [0, \theta_m]$  and  $int_2 = [\theta_m, \pi/2]$ . Let  $\mathcal{Z}_i(m, n)$  be defined by

$$\mathcal{Z}_i(m, n) = \int_{int_i} \cos((2n-2)\theta') [k(\cos \theta_m - \cos \theta'; \zeta) + k(\cos \theta_m + \cos \theta'; \zeta)] d\theta'. \quad (3.118)$$

On each interval  $int_i$ , we define a number of subintervals  $nsub\_int_i$  ( $\geq 2$ ) and a number of steps  $nsteps\_int_i$  ( $\geq 1$ ). We approximate  $\mathcal{Z}_i(m, n)$  as follows:



**Figure 3.3** The integrand of  $\mathcal{Z}(m, n)$  with  $\theta_m = 0.5$  and  $n = 1$  for  $\zeta = 50$  (upper left),  $\zeta = 1$  (upper right),  $\zeta = 0.05$  (lower left).

1.  $\mathcal{Z}_i(m, n) = 0$ .
2. The interval is divided into  $nsub\_int_i$  subintervals of equal length and  $nsteps\_int_i := nsub\_int_i - 1$ .
3. On  $nsub\_int_i - 1$  of these intervals the integrand of  $\mathcal{Z}_i(m, n)$  is not singular. The integrals over these intervals are approximated by the Newton Cotes rule on eight points, say with results  $R_1$  and  $R_2$ . Then,  $\mathcal{Z}_i(m, n) = \mathcal{Z}_i(m, n) + R_1 + R_2$ .
4. If  $nsteps\_int_i = 0$  the algorithm is terminated. If  $nsteps\_int_i \neq 0$ , the remaining interval (on which the integrand of  $\mathcal{Z}_i(m, n)$  is singular) is divided into  $nsub\_int_i$  subintervals of equal length and  $nsteps\_int_i := nsub\_int_i - 1$ . The algorithm returns to step 3.

The approximation for  $\mathcal{Z}(m, n)$  is  $\mathcal{Z}_1(m, n) + \mathcal{Z}_2(m, n)$ . We note that on the intervals  $int_1$ , the integrand of  $\mathcal{Z}_i(m, n)$  is always singular in the most right interval of the subdivision, while on the interval  $int_2$ , the integrand is always singular in the most left interval.

The system (3.116) is solved by LU-decomposition. Then, the current is given by (3.113) and the potential by

$$\phi(x, z; \zeta) = \sum_{n=1}^N \alpha_n(\zeta) \int_{\theta'=0}^{\pi/2} \cos((2n-2)\theta') [k(x - \cos \theta', z; \zeta) + k(x + \cos \theta', z; \zeta)] d\theta'. \quad (3.119)$$

To compute the impedance, we deduce

$$\begin{aligned} \int_{x=0}^1 J(x; \zeta) dx &= \sum_{n=1}^N \alpha_n(\zeta) \int_0^1 \frac{T_{2n-2}(x)}{\sqrt{1-x^2}} dx = \\ &= \sum_{n=1}^N \alpha_n(\zeta) \int_0^{\pi/2} \cos((2n-2)\theta) d\theta = \frac{\pi}{2} \alpha_1(\zeta). \end{aligned} \quad (3.120)$$

Then, the impedance is given by

$$Z = \frac{60}{\sqrt{\varepsilon_r}} \frac{\zeta}{\alpha_1(\zeta)}. \quad (3.121)$$

(see Appendix C, (C.6)).



## Numerical Results

In this chapter, we discuss some numerical results obtained by the numerical algorithm described in Section 3.5. We compare the results for impedance and current with the asymptotic results given in Section 3.4 and 3.5.

### 4.1 Results for Current and Impedance for Three Values of $\zeta$

Computing the current and the impedance, we use the following settings for the numerical algorithm of Section 3.5:  $nsub\_int_i = 4$ ,  $nsteps\_int_i = 10$  ( $i = 1, 2$ ). The collocation points will be chosen later. We consider the case  $\zeta = 0.05$  first. In Table 4.1, four collocation point sets and the corresponding results for the expansion coefficients  $\alpha_n$  and for  $Z\sqrt{\varepsilon_r}$  are given. From the expansion coefficients of the first three sets, we see that the set of three collocation

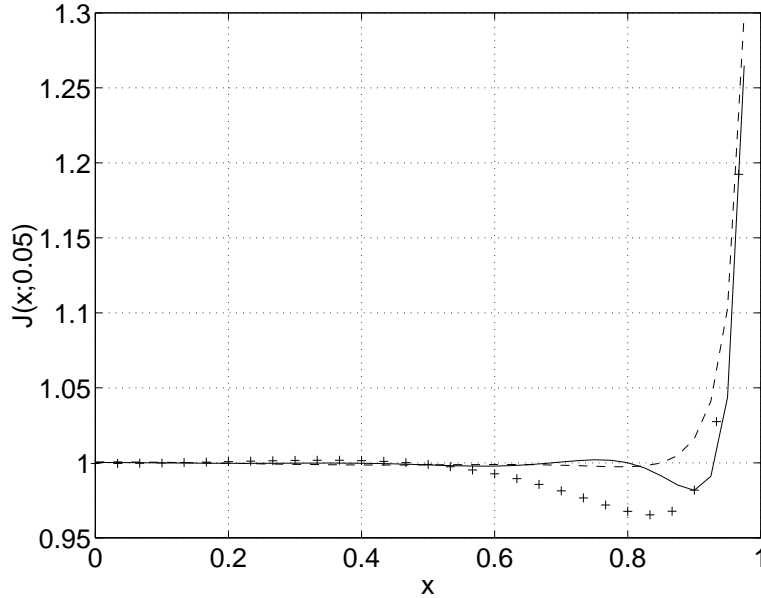
Set	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\alpha_4$	$\alpha_5$	$\alpha_6$	$\alpha_7$	$\alpha_8$	$\alpha_9$	$Z\sqrt{\varepsilon_r}$ (ohm)
1	0.656	-0.381	-0.038							4.575
2	0.660	-0.379	-0.046	-0.008						4.544
3	0.659	-0.380	-0.044	-0.001	0.008	0.006	0.002			4.553
4	0.856	0.014	0.341	0.351	0.296	0.208	0.116	0.046	0.010	3.503

**Table 4.1** Values of the expansion coefficients  $\alpha_n$  and the impedances for four collocation point sets  $x_m$  ( $\zeta = 0.05$ ). Set 1: 0.1, 0.5, 0.9; Set 2: 0.2, 0.4, 0.6, 0.8; Set 3: 0.1, 0.2, 0.35, 0.5, 0.65, 0.8, 0.9; Set 4: 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9.

points (and three expansion functions) yields almost as good approximations of current and impedance as the other two sets. Furthermore, we see that the algorithm is not stable for larger numbers of collocation points, because the expansion coefficients and impedance (i.e. the first expansion coefficient) of the last set differ significantly from the expansion coefficients and impedances of the other three sets. We calculate also the impedance for the set of collocation points  $x_m = i/20, i = 1, 2, \dots, 19$ , with result  $Z\sqrt{\varepsilon_r} = 0.0625$  ohm, which differs even in order from the values in the table. The reason for instability is that the rows in the matrix  $Z$  become almost equal. This can be avoided as follows. The number of collocation points and the number of expansion functions are chosen equal in the algorithm of Section 3.5. However, they are not interrelated. Therefore, it is better to choose these numbers unequal. To avoid instability on one hand, but to satisfy the integral equation (3.49) on the other hand, the number of collocation points should be higher than the number of expansion functions. A

solution of the resulting redundant system (3.116) can be obtained by the Penrose inverse of  $\mathcal{Z}$ .

Figure 4.1 shows the current for the first three sets of collocation points for  $\zeta = 0.05$ . The



**Figure 4.1** The current distribution on the strip ( $0 < x < 1$ ) for  $\zeta = 0.05$  for three sets of collocation points. Solid curve: collocation points 0.1, 0.2, 0.35, 0.5, 0.65, 0.8, 0.9; Dashed curve: collocation points 0.2, 0.4, 0.6, 0.8; Crosses: collocation points 0.1, 0.5, 0.9.

best approximation seems to be obtained with the set of four collocation points. Furthermore, the current is not uniform as in the asymptotic approximation for  $\zeta \ll 1$  in Subsection 3.4.1. However, it is almost uniform on the interval  $|x| \lesssim 0.9$ . This is in correspondence with the result of Figure 3.1, which shows that the asymptotic approximation satisfies the condition  $\phi = 1$  ( $z = 0^+$ ,  $|x| < 1$ ) for  $|x| \lesssim 0.9$ . The numerically obtained impedance equals almost (difference about 3.4%) the impedance of the asymptotic approximation (3.87), i.e.  $Z\sqrt{\varepsilon_r} \approx 4.71$  ohm.

We carry out the same kind of analysis for the case  $\zeta = 20$ . In Table 4.2, three collocation point sets and the corresponding results for the expansion coefficients  $\alpha_n$  and for  $Z\sqrt{\varepsilon_r}$  are given. We see that the first set yields as good approximations of current and impedance as the

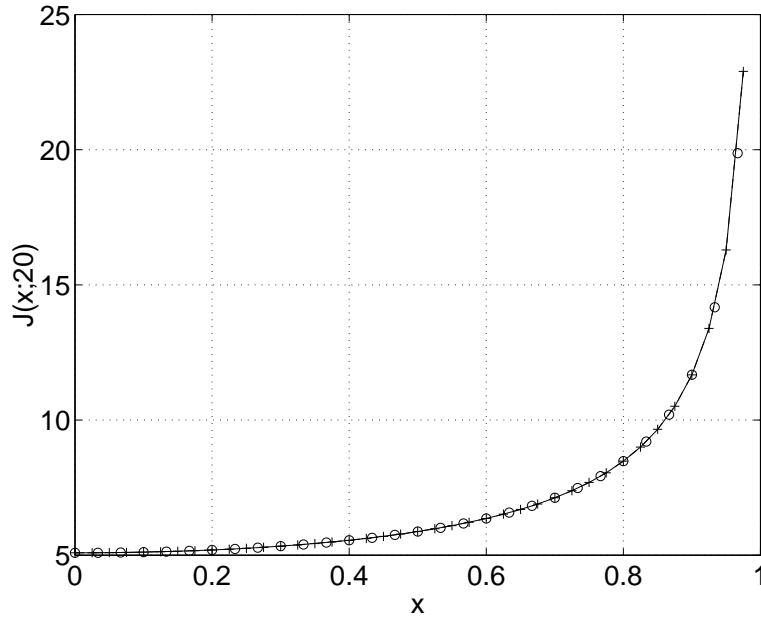
Set	$\alpha_1$	$\alpha_2$	$\alpha_3$	$Z\sqrt{\varepsilon_r}$ (ohm)
1	5.088			235.9
2	5.088	-0.002		235.9
3	5.088	-0.003	0.0002	235.9

**Table 4.2** Values of the expansion coefficients  $\alpha_n$  and the impedances for three collocation point sets  $x_m$  ( $\zeta = 20$ ). Set 1: 0.5; Set 2: 0.3, 0.7; Set 3: 0.1, 0.5, 0.9.

other two sets. For the first set, the current distribution equals a constant times  $1/\sqrt{1-x^2}$ ,

which represents the square root singularity at the edge of the stripline. This is the same current distribution as in the asymptotic approximation (3.107) of the current for  $\zeta \gg 1$  in Subsection (3.4.2). Therefore, the impedance (3.108) of the asymptotic approximation for  $\zeta \gg 1$ , i.e.  $Z\sqrt{\varepsilon_r} \approx 235.8$  ohm, matches perfectly the numerically obtained impedance.

Figure 4.2 shows the current for the three sets of collocation points for  $\zeta = 20$  and the asymptotic approximation (3.107) of the current for  $\zeta \gg 1$ . As seen from the results for the



**Figure 4.2** The current distribution on the strip ( $0 < x < 1$ ) for  $\zeta = 20$  for three sets of collocation points. Solid curve: collocation points 0.1, 0.5, 0.9; Dashed curve: collocation points 0.3, 0.7; Crosses: collocation point 0.5; Circles: asymptotic approximation (3.107).

expansion coefficients, the three sets yield almost the same current distribution.

Finally, we carry out the same kind of analysis for the case  $\zeta = 1$ . This case is not described by one of the asymptotic approximations. However,  $\zeta$  is of order 1 in many applications. Therefore, we consider also this case. In Table 4.3, four collocation point sets and the corresponding results for the expansion coefficients  $\alpha_n$  and for  $Z\sqrt{\varepsilon_r}$  are given. We

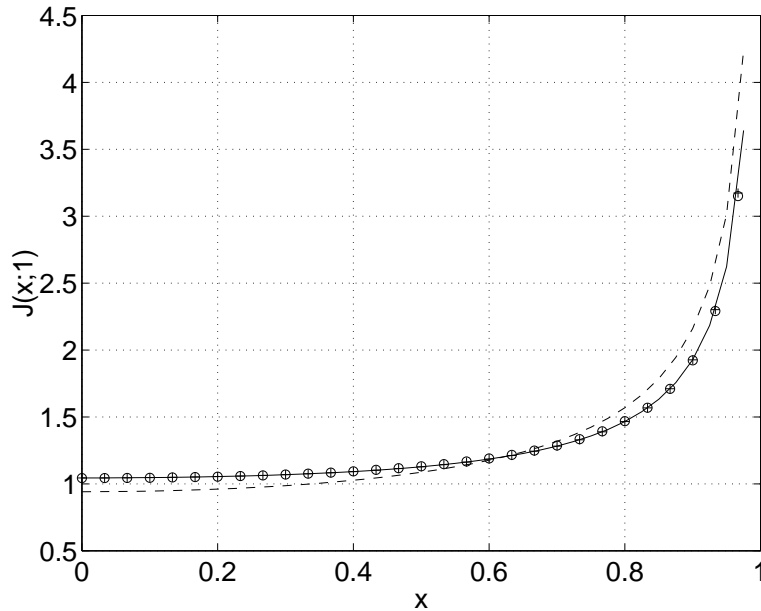
Set	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\alpha_4$	$Z\sqrt{\varepsilon_r}$ (ohm)
1	0.942				63.72
2	0.917	-0.127			65.46
3	0.918	-0.123	0.004		65.40
4	0.918	-0.123	0.004	0.000 ( $-2 \cdot 10^{-5}$ )	65.40

**Table 4.3** Values of the expansion coefficients  $\alpha_n$  and the impedances for four collocation point sets  $x_m$  ( $\zeta = 1$ ). Set 1: 0.5; Set 2: 0.3, 0.7; Set 3: 0.1, 0.5, 0.9; Set 4: 0.2, 0.4, 0.6, 0.8.

see that the approximations of current and impedance of the first set differ slightly from the

approximations of the second set, while the approximations of the second set differ hardly from those of the third and fourth set. As in the case  $\zeta = 20$ , the current distribution of the first set equals a constant times  $1/\sqrt{1-x^2}$ , which represents the square root singularity at the edge of the stripline. So, we see that the current equals its edge behaviour plus a smaller term. Because the asymptotic approximation (3.107) of the current for  $\zeta \gg 1$  equals its edge behaviour, the impedance (3.108) of this asymptotic approximation, i.e.  $Z\sqrt{\varepsilon_r} \approx 56.1$  ohm, differs from the numerically obtained impedance (about 14%). The impedance of the asymptotic approximation (3.87) for  $\zeta \ll 1$ , i.e.  $Z\sqrt{\varepsilon_r} \approx 94.2$  ohm, differs much more from the numerically obtained impedance (about 30%).

Figure 4.3 shows the current for the four sets of collocation points for  $\zeta = 1$ . As seen from

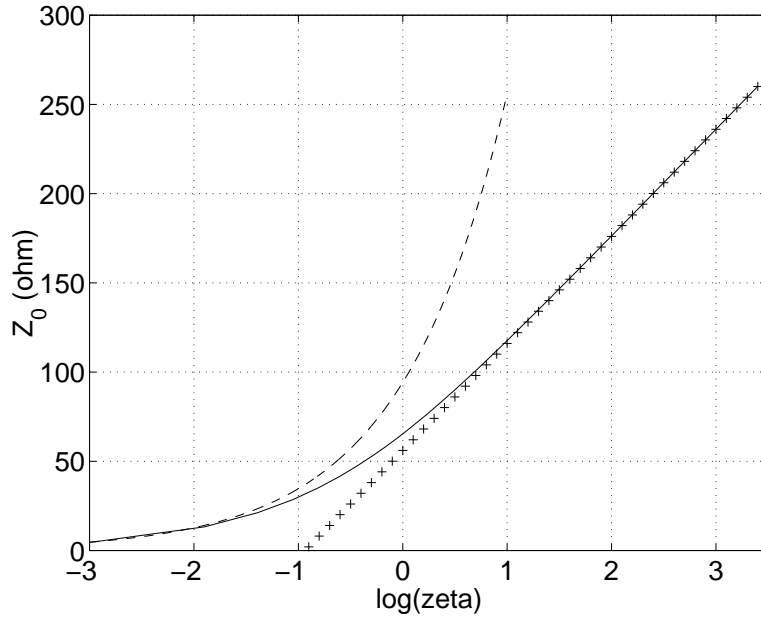


**Figure 4.3** The current distribution on the strip ( $0 < x < 1$ ) for  $\zeta = 1$ . Dashed line: collocation point 0.5; Circles: collocation points 0.3, 0.7; Crosses: collocation points 0.1, 0.5, 0.9; Solid line: collocation points 0.2, 0.4, 0.6, 0.8.

the results for the expansion coefficients, the current of the first set differs slightly from the currents of the other three sets.

## 4.2 The Impedance as a Function of $\zeta$

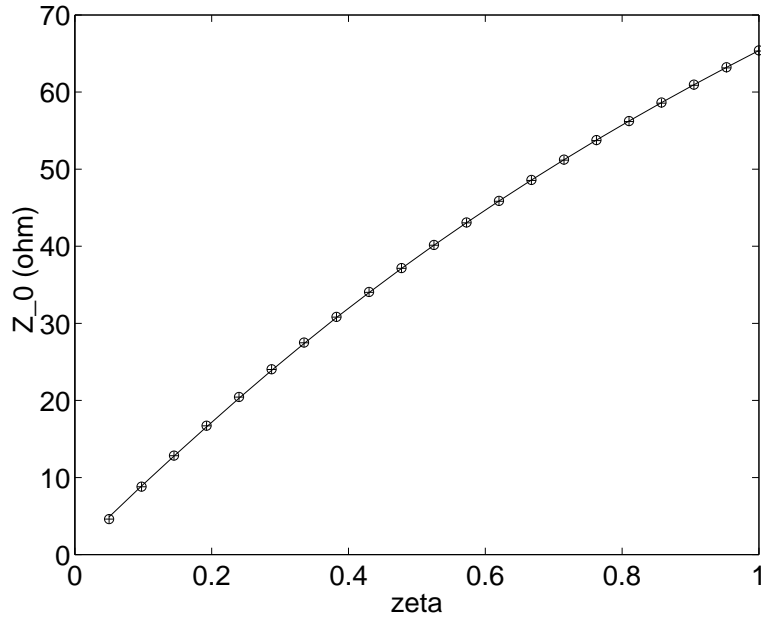
In this section, we compute the impedance with the numerical algorithm of Section 3.5. The following settings are used:  $nsub\_int_i = 4$ ,  $nsteps\_int_i = 10$  ( $i = 1, 2$ ). Furthermore, based on the observations of the previous section, we use three collocation points:  $x_1 = 0.1$ ,  $x_2 = 0.5$ ,  $x_3 = 0.9$ . To compare the numerically obtained impedance with the asymptotic approximations of Section 3.4, we plot  $Z_0 = Z\sqrt{\varepsilon_r}$  as a function of  $\log \zeta$  for  $0.05 \leq \zeta \leq 30$ . Furthermore, we plot the asymptotic approximations (3.87) ( $\zeta \ll 1$ ) and (3.108) ( $\zeta \gg 1$ ) as functions of  $\log \zeta$  for  $0.05 \leq \zeta \leq e$  and  $e^{-1} \leq \zeta \leq 30$ , respectively. Figure 4.4 shows the results. We see that the asymptotic approximation for  $\zeta \ll 1$  yields reasonable results



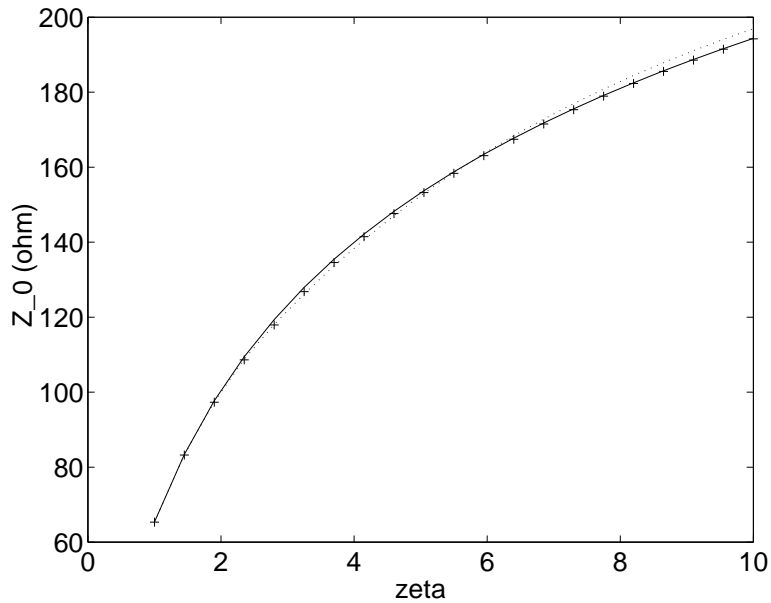
**Figure 4.4** The impedance  $Z_0 = Z\sqrt{\epsilon_r}$  as a function of  $\log \zeta$ . Solid line: Numerical approximation; Dashed line: Asymptotic approximation (3.87) for  $\zeta \ll 1$ ; Crosses: Asymptotic approximation (3.108) for  $\zeta \gg 1$ .

for  $\zeta \lesssim e^{-1} \approx 0.22$ . The asymptotic approximation for  $\zeta \gg 1$  yields reasonable results for  $\zeta \gtrsim e^{0.8} \approx 2.23$ .

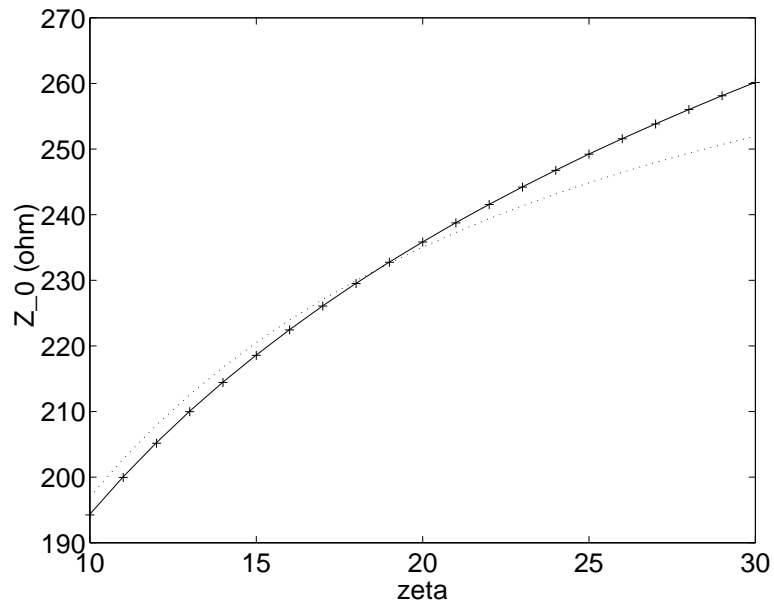
The figures 4.5, 4.6, and 4.7 show the impedance  $Z_0 = Z\sqrt{\epsilon_r}$  as a function of  $\zeta$ , i.e the ratio between the distance  $2h$  between the ground plates and the width  $2a$  of the stripline ( $\zeta = h/a$ ). The results of the formulae of Pozar [7, p. 156] and Howe [6, p. 34-35] are also shown in the figures. In all three figures, we see that the numerical approximation and the impedance formula of Howe match perfectly. The impedance formula of Pozar matches perfectly with the numerical approximation for  $\zeta \lesssim 2$ . For  $2 \lesssim \zeta \lesssim 8$ , there is a slight difference between the two, and for larger values of  $\zeta$  the difference increases. We conclude that for  $\zeta \gtrsim 8$  the impedance formula of Pozar is not a good approximation.



**Figure 4.5** The impedance  $Z_0 = Z\sqrt{\varepsilon_r}$  as a function of the ratio  $\zeta$ ,  $0.05 \leq \zeta \leq 1$ . Solid line: numerical approximation, Crosses: impedance formula of Howe, Circles: Impedance formula of Pozar.



**Figure 4.6** The impedance  $Z_0 = Z\sqrt{\varepsilon_r}$  as a function of the ratio  $\zeta$ ,  $1 \leq \zeta \leq 10$ . Solid line: numerical approximation, Crosses: impedance formula of Howe, Dotted line: Impedance formula of Pozar.



**Figure 4.7** The impedance  $Z_0 = Z\sqrt{\varepsilon_r}$  as a function of the ratio  $\zeta$ ,  $10 \leq \zeta \leq 30$ . Solid line: numerical approximation, Crosses: impedance formula of Howe, Dashed line: Impedance formula of Pozar.





## CHAPTER 5

## Survey and Conclusions

In this report, a long thin stripline is considered, which is symmetrically positioned in a stripline environment (see Figure 1.1). The stripline is characterized by a set of parameters specified in Chapter 1. As explained in this chapter, stripline and ground plates can be considered as infinitely long. In Chapter 2, it is shown that the stripline can be modeled as infinitely thin and perfectly conducting. For more details, we refer to the recapitulation of this chapter in Section 2.5.

On the stripline, a (surface) current is assumed, being a planar wave in the length direction of the stripline. Then, the electric and magnetic field in the dielectric, as well as the (surface) charge density on the stripline are also planar waves in length direction. In this way, each quantity is written as the multiplication of its wave behaviour, which depends only on time and length direction, and its ‘stationary’ part (with respect to the propagation of the wave), which depends only on the directions perpendicular to the length direction (or, propagation direction). Balance equations and boundary conditions for these ‘stationary’ quantities are deduced, whereby the ground plates are assumed perfectly conducting and the stripline is modeled as infinitely thin and perfectly conducting with the results of Chapter 2 (see Section 2.5). Due to symmetry with respect to the plane of the stripline, only the space between this plane and one of the ground plates is considered.

The wave in the stripline is TEM (transverse electromagnetic) if and only if the field components in propagation direction are zero. It is shown that the stripline supports a TEM wave, only if the wave number of the TEM wave equals the wave number of the dielectric. This is in correspondence with the analysis in Chapter 2, in which it is shown that these wave numbers become equal if a strip of finite thickness becomes perfectly conducting. Then, a TEM wave with the wave number of the dielectric is assumed on the stripline. It is shown that the electric and magnetic field in the dielectric are perpendicular. Furthermore, the ‘stationary’ electric and magnetic fields are conservative by which they can be written as gradients of scalar functions. These scalar functions, or potentials, are harmonic and satisfy the Cauchy-Riemann equations. A mixed boundary value problem for the electric potential and the current is deduced. The solution to this problem is unique only if the values of the electric potential at the stripline and the ground plates are specified. The difference between these potential values can be interpreted as the generator of the current on the stripline. From the resulting boundary value problem, it can be shown that the current has a square root singularity at the edges of the stripline.

The boundary value problem is solved in two ways: 1. First calculation of the current and then calculation of the potential, 2. First calculation of the potential and then calculation of the current. Both approaches lead to a Fredholm equation of the first kind, in the first

approach for the current and in the second approach for an auxiliary function. Consistency of both approaches is shown by deducing a relation between the current and the aforementioned auxiliary function. Furthermore, it is shown that the integral equation for the auxiliary function can be transformed into the integral equation for the current. In both approaches, it is shown that the current is symmetric with respect to the width of the stripline.

With both approaches, asymptotic approximations of the current and the (characteristic) impedance are deduced for the cases  $h/a \ll 1$  and  $h/a \gg 1$ . Here,  $2h$  is the distance between the ground plates and  $2a$  is the width of the stripline. The asymptotic approximations show that the current is almost uniform for  $h/a \ll 1$  and almost equal to its edge behaviour for  $h/a \gg 1$ . The asymptotic approximations of the impedance are exactly or almost equal to formulae of Pozar [7, p. 156] and Howe [6, p. 34-35]. The solution of the Fredholm equation of the first approach is also solved numerically by a collocation method. Hereby, the current is written as the multiplication of its (singular) edge behaviour and a regular function. The regular function is expanded into Chebyshev polynomials of even order. The number of collocation points is chosen equal to the number of expansion functions for the current. From the numerical results, we draw the following conclusions.

- For  $h/a \ll 1$ , three or four collocation points are needed for numerical convergence, while for  $h/a \gg 1$  only one is needed. For  $h/a \approx 1$ , about two collocation points are needed.
- The collocation method becomes unstable for large (about 9) numbers of collocation points. This can be avoided by choosing the number of collocation points not equal to the number of expansion functions.
- For  $h/a \ll 1$  ( $h/a = 0.05$ ), the current is almost uniform. Furthermore, the numerically obtained impedance differs about 4% from the asymptotic result.
- For  $h/a \gg 1$  ( $h/a = 20$ ), the current equals its edge behaviour as in the asymptotic approximation for this case. Furthermore, the impedance matches perfectly with the asymptotic approximation.
- The asymptotic approximation for  $h/a \ll 1$  is valid for  $h/a \lesssim e^{-1} \approx 0.22$  and the asymptotic approximation for  $h/a \gg 1$  for  $h/a \gtrsim e^{0.8} \approx 2.23$ .
- The numerical results for the impedance match perfectly with the formula of Howe [6, p. 34-35]. They match also perfectly with the formula of Pozar [7, p. 156] in the range  $h/a \lesssim 2$ . For  $2 \lesssim h/a \lesssim 8$ , they match almost, while for  $h/a \gtrsim 8$ , they do not match. Hence, the impedance formula of Pozar is not a good approximation for  $h/a \gtrsim 8$ .

# Bibliography

- [1] O.C. BALAN, *Antenna Design*, Final Report of the Post Graduate Programme “Mathematics for Industry”, Eindhoven University of Technology, Eindhoven (2000).
- [2] D.J. BEKERS, *Finitely Large Phased Arrays of Microstrip Antennas, Analysis and Design*, Final Report of the Post Graduate Programme “Mathematics for Industry”, Eindhoven University of Technology, Eindhoven (2001).
- [3] R.E. COLLIN, *Field Theory of Guided Waves*, IEEE Press, New York (1991).
- [4] A. ERDELYI, W. MAGNUS, F. OBERHETTINGER, F. TRICOMI, *Tables of Integral Transforms (vol. 1)*, McGraw-Hill, New York (1954).
- [5] I.S. GRADSHTEYN, I.M. RYZHIK, *Table of Integrals, Series and Products*, Academic Press, Orlando (1965).
- [6] H. HOWE, *Stripline Circuit Design*, Artech House, Burlington, Mass. (1974).
- [7] D.M. POZAR, *Microwave Engineering*, John Wiley, New York (1998).
- [8] T. ULICEVIC, *Skin Effect in a Gradient Coil*, Final Report of the Post Graduate Programme “Mathematics for Industry”, Eindhoven University of Technology, Eindhoven (2001).
- [9] H.M. WHEELER, *Transmission-Line Properties of Parallel Wide Strips by a Conformal-Mapping Approximation*, IEEE Trans., Vol. MTT-12, 1964, p. 280-289.



## APPENDIX A

## Calculation of $K(\alpha, z; \zeta)$

In this appendix, we calculate the integral  $K(\alpha, z; \zeta)$  as defined by (3.69) for  $0 < z < 1$ . Since  $K$  is odd in  $\alpha$ , we need to consider  $\alpha > 0$  only. The integral expression for  $K$  can be written as

$$\begin{aligned} K(\alpha, z; \zeta) &= \frac{1}{\pi\zeta} \int_0^\infty \frac{\sin(\alpha s) \sinh(\zeta s(1-z))}{s^2 \cosh(\zeta s)} ds = \\ &= \frac{1}{2\pi i \zeta} \lim_{\varepsilon \downarrow 0} \left( \int_\varepsilon^\infty + \int_{-\infty}^{-\varepsilon} \right) I(\alpha, z; s, \zeta) ds, \end{aligned} \quad (\text{A.1})$$

where  $I(\alpha, z; s, \zeta)$  is defined by

$$I(\alpha, z; s, \zeta) = \frac{\sinh(\zeta s(1-z)) e^{i\alpha s}}{s^2 \cosh(\zeta s)}. \quad (\text{A.2})$$

To calculate the integral

$$\left( \int_\varepsilon^\infty + \int_{-\infty}^{-\varepsilon} \right) I(\alpha, z; s, \zeta) ds, \quad \varepsilon < \frac{\pi}{2\zeta}, \quad (\text{A.3})$$

we close the path of integration by two semi-circles

$$\begin{aligned} C_\varepsilon &: |s| = \varepsilon, \operatorname{Im} s \geq 0, \\ C_N &: |s| = \frac{N\pi}{\zeta}, \operatorname{Im} s \geq 0. \end{aligned} \quad (\text{A.4})$$

We call the whole contour  $C_{N,\varepsilon}$ , which is positively oriented. The poles of  $I(\alpha, z; s, \zeta)$  are

$$s = 0, \quad s = \frac{1}{\zeta} \left( \frac{\pi}{2} + k\pi \right) i, \quad k \in \mathbb{Z}. \quad (\text{A.5})$$

All poles are simple. According to the residual theorem,

$$\begin{aligned} \int_{C_{N,\varepsilon}} I(\alpha, z; s, \zeta) ds &= 2\pi i \sum_{k=0}^{N-1} \operatorname{Res}_{s=(\frac{\pi}{2}+k\pi)i/\zeta} I(\alpha, z; s) = \\ &= -2\pi i \zeta \sum_{k=0}^{N-1} (-1)^k \frac{\sin((k+\frac{1}{2})\pi(1-z))}{(k+\frac{1}{2})^2 \pi^2} \exp\left(-\frac{\alpha}{\zeta} \left(k+\frac{1}{2}\right) \pi\right). \end{aligned} \quad (\text{A.6})$$

Let  $N \rightarrow \infty$ . Then, the integral along  $C_N$  vanishes by the lemma of Jordan. We obtain

$$\begin{aligned} & \left( \int_{\varepsilon}^{\infty} + \int_{-\infty}^{-\varepsilon} \right) I(\alpha, z; s, \zeta) ds = - \int_{C_{\varepsilon}} I(\alpha, z; s, \zeta) ds + \\ & -2\pi i \zeta \sum_{k=0}^{\infty} (-1)^k \frac{\sin((k + \frac{1}{2})\pi(1-z))}{(k + \frac{1}{2})^2 \pi^2} \exp\left(-\frac{\alpha}{\zeta} \left(k + \frac{1}{2}\right) \pi\right). \end{aligned} \quad (\text{A.7})$$

For  $\varepsilon \downarrow 0$ , it results

$$\int_{C_{\varepsilon}} I(\alpha, z; s, \zeta) ds \rightarrow -\pi i \operatorname{Res}_{s=0} I(\alpha, z; s, \zeta) = -\pi i \zeta (1-z). \quad (\text{A.8})$$

Then, it follows from (A.7) and (A.8) that

$$K(\alpha, z; \zeta) = \frac{1}{2}(1-z) - \frac{1}{\pi^2} \sum_{k=0}^{\infty} (-1)^k \frac{\sin((k + \frac{1}{2})\pi(1-z))}{(k + \frac{1}{2})^2} \exp\left(-\frac{\alpha}{\zeta} \left(k + \frac{1}{2}\right) \pi\right), \quad \alpha > 0, \quad (\text{A.9})$$

or,

$$K(\alpha, z; \zeta) = \frac{1}{2}(1-z) - \frac{1}{\pi^2} \sum_{k=0}^{\infty} \frac{\cos((k + \frac{1}{2})\pi z)}{(k + \frac{1}{2})^2} \exp\left(-\frac{\alpha}{\zeta} \left(k + \frac{1}{2}\right) \pi\right), \quad \alpha > 0. \quad (\text{A.10})$$

It can be shown that  $K(0, z; \zeta) = 0$  by expanding the function  $(1-|z|)/2$  into a Fourier cosine series on the interval  $|z| \leq 2$ . This is in correspondence with the fact that  $K$  is odd in  $\alpha$ . For  $\alpha \in \mathbb{R}$ ,  $\alpha \neq 0$ ,  $K$  is given by

$$\operatorname{sign}(\alpha) K(\alpha, z; \zeta) = \frac{1}{2}(1-z) - \frac{1}{\pi^2} \sum_{k=0}^{\infty} \frac{\cos((k + \frac{1}{2})\pi z)}{(k + \frac{1}{2})^2} \exp\left(-\frac{|\alpha|}{\zeta} \left(k + \frac{1}{2}\right) \pi\right). \quad (\text{A.11})$$

## APPENDIX B

## The Potential near the Boundary of the Plate

For numerical computation, it is important to know the behaviour of the current near the edges  $x = \pm 1$  of the infinitely thin, perfectly conducting stripline in Chapter 3. Because of symmetry, we need to consider the edge  $x = 1$  only. The potential and current near this edge is determined by the differential equation (3.31)<sup>1</sup> and the boundary conditions (3.31)<sup>3,4,5</sup>. With  $\eta = \zeta z$ , the differential equation turns into

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial \eta^2} = 0, \quad (\text{B.1})$$

and the boundary conditions turn into

$$\begin{aligned} \frac{\partial \phi}{\partial \eta} &= 0, & \eta = 0, \quad x > 1, \\ \zeta \frac{\partial \phi}{\partial \eta} &= -J(x; \zeta), & \eta = 0^+, \quad -1 < x < 1, \\ \phi &= 1, & \eta = 0^+, \quad -1 < x < 1. \end{aligned} \quad (\text{B.2})$$

To analyse the potential and the current near the edge, we introduce polar coordinates

$$x = 1 + r \cos \theta, \quad \eta = r \sin \theta. \quad (\text{B.3})$$

Then, the differential equation for  $\phi$  turns into

$$\Delta \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0. \quad (\text{B.4})$$

and the boundary conditions turn into

$$\begin{aligned} \frac{\partial \phi}{\partial \theta} &= 0, & \theta = 0, \\ \frac{1}{r} \frac{\partial \phi}{\partial \theta} &= \frac{1}{\zeta} J(1 - r; \zeta), & \theta = \pi, \\ \phi &= 1, & \theta = \pi. \end{aligned} \quad (\text{B.5})$$

The potential  $\phi(r, \theta)$  is determined by (B.4) and (B.5)<sup>1,3</sup>, and the current by (B.5)<sup>2</sup>. To obtain homogeneous boundary conditions, we introduce the function  $\varphi(r, \theta)$  by

$$\phi(r, \theta) = 1 + \varphi(r, \theta). \quad (\text{B.6})$$

The boundary problem for  $\varphi$  is the same as that for  $\phi$ , except for the condition (B.5)<sup>3</sup> which turns into

$$\varphi = 0, \quad \theta = \pi. \quad (\text{B.7})$$

To solve  $\varphi$  we apply separation of variables,

$$\varphi(r, \theta) = R(r)\Theta(\theta). \quad (\text{B.8})$$

We obtain

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda, \quad (\text{B.9})$$

where  $\lambda$  is a constant. For  $\lambda = 0$  it can easily be seen that  $\Theta(\theta) = 0$ . So,  $\lambda = 0$  is not an eigenvalue. We put  $\lambda = \mu^2$ ,  $\mu$  complex. Then,

$$\Theta''(\theta) + \mu^2 \Theta(\theta) = 0, \quad r^2 R''(r) + rR'(r) - \mu^2 R(r) = 0. \quad (\text{B.10})$$

From the boundary conditions for  $\varphi$ , it follows that  $\Theta'(0) = 0$  and  $\Theta(\pi) = 0$ . Then, from the differential equation for  $\Theta$ , we obtain  $\mu = \frac{1}{2} + m$  ( $m \in \mathbb{Z}$ ), and

$$\lambda = \left(\frac{1}{2} + m\right)^2, \quad \Theta(\theta) = \cos\left(\left(\frac{1}{2} + m\right)\theta\right), \quad m = 0, 1, 2, \dots \quad (\text{B.11})$$

Consequently, the differential equation for  $R(r)$  has solutions  $r^{1/2+m}$  ( $m \in \mathbb{Z}$ ), which yields the following collection of eigenvalues  $\lambda$  with eigenfunctions  $\varphi_m$ :

$$\lambda = \left(\frac{1}{2} + m\right)^2, \quad \varphi_m(r, \theta) = r^{1/2+m} \cos\left(\left(\frac{1}{2} + m\right)\theta\right), \quad m \in \mathbb{Z}. \quad (\text{B.12})$$

For the potential  $\phi$ , we obtain

$$\phi(r, \theta) = 1 + \sum_{m=-\infty}^{\infty} A_m r^{1/2+m} \cos\left(\left(\frac{1}{2} + m\right)\theta\right). \quad (\text{B.13})$$

The electric energy should be finite, which implies

$$\int_V |\mathbf{E}|^2 dV < \infty, \quad (\text{B.14})$$

where  $V$  a volume around the edge. Since  $\mathbf{E} = -\text{grad } \phi$ , we need to require that  $A_m = 0$  for  $m < 0$ . Then, it follows from (B.5)<sup>2</sup> that

$$J(1-r; \zeta) = r^{-1/2}, \quad r \downarrow 0, \quad (\text{B.15})$$

or,

$$J(x; \zeta) = \frac{1}{\sqrt{1-x}}, \quad x \uparrow 1. \quad (\text{B.16})$$



## APPENDIX C

## Calculation of the Characteristic Impedance

In this appendix, we calculate the characteristic impedance of the infinitely thin and long, perfectly conducting stripline as described in Chapter 1. The characteristic impedance is defined by

$$Z = \frac{\sqrt{\mu_0 \varepsilon}}{C} \quad (\text{C.1})$$

(see [7, p. 178]), where  $C$  is the capacitance of the stripline,

$$C = \frac{Q_S}{V}. \quad (\text{C.2})$$

Here,  $Q_S$  is the total charge on the strip per unit of length,

$$Q_S = \int_{x=-a}^a \rho_S(x) dx, \quad (\text{C.3})$$

and  $V$  is the potential difference between one of the ground plates and the stripline. The charge density  $\rho_S(x)$  on the strip per unit of length is defined by (3.4). To calculate  $Q_S$ , we use the boundary condition (3.13)<sup>1</sup> with  $\beta = k$ , which relates the charge density  $\rho_S(x)$  to the current density  $J_y(x)$ . We deduce

$$Q_S = \frac{\varepsilon \omega \mu_0}{k} \int_{x=-a}^a J_y(x) dx = \frac{4\varepsilon \phi_0}{\zeta} \int_{x=0}^1 J(x; \zeta) dx. \quad (\text{C.4})$$

Here, we have used the definitions (3.30) of  $\hat{J}(\hat{x})$  and  $j_y$ , and the fact that  $\hat{J}(\hat{x})$  is even in  $x$ . Note that  $\hat{J}(\hat{x})$  is also denoted by  $J(x; \zeta)$ . Taking  $V = \phi_0$ , we obtain

$$C = \frac{4\varepsilon}{\zeta} \int_{x=0}^1 J(x; \zeta) dx \quad (\text{C.5})$$

and

$$Z = \frac{\zeta}{4\sqrt{\varepsilon_r}} \underbrace{\sqrt{\frac{\mu_0}{\varepsilon_0}}}_{\approx 120\pi \text{ ohm}} \left( \int_{x=0}^1 J(x; \zeta) dx \right)^{-1}, \quad (\text{C.6})$$

where  $\varepsilon = \varepsilon_r \varepsilon_0$ .