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H_2 Optimal Controllers with Measurement Feedback for Discrete-time Systems – Flexibility in Closed-loop Pole Placement

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For a general H_2 optimal control problem, at first all H_2 optimal measurement feedback controllers are characterized and parameterized, and then attention is focused on controllers with estimator based architecture. The H_2 optimal control problem with strictly proper controllers and the H_2 optimal control problem with proper controllers are essentially different and hence clearly delineated. Next, estimator based H_2 optimal controllers are characterized and parameterized. Also, systematic methods of designing them are presented. Three different estimator structures, prediction, current, and reduced order estimators, are considered. Since in general there exist many H_2 optimal measurement feedback controllers, utilizing such flexibility and freedom, one can place the closed-loop poles at more desirable locations while still preserving H_2 optimality. All the design algorithms developed here are easily computer implementable.

Key words: H_2 Optimal Control, Multivariable Control, Discrete Time Systems.

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1 Introduction

For discrete-time systems, a general H_2 optimal control problem which utilizes measurement feedback is considered. The problem is to find an internally stabilizing controller which attains the infimum of the H_2 norm of a transfer function from an exogenous disturbance to a controlled output of a given linear shift invariant system, while utilizing the measured output.

In contrast to the continuous-time, for discrete-time systems, one basically encounters two different problems, firstly the minimization of the closed-loop H_2 norm over all strictly proper internally stabilizing controllers, and secondly the minimization of the closed-loop H_2 norm over all proper internally stabilizing controllers. This is so because, for discrete-time systems, the minima for these two problems are in general different. For each of these problems, two main aspects are addressed in this paper. The first one deals with the characterization and parameterization of all H_2 optimal measurement feedback controllers. The second aspect focuses attention on controllers with estimator based architecture, and for such controllers, it characterizes and develops methods for constructing all H_2 optimal controllers. Also, it investigates the freedom and constraints that arise in closed-loop pole placement while preserving H_2 optimality, and in so doing, it solves what can be coined as an H_2 optimal control problem with simultaneous pole placement. Note that this problem studies among H_2 optimal controllers, the available flexibility in the location of the closed-loop poles. It does not compromise H_2 performance in favour of better pole locations.

In recent years, there has been a renewed interest in discrete-time H_2 optimal control utilizing the state or measurement feedback. This initially was induced by an interest in sampled data H_2 control which can be transformed into a discrete-time H_2 control problem (see (Bamieh & Pearson 1992; Chen & Francis 1991)). In (Trentelman & Stoorvogel 1995; Chen & Francis 1992) the H_2 control problem for discrete-time systems was solved and it became clear that H_2 optimal controllers are in general non-strictly proper. In other words, the infimum over the class of strictly proper controllers is in general larger than the infimum over the class of proper controllers (in more classical references such as (Anderson & Moore 1979; Kwakernaak & Sivan 1972) this fact was not clearly presented). Also, (Trentelman & Stoorvogel 1995) developed the necessary and sufficient conditions under which an H_2 optimal measurement feedback controller exists.

In (Chen *et al.* 1994), a complete treatment of the H_2 optimal control problem was provided for the case when the entire state is available for feedback. More specifically, it characterized all H_2 optimal state feedback controllers including static as well as dynamic ones. Moreover, it solved the H_2 optimal control

problem with simultaneous pole placement when the entire state is available for feedback. In order to do so, for the set of H_2 optimal state feedback controllers, it constructed an associated set of complex numbers that points out explicitly the freedom and constraints one has in closed-loop pole placement. This set is called the set of H_2 *optimal fixed modes*. For any H_2 optimal state feedback, the closed-loop poles include the elements of this set. A significant aspect of the work in (Chen *et al.* 1994) is the development of a computationally feasible step by step algorithm called ‘*Optimal Gains and Fixed Modes*’, abbreviated as (*OGFM*). Given a matrix quintuple that specifies the given H_2 optimal state feedback control problem, (*OGFM*) algorithm computes, among other things, the set of all H_2 optimal static state feedback gains and the associated set of H_2 optimal fixed modes. A software package implementing the (*OGFM*) algorithm in *MATLAB* is given in (Lin *et al.* 1991) and (Lin *et al.* 1992).

A lot has been done but there still remains a gap regarding the complete characterization of all H_2 optimal controllers with estimator based architecture, and the investigation of freedom and constraints they offer in closed-loop pole placement. The intention of this paper is to fill this gap. In fact, the spirit of this paper is to capture, while using measurement feedback controllers rather than state feedback controllers, all the aspects of H_2 optimal control that were developed in (Chen *et al.* 1994). We characterize and parameterize estimator based H_2 optimal controllers while considering three different estimator structures: prediction, current, and reduced order estimators. We construct explicitly the set of all H_2 optimal measurement feedback controllers for each chosen estimator based architecture, and also certain associated sets of H_2 *optimal fixed modes*. All the theoretical aspects of these sets are developed in such a way that the explicit construction of these sets can be computationally accomplished by merely using the (*OGFM*) algorithm.

The above task of investigating all the aspects of H_2 optimal control while utilizing estimator based controllers, turns out to be complex and involved. The basic reason for complexity arises from the fact that the traditional separation principle does not hold in general. To expand on this, let us note that in the literature on control, the notion of a controller with estimator based architecture is very much tied with the notion of the separation principle. Two implications arise from the traditional separation principle. The first one relates to the existence of an H_2 optimal measurement feedback controller. It says that, whenever an H_2 optimal static state feedback controller and an H_2 optimal state estimator exist, there exists as well an H_2 optimal estimator based measurement feedback controller. This first implication of the traditional separation principle is in general *false* as pointed out in (Stoorvogel 1992). The second implication of the separation principle relates to the actual construction of an H_2 optimal measurement feedback controller. Suppose there exists an H_2 optimal measurement feedback controller. Then, the traditional

separation principle implies that an H_2 optimal measurement feedback controller can be obtained by cascading together *any* H_2 optimal estimator and *any* H_2 optimal static state feedback controller. It becomes obvious from the development given in this paper that the second implication of the separation principle is in general *not true* either.

This paper is the discrete-time version of the paper (Saber *et al.* 1994). Although there are some conceptual similarities between the H_2 optimal control problems for continuous- and discrete-time systems, there are several fundamental differences between them. These fundamental differences arise mainly from the fact that, in contrast to continuous-time, for discrete-time systems, the infimum of the H_2 norm over the class of strictly proper controllers is in general different from the infimum of the H_2 norm over the class of proper controllers.

This paper is organized as follows. In the next section, we recall some preliminary results needed for our development. Among several results, an important result we recall here is that the task of designing H_2 optimal controllers for a given system Σ , reduces to the task of designing controllers that solve the disturbance decoupling problem for one or the other of two new auxiliary systems.

Also, it turns out that the problem of simultaneous closed-loop pole placement for Σ while preserving H_2 optimality can be recast as a disturbance decoupling problem with closed-loop pole placement. In Section 3, we characterize and parameterize all H_2 optimal measurement feedback controllers for Σ . In fact, in order to do so, utilizing the results of Section 2, we characterize and parameterize all measurement feedback controllers that solve the disturbance decoupling problem. Next, we focus our attention on measurement feedback controllers with estimator based architecture. In this regard, in Section 4, we review the architecture of prediction, current, and reduced order estimator based controllers. In Section 5, we characterize and parameterize all prediction, current, and reduced order estimator based controllers that solve the disturbance decoupling problem for the auxiliary system Σ_{PQ} . We also characterize here the flexibility one has in the closed-loop pole placement while utilizing such controllers. This is done by explicitly constructing the set of fixed modes of such controllers. By modifying the direct feedthrough term of these proper controllers, one can achieve additional flexibility in the estimator gain. This is worked out in Section 6 for the case of prediction estimators. Finally, in Section 7, we draw the conclusions of our work.

Throughout the paper, A' denotes the transpose of A , I denotes an identity matrix, while I_k denotes the identity matrix of dimension $k \times k$. \mathbb{C} , \mathbb{C}° , and \mathbb{C}^\ominus respectively denote the whole complex plane, the unit circle and the open unit disc. $\lambda(A)$ denotes the set of eigenvalues of A . A^\dagger denotes the Moore-Penrose

generalized inverse of the matrix A . A matrix is said to be stable if all its eigenvalues are in \mathbb{C}° . Similarly, a transfer function $G(z)$ is said to be stable if all its poles are in \mathbb{C}° . $\ker V$ and $\text{im } V$ respectively denote the kernel and the image of V . Given \mathcal{X} a subspace of \mathbb{R}^n or \mathbb{C}^n , and a matrix $N \in \mathbb{R}^{n \times m}$, we define

$$N^{-1}\mathcal{X} := \{z \in \mathbb{R}^m \mid Nz \in \mathcal{X}\}.$$

Given a stable transfer function $G(z)$, as usual, its H_2 norm is defined by

$$\|G\|_2 = \left(\frac{1}{2\pi} \text{trace} \int_{-\pi}^{\pi} G(e^{j\varphi})G'(e^{-j\varphi})d\varphi \right)^{1/2}.$$

2 Preliminaries

We consider the following system Σ characterized by,

$$\Sigma : \begin{cases} \sigma x = Ax + Bu + Ew \\ y = C_1x + D_1w \\ z = C_2x + D_2u, \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ is a state, $u \in \mathbb{R}^m$ is a control input, $w \in \mathbb{R}^l$ is an exogenous disturbance input, $y \in \mathbb{R}^p$ is a measured and $z \in \mathbb{R}^q$ is a controlled output. Moreover, σ denotes the time shift,

$$(\sigma x)(k) = x(k+1).$$

Next, we consider proper controllers Σ_c of the form

$$\Sigma_c : \begin{cases} \sigma v = Jv + Ly \\ u = Mv + Ny. \end{cases} \quad (2)$$

Clearly Σ_c , as given in (2), is strictly proper when $N = 0$.

We use the following notations. The closed-loop system consisting of the plant Σ and a controller Σ_c is denoted by $\Sigma \times \Sigma_c$. A controller Σ_c is said to be internally stabilizing the system Σ , if the closed-loop system $\Sigma \times \Sigma_c$ is internally stable. Also, a controller Σ_c is said to be admissible if it provides internal stability for the closed-loop system $\Sigma \times \Sigma_c$. The transfer matrix from w to z of $\Sigma \times \Sigma_c$ is denoted by $T_{zw}(\Sigma \times \Sigma_c)$.

If we say that a system or a subsystem Σ_* is characterized by a quadruple (A, B, C, D) , we mean that the dynamic equations of the system are given by,

$$\Sigma_* : \begin{cases} \sigma x = Ax + Bu \\ y = Cx + Du, \end{cases} \quad (3)$$

where u and y are respectively some input (control input or disturbance) and output (measured or controlled output) of Σ_* .

Definition 1 Consider a linear system Σ_* characterized by the matrix quadruple (A, B, C, D) . Then,

- (i) The \mathbb{C}_g -stabilizable weakly unobservable subspace $\mathcal{V}_g(\Sigma_*)$ is defined as the largest subspace \mathcal{V} of \mathbb{R}^n for which there exists F such that \mathcal{V} is $(A + BF)$ -invariant and contained in $\ker(C + DF)$ and such that the eigenvalues of $(A + BF)|_{\mathcal{V}_g}$ are contained in $\mathbb{C}_g \subseteq \mathbb{C}$ for some F .
- (ii) The \mathbb{C}_g -detectable strongly controllable subspace $\mathcal{S}_g(\Sigma_*)$ is defined as the smallest subspace \mathcal{S} of \mathbb{R}^n for which there exists K such that \mathcal{S} is $(A + KC)$ -invariant and contains $\text{im}(B + KD)$ and such that the eigenvalues of the map which is induced by $(A + KC)$ on the factor space $\mathbb{R}^n/\mathcal{S}_g$ are contained in $\mathbb{C}_g \subseteq \mathbb{C}$ for some K .

For the case when $\mathbb{C}_g = \mathbb{C}$, \mathcal{V}_g and \mathcal{S}_g are respectively denoted by \mathcal{V}^* and \mathcal{S}^* . Similarly, for the case when $\mathbb{C}_g = \mathbb{C}^\odot$, \mathcal{V}_g and \mathcal{S}_g are respectively denoted by \mathcal{V}^\odot and \mathcal{S}^\odot .

Next, we have the following definitions regarding H_2 optimal control.

Definition 2 Let a system Σ of the form (1) be given. Then the H_2 optimal control problem by proper (strictly proper) controllers is defined as the problem of finding, if it exists, a proper (respectively, strictly proper) admissible controller Σ_c which minimizes the H_2 norm of the closed-loop transfer matrix (i.e. $\|T_{zw}(\Sigma \times \Sigma_c)\|_2$) over all the proper (respectively, strictly proper) admissible controllers. The infimum of $\|T_{zw}(\Sigma \times \Sigma_c)\|_2$ over the class of proper admissible controllers is denoted by γ_p^* , i.e.

$$\gamma_p^* := \inf \{ \|T_{zw}(\Sigma \times \Sigma_c)\|_2 \mid \Sigma_c \text{ is proper and internally stabilizes } \Sigma \}. \quad (4)$$

Similarly, the infimum of $\|T_{zw}(\Sigma \times \Sigma_c)\|_2$ over the class of strictly proper admissible controllers is denoted by γ_{sp}^* , i.e.

$$\gamma_{sp}^* := \inf \{ \|T_{zw}(\Sigma \times \Sigma_c)\|_2 \mid \Sigma_c \text{ strictly proper and internally stabilizes } \Sigma \}. \quad (5)$$

The conditions for the existence of a *strictly proper* H_2 optimal controller are

different from those of a *non-strictly proper* H_2 optimal controller. Moreover, as said earlier, it turns out that in the case of discrete-time systems (but not for continuous-time systems) γ_p^* is in general smaller than γ_{sp}^* (see for details (Trentelman & Stoorvogel 1995)).

As discussed in detail in (Trentelman & Stoorvogel 1995), the strictly proper H_2 optimal control problem for a given system Σ can be reformulated as a disturbance decoupling problem via strictly proper measurement feedback with internal stability (DDPMS) for an auxiliary system denoted here by Σ_{PQ} . In fact, a strictly proper controller that is H_2 optimal for Σ solves the DDPMS for the auxiliary system Σ_{PQ} and vice versa. Next, the proper H_2 optimal control problem can also be reformulated as a DDPMS for an auxiliary system denoted here by Σ_{PQ}^N . In the latter case, we first choose a preliminary static output feedback. Then, there is a one to one relationship between the H_2 optimal controllers for Σ and the controllers that solve the DDPMS for the auxiliary system Σ_{PQ}^N . However in this case, the controllers are not identical but are related via this preliminary feedback.

In what follows, we first state the dynamic equations of Σ_{PQ} and Σ_{PQ}^N . Then we give a definition of DDPMS and, finally, we recall lemmas that connect the strictly proper and proper H_2 optimal control problems for Σ respectively to DDPMS for Σ_{PQ} and Σ_{PQ}^N .

In order to define the auxiliary system Σ_{PQ} , we first note that if (A, B) is stabilizable and (C_1, A) is detectable, there exist matrices P and Q such that

$$P = A'PA - (A'PB + C_2'D_2)(B'PB + D_2'D_2)^\dagger(B'PA + D_2'C_2) + C_2'C_2 \quad (6)$$

$$Q = AQA' - (AQC_1' + ED_1')(C_1QC_1' + D_1D_1')^\dagger(C_1QA' + D_1E') + E'E \quad (7)$$

and such that the matrix pencils

$$\begin{pmatrix} zI - A & -B \\ C_2'C_2 + A'PA - P & A'PB + C_2'D_2 \\ B'PA + D_2'C_2 & B'PB + D_2'D_2 \end{pmatrix} \quad (8)$$

and

$$\begin{pmatrix} zI - A & EE' + AQA' - Q & AQC_1' + ED_1' \\ -C_1 & C_1QA' + D_1E' & C_1QC_1' + D_1D_1' \end{pmatrix} \quad (9)$$

have no zeros outside the closed unit circle. Then, the auxiliary system Σ_{PQ} is

described by

$$\Sigma_{\text{PQ}} : \begin{cases} \sigma x_{\text{PQ}} = Ax_{\text{PQ}} + Bu_{\text{PQ}} + E_{\text{Q}}w_{\text{PQ}} \\ y_{\text{PQ}} = C_1x_{\text{PQ}} + D_{\text{Q}}w_{\text{PQ}} \\ z_{\text{PQ}} = C_{\text{P}}x_{\text{PQ}} + D_{\text{P}}u_{\text{PQ}}. \end{cases} \quad (10)$$

Here C_{P} , D_{P} , E_{Q} and D_{Q} are defined by:

$$\begin{aligned} D_{\text{P}} &:= (D_2'D_2 + B'PB)^{\frac{1}{2}}, \\ C_{\text{P}} &:= D_{\text{P}}^\dagger(D_2'C_2 + B'PA), \\ D_{\text{Q}} &:= (D_1D_1' + C_1QC_1')^{\frac{1}{2}}, \\ E_{\text{Q}} &:= (AQC_1' + ED_1')D_{\text{Q}}^\dagger. \end{aligned}$$

Often we use two subsystems $\Sigma_{1\text{PQ}}$ and $\Sigma_{2\text{PQ}}$. These subsystems $\Sigma_{1\text{PQ}}$ and $\Sigma_{2\text{PQ}}$ are characterized by the matrix quadruples $(A, E_{\text{Q}}, C_1, D_{\text{Q}})$ and $(A, B, C_{\text{P}}, D_{\text{P}})$, respectively.

In order to define the auxiliary system Σ_{PQ}^N , we first define a set \mathcal{N}^* as

$$\mathcal{N}^* := \{ N \in \mathbb{R}^{m \times p} \mid N \text{ satisfies the equation (12)} \}, \quad (11)$$

$$D_{\text{P}}ND_{\text{Q}} = -R^*, \quad (12)$$

where

$$R^* = D_{\text{P}}^\dagger(D_{\text{P}}'C_{\text{P}}QC_1' + B'PED_1')D_{\text{Q}}^\dagger. \quad (13)$$

We note that \mathcal{N}^* is non-empty. In fact, a particular member of \mathcal{N}^* is given by

$$N^* = -(D_{\text{P}}^\dagger)^2(D_{\text{P}}'C_{\text{P}}QC_1' + B'PED_1')(D_{\text{Q}}^\dagger)^2. \quad (14)$$

We now define³ the auxiliary system Σ_{PQ}^N . For any given $N \in \mathcal{N}^*$, let

$$\Sigma_{\text{PQ}}^N : \begin{cases} \sigma x_{\text{PQ}} = A^N x_{\text{PQ}} + B\tilde{u}_{\text{PQ}} + E_{\text{Q}}^N w_{\text{PQ}} \\ y_{\text{PQ}} = C_1 x_{\text{PQ}} + D_{\text{Q}} w_{\text{PQ}} \\ z_{\text{PQ}} = C_{\text{P}}^N x_{\text{PQ}} + D_{\text{P}} \tilde{u}_{\text{PQ}}, \end{cases} \quad (15)$$

³ To obtain Σ_{PQ}^N , we apply to the auxiliary system Σ_{PQ} the static output feedback $u_{\text{PQ}} = Ny_{\text{PQ}} + \tilde{u}_{\text{PQ}}$ with \tilde{u}_{PQ} as the new control signal, and then delete in it the feedthrough term from w_{PQ} to z_{PQ} .

where \tilde{u}_{PQ} is a new control signal, and where

$$A^N = A + BNC_1, E_Q^N = E_Q + BND_Q, C_P^N = C_P + D_PNC_1. \quad (16)$$

We now proceed to define the DDPMS for Σ_{PQ} .

Definition 3 Consider a system Σ_{PQ} as in (10). The disturbance decoupling problem with measurement feedback and internal stability (DDPMS) for Σ_{PQ} is the problem of finding a proper controller Σ_C of the form (2) such that the closed-loop system $\Sigma_{PQ} \times \Sigma_C$ is internally stable, while the resulting closed-loop transfer function is identical to 0.

We say that the strictly proper disturbance decoupling problem with measurement feedback and internal stability for Σ_{PQ} is solvable if there exists a strictly proper controller Σ_C of the form (2) with $N = 0$ such that the closed-loop system $\Sigma_{PQ} \times \Sigma_C$ is internally stable, while the resulting closed-loop transfer function is identical to 0.

Moreover, for the special case when the entire state of Σ_{PQ} is available for feedback, the corresponding DDPMS is referred to as DDPS which represents disturbance decoupling problem with state feedback and internal stability. We note that DDPS for Σ_{PQ} is characterized by the matrix quintuple (A, B, E_Q, C_P, D_P) .

The following lemma recalled from (Trentelman & Stoorvogel 1995) connects the strictly proper H_2 optimal control problem for Σ with the strictly proper DDPMS for Σ_{PQ} . Such a reformulation plays a significant role in the rest of this paper.

Lemma 4 Consider an H_2 optimal control problem by strictly proper controllers as defined by Definition 2 for a system Σ as in (1). Assume that (A, B) is stabilizable and (C_1, A) is detectable. Also, consider the auxiliary system Σ_{PQ} as given in (10), and a strictly proper controller Σ_C as in (2) with $N = 0$. Then, the controller Σ_C is a strictly proper H_2 optimal controller for Σ if and only if it solves the DDPMS for Σ_{PQ} .

The following lemma recalled from (Trentelman & Stoorvogel 1995) connects the proper H_2 optimal control problem for Σ with the proper DDPMS for Σ_{PQ}^N .

Lemma 5 Consider a proper H_2 optimal control problem as defined by Definition 2 for a system Σ as in (1). Assume that (A, B) is stabilizable and (C_1, A) is detectable. Also, consider the auxiliary system Σ_{PQ}^N defined in (15) for some $N \in \mathcal{N}^*$. Then the following statements are equivalent:

- (i) A proper controller Σ_C with state space realization (J, L, M, \tilde{N}) solves the DDPMS for Σ_{PQ}^N .
- (ii) A proper controller Σ_C with state space representation $(J, L, M, N + \tilde{N})$

is a proper H_2 optimal controller for the given system Σ .

Moreover, whenever the DDPMS for Σ_{PQ}^N is solvable via a proper controller, it is also solvable via a strictly proper controller, and the solvability conditions of the DDPMS for Σ_{PQ}^N are independent of the particular choice of N as long as $N \in \mathcal{N}^*$.

Lemmas 4 and 5 convert the task of finding an H_2 optimal controller for a given system Σ to the task of finding a controller that solves the DDPMS for an auxiliary system constructed from the data of the given system Σ . As such, these lemmas are the vehicles by which the goals of this paper are carried out.

We will now define some notations that will be used throughout the paper. Since in the case of state feedback $C_1 = I$ and $D_1 = 0$, we note that an H_2 optimal static state feedback control problem, say for Σ , is characterized by the quintuple (A, B, E, C_2, D_2) . Then, we denote by $\mathbf{F}_s^*(A, B, E, C_2, D_2)$ the set of all H_2 optimal static state feedback controllers (or gains) for Σ . It can be checked that the set $\mathbf{F}_s^*(A, B, E, C_2, D_2)$ equals the set $\mathbf{F}_s^*(A, B, E, C_P, D_P)$ (see (Chen *et al.* 1994)). Also, throughout the paper, we use another set of gains defined as

$$\mathbf{K}_s^*(A, E_Q, C_1, C_2, D_Q) := \{K \mid K' \in \mathbf{F}_s^*(A', C_1', C_2', E_Q', D_Q')\}.$$

We have the following additional definition.

Definition 6 A scalar $\lambda \in \mathbb{C}^\circ$ is said to be an H_2 **optimal fixed mode** if λ is an eigenvalue of $A + BF$ for all $F \in \mathbf{F}_s^*(A, B, E, C_2, D_2)$. We will denote by $\Omega^*(A, B, E, C_P, D_P)$ the set of H_2 optimal fixed modes with respect to $\mathbf{F}_s^*(A, B, E, C_P, D_P)$. Likewise, we will denote by $\Psi^*(A, E_Q, C_1, C_2, D_Q)$ the set of H_2 optimal fixed modes with respect to $\mathbf{K}_s^*(A, E_Q, C_1, C_2, D_Q)$.

Utilization of the sets \mathbf{F}_s^* , \mathbf{K}_s^* , Ω^* , and Ψ^* to form an appropriate H_2 optimal measurement feedback controller is discussed in the later sections. However, at this time, we like to emphasize that an algorithm called (*OGFM*) is developed in (Chen *et al.* 1994) to characterize and construct explicitly the set of state feedbacks \mathbf{F}_s^* , and its associated set of fixed modes Ω^* . By duality this algorithm can also be used to characterize and construct the set \mathbf{K}_s^* , and its associated fixed modes Ψ^* .

3 Characterization and Parameterization of All H_2 Optimal Dynamic Measurement Feedback Controllers

In this section, we characterize and parameterize all H_2 optimal dynamic measurement feedback controllers of proper as well as strictly proper type for the given system Σ . In fact, in order to do so, in view of Lemmas 4 and 5, we characterize and parameterize all controllers that solve the DDPMS for Σ_{PQ} or Σ_{PQ}^N .

Our characterization and parameterization of all H_2 optimal *proper* dynamic measurement feedback controllers involves the following steps.

- (i) Find a matrix $F \in \mathbb{R}^{m \times n}$ such that $\mathcal{V}^\circ(\Sigma_{2\text{PQ}})$ is $(A + BF)$ -invariant and contained in $\ker(C + DF)$ and such $A + BF$ is asymptotically stable.
Similarly, find a matrix $K \in \mathbb{R}^{n \times p}$ such that $\mathcal{S}^\circ(\Sigma_{1\text{PQ}})$ is $(A + KC)$ -invariant and contains $\text{im}(B + KD)$ and such that $A + KC$ is asymptotically stable.
- (ii) Define a set \mathcal{N}^* as in (11).
- (iii) Define the set \mathbf{Q}_s as all $Q_s \in \mathcal{RH}_2$ such that

$$\begin{aligned} & [(C_P + D_P F)(zI - A - BF)^{-1}B + D_P]Q_s(z) \\ & \times [C_1(zI - A - KC_1)^{-1}(E_Q + KD_Q) + D_Q] = 0, \end{aligned} \quad (17)$$

where \mathcal{RH}_2 denotes the set of strictly proper and stable rational matrices.

- (iv) Define a set \mathbf{Q} as

$$\mathbf{Q} := \{ Q = Q_s + N \mid Q_s \in \mathbf{Q}_s \text{ and } N \in \mathcal{N}^* \}. \quad (18)$$

- (v) One can now define now a set of *proper* dynamic measurement feedback controllers parameterized by Q as

$$\Sigma_c : \begin{cases} \sigma\xi = (A + BF + KC_1)\xi - Ky + By_1 \\ u = F\xi + y_1 \\ y_1 = \mathcal{Q}(y - C_1\xi), \end{cases} \quad (19)$$

where \mathcal{Q} is an input-output operator with transfer matrix $Q \in \mathbf{Q}$ with \mathbf{Q} as defined in (18).

We have the following theorem.

Theorem 7 *Consider an H_2 optimal control problem as defined by Definition 2 for a system Σ as in (1). Assume⁴ that there exists an H_2 optimal*

⁴ The necessary and sufficient conditions under which a given system Σ has an H_2

proper measurement feedback controller for Σ . Then the parameterized set of controllers, each element of which is of the form Σ_C given in (19) with $Q \in \mathbf{Q}$ as given by (18), coincides with the set of all H_2 optimal proper dynamic measurement feedback controllers.

PROOF. See Appendix A. \square

Next, we characterize and parameterize the set of all *strictly proper* dynamic measurement feedback controllers. Consider

$$\Sigma_C : \begin{cases} \sigma\xi = (A + BF + KC_1)\xi - Ky + By_1 \\ u = F\xi + y_1 \\ y_1 = \mathcal{Q}_s(y - C_1\xi), \end{cases} \quad (20)$$

where \mathcal{Q}_s is the input-output operator associated to the transfer matrix $Q_s \in \mathbf{Q}_s$. Here F, K and \mathbf{Q}_s are as defined on page 11.

We have the following theorem.

Theorem 8 Consider an H_2 optimal control problem as defined by Definition 2 for a system Σ as in (1). Assume that there exists an H_2 optimal strictly proper measurement feedback controller for Σ . Then the parameterized set of controllers, each element of which is of the form Σ_C given in (20) with $Q_s \in \mathbf{Q}_s$, coincides with the set of all H_2 optimal strictly proper dynamic measurement feedback controllers.

PROOF. It follows easily from the proof of Theorem 7 as given in Appendix A. \square

We would like to point out an important characteristic of the closed-loop transfer function $T_{zw}(\Sigma \times \Sigma_C)$ from w to z for any H_2 optimal controller Σ_C . It turns out that, for any H_2 optimal *proper* controller, the closed loop transfer matrix $T_{zw}(\Sigma \times \Sigma_C)$ is the same. Similarly, $T_{zw}(\Sigma \times \Sigma_C)$ is the same for any H_2 optimal *strictly proper* controller. Let us consider either the set of H_2 optimal *proper* controllers or the set of H_2 optimal *strictly proper* controllers. The fact that the closed-loop transfer function is unique implies that we cannot use any available freedom in selecting an H_2 optimal controller Σ_C to shape the closed-loop transfer function $T_{zw}(\Sigma \times \Sigma_C)$. However, the available freedom can

optimal proper (similarly, a strictly proper) measurement feedback controller are developed in (Trentelman & Stoorvogel 1995).

be used to change the internal dynamics of the closed-loop system either by assigning the closed-loop poles appropriately or by changing the closed-loop transfer matrices between some other sets of signals. In this paper, we focus on utilizing the freedom in assigning the closed-loop poles. One other factor worth mentioning is this. Although the above theorems completely characterize all H_2 optimal controllers, the parameterization is not very transparent in its effect on closed-loop poles. Moreover, the structure of the controller is not very clear either. To remedy this to some extent, in the following sections we will study estimator based controllers. This class of controllers has a desirable and clear structure and we will completely characterize the freedom we have to place the closed-loop poles. As said earlier, we investigate three classes of estimators, prediction, current, and reduced order ones. In the next section, we briefly review the structure of these controllers.

4 Review of Controllers with Estimator Based Architecture

Traditionally, observer or estimator based controller design is done using a sequential design philosophy. In the first stage, a state feedback control law $u = Fx$, or equivalently a static state feedback gain F , is designed. In the second stage, an estimator is designed to implement the state feedback control law $u = Fx$; that is, the state x is estimated as \hat{x} while using the measured output y , and then the control law $u = F\hat{x}$ is implemented. Thus the job of the estimator is to produce \hat{x} by utilizing y as its input. We review here three commonly used estimator based controllers: (1) prediction, (2) current, and (3) reduced order estimator based controllers. Estimators can be developed for any specified system, in what follows, however, we use Σ_{PQ} as the given system.

4.1 Prediction estimator based controllers

Consider a prediction estimator which is of dynamic order n ,

$$\sigma \hat{x}_{PQ} = A\hat{x}_{PQ} + Bu_{PQ} + K_p(C_1\hat{x}_{PQ} - y_{PQ}). \quad (21a)$$

The above estimator is characterized by K_p which is referred to as the estimator gain. The gain K_p is selected such that $(A + K_p C_1)$ is stable, i.e. has all its eigenvalues in \mathbb{C}° . We note that the poles of the above estimator are given by $\lambda(A + K_p C_1)$. A static state feedback control law $u_{PQ} = Fx_{PQ}$ is implemented as

$$u_{PQ} = F\hat{x}_{PQ}. \quad (21b)$$

Without loss of generality, we assume that the matrices C_1 and D_Q have already been transformed to the form,

$$C_1 = \begin{pmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{pmatrix} \quad \text{and} \quad D_Q = \begin{pmatrix} D_0 \\ 0 \end{pmatrix}. \quad (22)$$

Let us partition,

$$x_{PQ} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad y_{PQ} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}.$$

Then, the system Σ_{PQ} as in (10) can be rewritten as follows,

$$\left\{ \begin{array}{l} \begin{pmatrix} \sigma x_1 \\ \sigma x_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u_{PQ} + \begin{pmatrix} E_{1Q} \\ E_{2Q} \end{pmatrix} w_{PQ} \\ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} D_0 \\ 0 \end{pmatrix} w_{PQ} \\ z_{PQ} = \begin{pmatrix} C_{P1} & C_{P2} \end{pmatrix} x_{PQ} + D_P u_{PQ}. \end{array} \right. \quad (23)$$

In a current estimator, one uses $y_1(k+1)$ rather than $y_1(k)$ to estimate $\hat{x}_{PQ}(k+1)$. Then, the dynamic equations of a current estimator are given by,

$$\sigma \hat{x}_{PQ} = A \hat{x}_{PQ} + B u_{PQ} + K_c \left(C_c \hat{x}_{PQ} + B_c u_{PQ} - \begin{pmatrix} y_0 \\ \sigma y_1 \end{pmatrix} \right). \quad (24)$$

where

$$C_c = \begin{pmatrix} 0 & C_{02} \\ A_{11} & A_{12} \end{pmatrix}, \quad B_c = \begin{pmatrix} 0 \\ B_1 \end{pmatrix} \quad \text{and} \quad D_c = \begin{pmatrix} D_0 \\ E_{1Q} \end{pmatrix}. \quad (25)$$

It can be verified that the pair (C_c, A) is detectable whenever the given system Σ_{PQ} is detectable (see (Saberi *et al.* 1993)).

The current estimator is of dynamic order n and is characterized by K_c which is referred to as the current estimator gain. To implement the above estimator, we partition $K_c = \begin{pmatrix} K_{c0} & K_{c1} \end{pmatrix}$ in conformity with the partitioning of y_{PQ} , and

also define a variable $v(k)$ as,

$$v = \hat{x}_{\text{PQ}} + K_{c1}y_1. \quad (26)$$

Then, (24) can be rewritten as

$$\begin{cases} \sigma v = (A + K_c C_c)v + (B + K_c B_c)u_{\text{PQ}} \\ \quad - [K_{c0}, (A + K_c C_c)K_{c1}]y_{\text{PQ}} \\ \hat{x}_{\text{PQ}} = v - K_{c1}y_1. \end{cases} \quad (27a)$$

The gain K_c is selected such that $(A + K_c C_c)$ is stable. A static state feedback control law $u_{\text{PQ}}(k) = Fx_{\text{PQ}}(k)$ is implemented as,

$$u_{\text{PQ}}(k) = F\hat{x}_{\text{PQ}}(k). \quad (27b)$$

Thus, equations (27a) and (27b) together define a current estimator based controller. Finally we would like to note that, as expected, the closed-loop poles are the eigenvalues of $A + K_c C_c$ and $A + BF$.

4.3 Reduced order estimator based controllers

We now proceed with the development of a reduced order estimator based controller of dynamic order $n - \text{rank}[C_1, D_Q] + \text{rank}[D_Q]$ where n as usual is the dynamic order of Σ . At first, as in the previous subsection, we assume that the matrices C_1 and D_Q have already been transformed to the form (22) and the system Σ_{PQ} is partitioned as in (23). The idea behind the construction of a reduced order estimator based controller is that we only need to build an estimator for x_2 as x_1 (or equivalently y_1) is available as a measurement. Our techniques to do so are based on the method discussed in Section 7.2 of (Anderson & Moore 1979). x_1 is known and observations of x_2 are made via y_1 and \tilde{y} , where

$$\tilde{y} := A_{12}x_2 + E_{1Q}w_{\text{PQ}} = \sigma y_1 - A_{11}x_1 - B_1 u_{\text{PQ}}. \quad (28)$$

If we do not worry about causality for a moment, we find the following estimator which utilizes a gain K_r ,

$$\sigma \hat{x}_2 = A_{22}\hat{x}_2 + A_{21}y_1 + B_2 u_{\text{PQ}} + K_r \left[\begin{pmatrix} C_{02} \\ A_{12} \end{pmatrix} \hat{x}_2 - \begin{pmatrix} y_0 \\ \sigma y_1 - A_{11}x_1 - B_1 u_{\text{PQ}} \end{pmatrix} \right].$$

We partition $K_r = [K_{r0} \ K_{r1}]$ so as to be compatible with the sizes of (y_0, \tilde{y}) . Then, the use of change of variables $v := \hat{x}_2 + K_{r1}y_1$ results in a causal reduced

order estimator,

$$\left\{ \begin{array}{l} \sigma v = (A_{22} + K_{r0}C_{02} + K_{r1}A_{12})v + (B_2 + K_{r1}B_1)u_{PQ} \\ \quad \quad \quad + [-K_{r0}, A_{21} + K_{r1}A_{11} - (A_{22} + K_{r0}C_{02} + K_{r1}A_{12})K_{r1}]y_{PQ} \\ \hat{x}_{PQ} = \begin{pmatrix} 0 \\ I_r \end{pmatrix} v + \begin{pmatrix} 0 & I_{n-r} \\ 0 & -K_{r1} \end{pmatrix} y_{PQ}, \end{array} \right. \quad (29a)$$

where r is the dimension of x_2 or equivalently the dimension of v . We use this reduced order estimator to obtain the control law as,

$$u_{PQ} = F\hat{x}_{PQ}. \quad (29b)$$

Equations (29a) and (29b) together define the reduced order estimator based controller. Note that the poles of the closed loop system are given by the eigenvalues of $A + BF$ and the eigenvalues of $A_{22} + K_{r0}C_{02} + K_{r1}A_{12}$.

5 Estimator based controllers that solve DDPMS – flexibility in assigning the closed-loop poles

Our next task is to characterize and parameterize all the prediction, current, and reduced order estimator based H_2 optimal controllers for Σ , and the flexibility they have in simultaneous closed-loop pole placement. However, in view of Lemmas 4 and 5 such a task translates into a task of characterizing and parameterizing all prediction, current, and reduced order estimator based controllers that solve the DDPMS for Σ_{PQ} or Σ_{PQ}^N , and the investigation of the flexibility they have in closed-loop pole placement.

Thus, in this section, we pursue the problem of characterizing, parameterizing and constructing the prediction, current, and reduced order estimator based controllers that achieve DDPMS for Σ_{PQ} while assigning the closed-loop poles at the desired locations whenever possible (clearly this analysis applies equally well to achieving DDPMS for Σ_{PQ}^N with some obvious modifications).

The design methodology to construct such controllers is straightforward and follows the conventional sequential design philosophy with certain care in designing the estimators. It is clear that the idea behind the estimator based controllers is to implement a “desirable” static state feedback law via an estimator of a given type that provides an estimate of the state and consequently forms a measurement feedback law hopefully having the same “desirable” feature as the original state feedback law does. In the context of DDPS, the “desirable” feature of the state feedback law is that it solves the DDPS for

the given system. Thus the design of a measurement feedback controller is divided into two stages, (1) design of a static state feedback law that solves the DDPS for the given system, and (2) the design of a “suitable” estimator to ensure that the resulting measurement feedback controller would solve the DDPMS for the given system. We note that to implement a given state feedback law that solves the DDPS for a given system via an estimator, the candidate estimator must satisfy certain conditions in order to ensure that the resulting measurement feedback controller also solves the DDPMS for the given system. That is, an estimator must be designed judiciously and must be “suitable” for the given particular state feedback law that solves the DDPS for the given system. It turns out that the “suitability” of an estimator depends on the choice of the state feedback law. In other words, an estimator that is “suitable” for a particular state feedback law might not be “suitable” for some other one. Also, a particular state feedback law might have many “suitable” estimators. All this discussion implies that the traditional separation principle does not hold and hence we cannot separate the choice of an estimator or estimator gain from the choice of a state feedback gain when constructing a measurement feedback controller that solves the DDPMS for the given system. Thus, two of our main goals in this section are (1) to identify the set of conditions that an estimator of a given type must satisfy in order to be “suitable” for a given particular state feedback law that solves the DDPS for the given system, and (2) to characterize as well as to produce algorithms for constructing the set of all such “suitable” estimators associated with a given state feedback law.

Another important goal of this section is to identify the constraints and freedom associated with an estimator based architecture regarding the simultaneous assignment of poles of the resulting closed-loop system. More specifically, since there are in general many “suitable” estimators associated with a given static state feedback law that solves the DDPS for a given system, one can formulate a design problem of utilizing such a freedom to assign estimator poles to desired locations whenever such an assignment is possible. We note that the poles of a closed-loop system comprising the given system and an estimator based controller are the union of the estimator poles and the poles of the closed-loop system under the state feedback control law alone. In view of this, the problem of assigning the poles of the closed-loop system under an estimator based controller translates to two problems which must be treated sequentially. The first problem is to design a “desired” state feedback control law that solves the DDPS while yielding a closed-loop system with poles in the desired locations whenever it is possible. The second problem is to design a “suitable” estimator associated with the state feedback control law obtained in the first problem such that its poles are in desired locations. However, as we shall see shortly, one cannot in general assign all the poles of a “suitable” estimator associated with the given state feedback control law arbitrarily in \mathbb{C}^{\odot} . Some of the poles must be located in certain locations in \mathbb{C}^{\odot} in order

to guarantee the “suitability” of the estimator. Obviously, such poles can be referred to as fixed modes of the “suitable” estimator associated with the given state feedback control law. One needs to obtain the set of all such fixed modes. Here we find an algorithm for constructing such a set. This leads us to a procedure of designing a measurement feedback controller that solves the DDPMS for a given system while placing the closed-loop poles at the desired locations whenever possible.

5.1 Prediction estimator based controllers

We first consider prediction estimator based controllers. We first choose a state feedback gain F from the set $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$. As shown in (Chen *et al.* 1994), such a state feedback gain F solves the DDPS for Σ_{PQ} . This leads us at first to study a basic question. If we have a state feedback gain F from the set $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$, does there exist a prediction estimator such that the interconnection of it and F solves the DDPMS for Σ_{PQ} ? Once this question is answered affirmatively, we then proceed to characterize and to construct the prediction estimator based controllers that solve the DDPMS for Σ_{PQ} while placing, whenever possible, the closed-loop poles at the desired locations.

To start with, for any given $F \in \mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$, let $\mathbf{K}_p^*(F)$ denote the set of all prediction estimator gains such that, for the given F and any $K_p \in \mathbf{K}_p^*(F)$, the prediction estimator based controller given in (21) solves the DDPMS for Σ_{PQ} . Also, let $\Psi_p^*(F)$ be the prediction estimator fixed modes with respect to $\mathbf{K}_p^*(F)$.

We have the following lemmas which characterize $\mathbf{K}_p^*(F)$.

Lemma 9 *Consider a system Σ as in (1), and its auxiliary system Σ_{PQ} as in (10). Assume that the subsystem Σ_{2PQ} characterized by the matrix quadruple (A, B, C_P, D_P) is left-invertible, or equivalently, assume that the subsystem Σ_2 of Σ , characterized by (A, B, C_2, D_2) , is left invertible. Let $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$ be non-empty. Then, for each $F \in \mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$, the prediction estimator based controller given in (21) solves the DDPMS for Σ_{PQ} , if and only if its gain K_p is such that $A + K_p C_1$ is stable and $\Phi(z) = 0$, where*

$$\Phi(z) = F(zI - A - K_p C_1)^{-1}(E_Q + K_p D_Q). \quad (30)$$

PROOF. See Appendix B. \square

Lemma 10 *Consider a system Σ as in (1), and its auxiliary system Σ_{PQ} as in (10). Assume that the subsystem Σ_{2PQ} characterized by the matrix quadruple (A, B, C_P, D_P) is left-invertible, or equivalently, assume that the subsystem Σ_2*

of Σ is left invertible. Let $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$ be non-empty. Then, for each $F \in \mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$,

$$K_p \in \mathbf{K}_p^*(F) = \mathbf{K}_s^*(A, E_Q, C_1, F, D_Q).$$

Moreover, $\mathbf{K}_p^*(F)$ is non-empty if and only if the following condition is true:

$$\mathcal{S}^\odot(\Sigma_{1PQ}) \subseteq \ker F. \quad (31)$$

PROOF. It is a consequence of Lemma 9 and the results of (Stoorvogel & van der Woude 1991). \square

Lemma 10 characterizes the set $\mathbf{K}_p^*(F)$. Also, it formulates a condition, namely (31), to test whether $\mathbf{K}_p^*(F)$ is non-empty or not. We note that Lemma 10, to start with, does not assume the existence of a strictly proper measurement feedback controller that solves the DDPMS for Σ_{PQ} . Suppose there exists a strictly proper measurement feedback controller that solves the DDPMS for Σ_{PQ} . Does this imply that, for every $F \in \mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$, $\mathbf{K}_p^*(F)$ is non-empty? The answer to such a question, as shown in the following lemma, is affirmative.

Lemma 11 *Consider a system Σ as in (1), and its auxiliary system Σ_{PQ} as in (10). Assume that the subsystem Σ_{2PQ} is left-invertible. Also, assume that there exists a strictly proper measurement feedback controller that solves the DDPMS for Σ_{PQ} , implying that $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$ is non-empty. Then, $\mathbf{K}_p^*(F)$ is non-empty for all $F \in \mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$.*

PROOF. See Appendix C. \square

The next lemma focuses on the flexibility one has in placing the closed-loop poles while solving the DDPMS for Σ_{PQ} .

Lemma 12 *Consider a system Σ as in (1), and its auxiliary system Σ_{PQ} as in (10). Assume that the subsystem Σ_{2PQ} is left-invertible. Also, assume that there exists a strictly proper measurement feedback controller that solves the DDPMS for Σ_{PQ} . Let F be any element of $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$. The set of prediction estimator fixed modes associated with F is given by*

$$\Psi_p^*(F) = \Psi^*(A, E_Q, C_1, F, D_Q).$$

PROOF. It is obvious. \square

It is important to note that both the sets $\mathbf{K}_p^*(F)$ and $\Psi_p^*(F)$ can be constructed by the (*OGFM*) algorithm developed in (Chen *et al.* 1994).

Next, we move on to consider the case when the subsystem Σ_{2PQ} of Σ_{PQ} (or equivalently, the subsystem (A, B, C_2, D_2) of Σ) is not left-invertible. For such a general case, the problem of characterizing all controllers with estimator based architecture that solve the DDPMS for Σ_{PQ} becomes complicated and challenging. Nevertheless, for a large subset of the set of all state feedback gains $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$, the problem of characterizing such controllers is not very difficult; and, moreover, it can be worked out using the same machinery that we developed for the case when Σ_{2PQ} is left invertible. Also, by focusing only on this subset rather than the whole set $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$, one does not lose much freedom in problems such as the simultaneous pole placement problem.

We now proceed to develop the subset of $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$ mentioned above. In the following development, without loss of generality, to start with we choose an appropriate basis for the control u such that the matrices B and D_P have the following form,

$$B = (\bar{B}_1 \ \bar{B}_2), \quad D_P = (D_{P1} \ 0) \quad (32)$$

such that $B \ker D_P \cap \mathcal{V}^*(\Sigma_{2PQ}) = \text{im } \bar{B}_2$ and \bar{B}_1 has full row rank and satisfies $\bar{B}_1 \ker D_{P1} \cap \mathcal{V}^*(\Sigma_{2PQ}) = \{0\}$. We define \bar{E}_Q and Γ by,

$$\bar{E}_Q = (E_Q \ \bar{B}_2), \quad \Gamma = (I_{\ell_0} \ 0_{m-\ell_0}), \quad (33)$$

where ℓ_0 is the normal rank of Σ_{2PQ} (or equivalently, the normal rank of (A, B, C_2, D_2)) and m the number of inputs; in other words I_{ℓ_0} is an identity matrix with the same number of rows as \bar{B}_1 and $0_{m-\ell_0}$ is a zero matrix with the same number of rows as \bar{B}_2 . Note that $\Gamma = I$ if Σ_{2PQ} is left-invertible.

The set $\mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ has the following properties.

Lemma 13 *The set $\mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ satisfies the following properties:*

- (i) $\mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P) \subseteq \mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$.
- (ii) $\mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ is non-empty if and only if $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$ is non-empty.
- (iii) $\mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ equals $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$ if Σ_{2PQ} is left invertible.
- (iv) $\Omega^*(A, B, \bar{E}_Q, C_P, D_P) = \Omega^*(A, B, E_Q, C_P, D_P)$.

PROOF. Parts (i) to (iii) are straightforward to check. Part (iv) is the tricky part. However, looking at the construction of the set Ω^* in the (*OGFM*) algorithm given in (Chen *et al.* 1994), it is straightforward to establish this fact. \square

Part (iv) of the above lemma assures us that the set $\mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ is sufficiently large in the sense that by focusing on this smaller set rather than on the entire set $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$ one does not lose any freedom available in choosing an estimator based controller that solves the DDPMS for Σ_{PQ} while simultaneously placing the closed-loop poles at desired locations whenever possible.

Remark: Recently, (Dórea & Milani 1995) characterized all state feedback laws that achieve DDPS for a class of non-left invertible plants which satisfy, in our notation, the condition $\text{im } B \cap \mathcal{V}^\circ(\Sigma_{2PQ}) \subseteq \text{im } E_Q$. Part (iii) of Lemma 13 covers the result of (Dórea & Milani 1995) as well. Note that, under the condition $\text{im } B \cap \mathcal{V}^\circ(\Sigma_{2PQ}) \subseteq \text{im } E_Q$, we have $\text{im } \bar{E}_Q = \text{im } E_Q$. Thus, it is easy to see that the condition (iii) of Lemma 13 is satisfied if we replace left-invertibility with the condition $\text{im } B \cap \mathcal{V}^\circ(\Sigma_{2PQ}) \subseteq \text{im } E_Q$. In view of this, under the assumption $\text{im } B \cap \mathcal{V}^\circ(\Sigma_{2PQ}) \subseteq \text{im } E_Q$, the set $\mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ characterizes all H_2 optimal state feedback laws.

We now can state the following theorem.

Theorem 14 *Consider a system Σ as in (1), and its auxiliary system Σ_{PQ} as in (10). Assume that there exists a strictly proper measurement feedback controller that solves the DDPMS for Σ_{PQ} . For any $F \in \mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$, we have*

$$\mathbf{K}_p^*(F) = \mathbf{K}_s^*(A, E_Q, C_1, \Gamma F, D_Q) \quad (34)$$

and

$$\Psi_p^*(F) = \Psi^*(A, E_Q, C_1, \Gamma F, D_Q). \quad (35)$$

Moreover, for each $F \in \mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$, the set $\mathbf{K}_p^*(F)$ is non-empty.

PROOF. See Appendix D. \square

For any $F \in \mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$, we have a method for constructing $\mathbf{K}_p^*(F)$ and $\Omega_p^*(F)$. Hence we are in a position to present a computationally implementable step by step sequential design procedure to design, whenever possible, a prediction estimator based measurement feedback controller that solves the DDPMS for Σ_{PQ} and simultaneously places the closed-loop poles at desired locations whenever it can be done. We describe below such a design method.

An algorithm for designing a prediction estimator based controller that solves the DDPMS for Σ_{PQ} with simultaneous closed-loop pole placement:

We have the following steps.

Step 1: Consider the system Σ_{PQ} . Select a set Λ_s of n desired self-conjugate poles. Using $(A, B, \bar{E}_Q, C_P, D_P)$ as the input to (*OGFM*) algorithm of (Chen *et al.* 1994), determine a state feedback gain $F \in \mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ such that $\lambda(A + BF) = \Lambda_s$. We note that such a gain $F \in \mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ always exists provided $\Omega^*(A, B, \bar{E}_Q, C_P, D_P) \subseteq \Lambda_s$.

Step 2: Select a set Λ_e of n desired self-conjugate estimator poles. Let the gain F be as chosen in Step 1. Using the quintuple $(A, E_Q, C_1, \Gamma F, D_Q)$ as the input to the dual of (*OGFM*) algorithm of (Chen *et al.* 1994), determine a gain $K_p \in \mathbf{K}_s^*(A, E_Q, C_1, \Gamma F, D_Q)$ such that $\lambda(A + K_p C_1) = \Lambda_e$. We note that such a gain $K_p \in \mathbf{K}_s^*(A, E_Q, C_1, \Gamma F, D_Q)$ always exists provided $\Psi^*(A, E_Q, C_1, \Gamma F, D_Q) \subseteq \Lambda_e$.

Step 3: Form a prediction estimator based controller as in (21) with F and K_p selected as in Steps 1 and 2.

It is obvious that the prediction estimator based controller formed in Step 3 indeed solves the DDPMS for Σ_{PQ} while placing the closed-loop poles at the locations given by the elements of the sets Λ_s and Λ_e , and thus at $2n$ desired locations.

5.2 Current estimator based controllers

As mentioned earlier, the set $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$ describes all stabilizing state feedback gains that solve the DDPMS for Σ_{PQ} . As before, we take an element out of this set which has desirable properties (for instance with respect to pole location) and this time we look for a current estimator such that the interconnection of this estimator and the chosen state feedback yields a stabilizing dynamic controller that solves the DDPMS for Σ_{PQ} . Also, as in the prediction estimators, we would like to know our flexibility in choosing the gain K_c for a given state feedback in the set $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$. To do so, to start with, for any given $F \in \mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$, we denote by $\mathbf{K}_c^*(F)$ the set of all current estimator gains such that, for the given F and any $K_c \in \mathbf{K}_c^*(F)$, the current estimator based controller given in (27) solves the DDPMS for Σ_{PQ} . Also, we denote by $\Psi_c^*(F)$ the current estimator fixed modes with respect to $\mathbf{K}_c^*(F)$.

We would like to proceed now with the characterization of $\mathbf{K}_c^*(F)$ and $\Psi_c^*(F)$ for any given $F \in \mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$. However, as in the case of prediction estimator based controllers when Σ_{2PQ} is not left invertible, such a characterization for all $F \in \mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$ is complicated and challenging.

Nevertheless, as in the case of prediction estimator based controllers, for any arbitrarily given F in the subset $\mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ of $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$, such a characterization can be done along the same lines as in the prediction estimator based controllers. We emphasize again that, for the case when Σ_{2PQ} is left invertible, the characterization given here is complete since $\Gamma = I$ in this case.

We have the following theorem.

Theorem 15 *Consider a system Σ as in (1), and its auxiliary system Σ_{PQ} as in (10). Assume that there exists a strictly proper measurement feedback controller that solves the DDPMS for Σ_{PQ} implying that $\mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ is non-empty. Then, for each $F \in \mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$, we have*

$$\mathbf{K}_c^*(F) = \mathbf{K}_s^*(A, E_Q, C_c, \Gamma F, D_c) \quad (36)$$

and

$$\Psi_c^*(F) = \Psi^*(A, E_Q, C_c, \Gamma F, D_c). \quad (37)$$

Moreover, for each $F \in \mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$, the set $\mathbf{K}_c^*(F)$ is non-empty.

PROOF. For a given $F \in \mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$, it can be shown that a current estimator based controller of the form (27) solves the DDPMS for Σ_{PQ} if and only if $A + K_c C_c$ is stable and

$$0 = F(zI - A - K_c C_c)^{-1}(E_Q + K_c D_c). \quad (38)$$

In view of the above equation, the rest of the proof follows along the same lines as the proof of Theorem 14. \square

Again, we note that the sets $\mathbf{K}_c^*(F)$ and $\Psi_c^*(F)$ can be constructed by utilizing the dual of the (OGFM) algorithm developed in (Chen *et al.* 1994). Then, as in Subsection 5.1, one can easily develop an algorithm for designing a current estimator based controller that solves the DDPMS for Σ_{PQ} with simultaneous closed-loop pole placement.

5.3 Reduced order estimator based controllers

As in the previous subsections, the set $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$ describes all stabilizing state feedback gains that solve the DDPS for Σ_{PQ} . As before, we take an element out of this set which has desirable properties, and this time we

look for a reduced order estimator such that the interconnection of this estimator and the chosen state feedback yields a stabilizing dynamic controller that solves the DDPMS for Σ_{PQ} . Also, as in the previous subsections, we would like to know our flexibility in choosing the reduced order estimator gain K_r for a given state feedback in the set $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$. To do so, to start with, for any given $F \in \mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$, we denote by $\mathbf{K}_r^*(F)$ the set of all reduced order estimator gains such that, for the given F and any $K_r \in \mathbf{K}_r^*(F)$, the reduced order estimator based controller given in (29) solves the DDPMS for Σ_{PQ} . Also, we denote by $\Psi_r^*(F)$ the reduced order estimator fixed modes with respect to $\mathbf{K}_r^*(F)$.

We would like to proceed now with the characterization of $\mathbf{K}_r^*(F)$ and $\Psi_r^*(F)$ for any given $F \in \mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$. However, as in the previous subsections, for the general case when Σ_{2PQ} is not necessarily left invertible, such a characterization for any given $F \in \mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$ is difficult. Nevertheless, as in the case of prediction, and current estimator based controllers, for any arbitrarily given F in the subset $\mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ of $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$, such a characterization can be done along the same lines as in the prediction, and current estimator based controllers. We emphasize again that, for the case when Σ_{2PQ} is left invertible, the characterization given here is complete since $\Gamma = I$ in this case.

Before we state our results, let us partition $\Gamma F := \bar{F}$ as $\begin{pmatrix} \bar{F}_1 & \bar{F}_2 \end{pmatrix}$ in conformity with the partitioning of x into $(x_1', x_2)'$ in Section 4.3. We have the following theorem.

Theorem 16 *Consider a system Σ as in (1), and its auxiliary system Σ_{PQ} as in (10). Assume that there exists a strictly proper measurement feedback controller that solves the DDPMS for Σ_{PQ} implying that $\mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ is non-empty. Then, for each $F \in \mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$, we have*

$$\mathbf{K}_r^*(F) = \mathbf{K}_s^*(A, E_{2Q}, \begin{pmatrix} C_{02} \\ A_{12} \end{pmatrix}, \bar{F}_2, \begin{pmatrix} D_0 \\ E_{1Q} \end{pmatrix}) \quad (39)$$

and

$$\Psi_r^*(F) = \Psi^*(A, E_{2Q}, \begin{pmatrix} C_{02} \\ A_{12} \end{pmatrix}, \bar{F}_2, \begin{pmatrix} D_0 \\ E_{1Q} \end{pmatrix}). \quad (40)$$

Moreover, for each $F \in \mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$, the set $\mathbf{K}_r^*(F)$ is non-empty.

PROOF. Using the fact that the system characterized by (A, B_1, C_P, D_{P1}) is left invertible, it can be shown that a reduced order estimator based controller

of the form (29) solves the DDPMS for Σ_{PQ} if and only if $A_{22} + K_{r0}C_{02} + K_{r1}A_{12}$ is stable and

$$0 = \bar{F}_2[(zI - A_{22} - K_{r0}C_{02} - K_{r1}A_{12})^{-1}(E_{2\text{Q}} + K_{r0}D_0 + K_{r1}E_{1\text{Q}})]. \quad (41)$$

Since $\mathcal{S}^\circ(\Sigma_{1\text{PQ}}) \subseteq \ker \Gamma F$ (as evident from Appendix B), we find in combination with the proposition 6.3.2 on p.172 of (Saber *et al.* 1993) that $\mathcal{S}^\circ(\Sigma_{re}) \subseteq \ker \bar{F}_2$. In view of this, the rest of the proof follows along the same lines as the proof of Theorem 14. \square

Again, we note that the sets $\mathbf{K}_r^*(F)$ and $\Psi_r^*(F)$ can be constructed by utilizing a dual version of the (*OGFM*) algorithm developed in (Chen *et al.* 1994). Then, as in Subsection 5.1, one can easily develop an algorithm for designing a reduced order estimator based controller that solves the DDPMS for Σ_{PQ} with simultaneous closed-loop pole placement.

6 The impact of the choice of the feedthrough term N

In the previous sections we developed methodologies to design prediction, current and reduced order estimators for the system Σ_{PQ} or Σ_{PQ}^N and this generated classes of stabilizing controllers which achieved disturbance decoupling (In doing so, we produced a methodology to design H_2 optimal estimator based measurement feedback controllers for the given system Σ). In the case of proper controllers, we first had to choose an arbitrary $N \in \mathcal{N}^*$ and then constructed the system Σ_{PQ}^N for which we had to achieve disturbance decoupling. This leads us now to an important enquiry as to the impact of the choice of N in our design methodology, and in particular the impact of N on the flexibility one has in closed-loop pole assignment. The goal of this section is to answer this enquiry when prediction estimator based controllers are used. Our analysis in this section leads to a design methodology which uses the development of previous sections but has a different sequential nature in selecting the design parameters N , F , and K , namely choosing F first and then choosing K and N together rather than choosing N , F , and K in that order as in the previous sections.

Proceeding with our analysis, we first note that if we have the state (or an estimate) available for feedback then we also have (or an estimate of) $D_{\text{Q}}w_{\text{PQ}} = y_{\text{PQ}} - C_1x_{\text{PQ}}$. Hence part or all of w might be available for feedback. Therefore, it is natural to consider full-information feedbacks of the form (instead of mere state feedbacks),

$$u_{\text{PQ}} = Fx_{\text{PQ}} + ND_{\text{Q}}w_{\text{PQ}} = Ny_{\text{PQ}} + (F - NC_1)x_{\text{PQ}}. \quad (42)$$

Note that clearly with full-information feedbacks the closed-loop poles are completely determined by F . Consider the sets

$$\mathbf{F}_s^*(A, B, E_{\text{Q}} + BND_{\text{Q}}, C_{\text{P}}, D_{\text{P}}), \quad \Omega^*(A, B, E_{\text{Q}} + BND_{\text{Q}}, C_{\text{P}}, D_{\text{P}})$$

for some $N \in \mathcal{N}^*$. These sets describe our flexibility in the state feedback F for a given direct feedthrough matrix N . It is a priori not clear whether these sets depend on the specific choice for N in \mathcal{N}^* . We have the following important result:

Lemma 17 *We have*

$$\Omega^*(A, B, E + BN_1D_{\text{Q}}, C_{\text{P}}, D_{\text{P}}) = \Omega^*(A, B, E + BN_2D_{\text{Q}}, C_{\text{P}}, D_{\text{P}})$$

for any $N_1, N_2 \in \mathcal{N}^*$.

PROOF. Without loss of generality we assume that we have an appropriate basis for the control u_{PQ} and we have the decompositions as described in (32). We note that by Lemma 13 we have

$$\begin{aligned} \Omega^*(A, B, E_{\text{Q}} + BN_1D_{\text{Q}}, C_{\text{P}}, D_{\text{P}}) &= \Omega^*(A, B, [E_{\text{Q}} + BN_1D_{\text{Q}} \quad \bar{B}_1], C_{\text{P}}, D_{\text{P}}) \\ \Omega^*(A, B, E_{\text{Q}} + BN_2D_{\text{Q}}, C_{\text{P}}, D_{\text{P}}) &= \Omega^*(A, B, [E_{\text{Q}} + BN_2D_{\text{Q}} \quad \bar{B}_1], C_{\text{P}}, D_{\text{P}}). \end{aligned}$$

We note that $D_{\text{P}}(N_1 - N_2)D_{\text{Q}} = 0$ and $B \ker D_{\text{Q}} \subset \mathcal{V}^{\odot}(\Sigma_{2\text{PQ}})$. The latter follows from the special structure of D_{P} . Combined with the definition of \bar{B}_1 we find that

$$\text{im } B(N_1 - N_2)D_{\text{Q}} \subset \text{im } \bar{B}_1.$$

This yields

$$\begin{aligned} \text{im}[E_{\text{Q}} + BN_1D_{\text{Q}} \quad \bar{B}_1] &= \text{im}[E_{\text{Q}} + BN_2D_{\text{Q}} + B(N_1 - N_2)D_{\text{Q}} \quad \bar{B}_1] \\ &= \text{im}[E_{\text{Q}} + BN_2D_{\text{Q}} \quad \bar{B}_1]. \end{aligned}$$

It is immediate that $\Omega^*(A, B, E, C, D)$ only depends on the image of E and hence the result follows. \square

The above result tells us that our flexibility in assigning closed-loop poles by state feedback does not depend on our particular choice of $N \in \mathcal{N}^*$. What about the flexibility in the feedback gain F ? To answer this question, as before

we first restrict the set of feedback gains in the case of non-left invertible systems. Then, obviously from the proof of the above lemma we observe another important result, namely

$$\mathbf{F}_s^*(A, B, [E_Q + BN_1D_Q \ \bar{B}_1], C_P, D_P) = \mathbf{F}_s^*(A, B, [E_Q + BN_2D_Q \ \bar{B}_1], C_P, D_P).$$

To indicate that the set $\mathbf{F}_s^*(A, B, [E_Q + BND_Q \ \bar{B}_1], C_P, D_P)$ does not depend on our particular choice for $N \in \mathcal{N}^*$, we denote it by

$$\bar{\mathbf{F}}_s^*(A, B, E_Q, C_P, D_P, D_Q)$$

which is a suitable notation since we have seen that this set is completely characterized by the 6 matrices given in its argument.

Now we need to design a suitable estimator gain K and a suitable direct feedthrough matrix N . As seen from the above development, we note that our particular choice for the state feedback gain does not limit in any sense our flexibility in choosing the direct feedthrough matrix $N \in \mathcal{N}^*$. Also, we note that, if we consider proper controllers, we have in general quite a bit of freedom in choosing the matrix N of the controller. Thus, we next consider a generalization of prediction estimators which clarifies the freedom we have in choosing the matrix N .

We consider controllers of the form:

$$\Sigma_C : \begin{cases} \sigma\xi = A\xi + Bu_{PQ} + K(C_1\xi - y_{PQ}) \\ u_{PQ} = F\xi - N(C_1\xi - y_{PQ}) \end{cases} \quad (43)$$

for the system Σ_{PQ} . Note that if we apply such a controller to Σ_{PQ} then the closed-loop poles are determined by the poles of $A + BF$ and $A + KC_1$.

The next theorem gives the flexibility one has in selecting the estimator gain and the direct feedthrough matrix N for a given state feedback gain $F \in \bar{\mathbf{F}}_s^*(A, B, E_Q, C_P, D_P, D_Q)$.

Theorem 18 *Assume the disturbance decoupling problem for Σ_{PQ} is solvable by a proper controller. Also, let $F \in \bar{\mathbf{F}}_s^*(A, B, E_Q, C_P, D_P, D_Q)$ be given. Then, the set $\mathbf{K}_s^*(F)$ of output injections K and direct feedthrough matrices N for which (43) achieves disturbance decoupling when applied to Σ_{PQ} is given by*

$$\begin{pmatrix} K \\ -N_1 \end{pmatrix} \in \mathbf{K}_s^* \left(\begin{pmatrix} A & 0 \\ \Gamma F & 0 \end{pmatrix}, \begin{pmatrix} E_Q \\ 0 \end{pmatrix}, \begin{pmatrix} C_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & I \end{pmatrix}, D_Q \right) \quad (44)$$

with N such that $\Gamma N = N_1$. Moreover, given F , the set of H_2 optimal prediction estimator fixed modes associated with F , denoted by $\Psi^*(F)$, is given

by

$$\Psi^* \left(\begin{pmatrix} A & 0 \\ \Gamma F & 0 \end{pmatrix}, \begin{pmatrix} E_Q \\ 0 \end{pmatrix}, (C_1 \ 0), (0 \ I), D_Q \right) \quad (45)$$

with l modes at 0 subtracted from it.

PROOF. Using some standard manipulations and since (A, B_1, C_P, D_{P1}) is left-invertible we obtain that (K, N) yield a controller which achieves a stable closed-loop system when applied to Σ_{PQ} with closed-loop transfer matrix equal to $-R^*$ if and only if the system

$$\begin{cases} \sigma \bar{\xi} = (A + KC_1)\bar{\xi} + (E_Q + KD_Q)w \\ s = (\Gamma F + N_1C_1)\bar{\xi} - N_1D_Qw \end{cases} \quad (46)$$

is stable and has transfer matrix 0 where $N_1 = \Gamma N$ provided $N \in \mathcal{N}^*$. It is crucial to note that for any N_1 there exists $N \in \mathcal{N}^*$ such that $N_1 = \Gamma N$ since $\ker D_P = \ker \Gamma$. The system (46) is stable and has a closed-loop transfer matrix equal to 0 if and only if the following system is stable and has a transfer matrix equal to 0,

$$\begin{cases} \sigma \bar{\xi} = (A + KC_1)\bar{\xi} + (E_Q + KD_Q)w \\ \sigma \xi_e = (\Gamma F - N_1C_1)\bar{\xi} - N_1D_Qw \\ \bar{s} = \xi_e. \end{cases}$$

We know that this is equivalent to (44). Regarding the fixed modes we have to remember that the above trick added l integrations and hence the optimal fixed modes are equal to the set (45) minus the l integrations we added. \square

7 Conclusions

At first we characterized and parameterized all H_2 optimal measurement feedback controllers. Then our attention is focused on controllers with estimator based architecture. Three different estimator structures, prediction, current, and reduced order estimators, are considered. All the prediction, current, and reduced order estimator based H_2 optimal measurement feedback controllers are characterized. Also, the flexibility they have in simultaneously placing the closed-loop poles at the desired locations is explicitly pointed out. The development given here is complete for the case when a certain subsystem Σ_2 of the given system Σ is left invertible. Nevertheless, for the general case when

the subsystem Σ_2 is not necessarily left invertible, a fairly large subset of estimator based H_2 optimal measurement feedback controllers is characterized. Actually, we claim that there is no loss of flexibility in placing the closed-loop poles at the desired locations whenever one works only with such a subset. For the case of prediction estimators we also clearly indicated the effect of the flexibility of the direct feedthrough matrix on the flexibility in the filter gain.

Appendix

A Proof of Theorem 7

In view of Lemma 5, it follows that obtaining all the H_2 optimal controllers for Σ is equivalent to obtaining all the controllers that achieve a constant closed-loop transfer function equal to 0 for Σ_{PQ}^N where N is an arbitrary element of \mathcal{N}^* . Then it is straightforward to show that any controller Σ_c given in (19) with $Q \in \bar{\mathbf{Q}}$, where

$$\bar{\mathbf{Q}} := \left\{ Q = Q_s + \bar{N} \mid Q_s \in \mathbf{Q}_s \text{ and } \bar{N} \text{ such that } D_P \bar{N} D_Q = 0 \right\}$$

when applied to Σ_{PQ}^N achieves disturbance decoupling and internal stability. This is obviously equivalent to the fact that any controller Σ_c given in (19) with $Q \in \mathbf{Q}$ achieves disturbance decoupling and internal stability.

Next, in order to show that any H_2 optimal proper controller for Σ can be written in the form (19) for some $Q \in \mathbf{Q}$, we proceed as follows. Utilizing the well known Youla parameterization, the general class of admissible proper controllers for Σ_{PQ} can be written as

$$\begin{cases} \sigma\zeta = (A + BF + KC_1)\zeta + By_1 - Ky \\ u = F\zeta + y_1, \end{cases} \quad (\text{A.1})$$

and

$$y_1 = \mathcal{Q}[y - C_1\zeta]. \quad (\text{A.2})$$

Moreover, \mathcal{Q} is a bounded operator with transfer matrix $Q \in \mathcal{RH}_\infty$ being a free parameter with appropriate dimensions, and \mathcal{RH}_∞ denotes the set of proper and stable rational matrices. In order that the controller (A.1) and (A.2) achieves a constant closed-loop transfer function equal to 0 for Σ_{PQ}^N , the free parameter Q must satisfy some additional conditions.

It is well known that the closed-loop system is internally stable if and only if

$Q \in \mathcal{RH}_\infty$. It is also simple to verify that

$$T_{z_{PQ} w_{PQ}}(\Sigma_{PQ}^N \times \Sigma_C) = T_0 - T_q + D_P \bar{N} D_Q$$

where

$$\begin{aligned} T_0 = & (C_P + D_P F)(zI - A - BF)^{-1}(E_Q + B(N + \bar{N})D_Q) \\ & + (C_P + D_P(N + \bar{N})C_1)(zI - A - KC_1)^{-1}(E_Q + KD_Q) \\ & - (C_P + D_P F)(zI - A - BF)^{-1}(zI - A - B(N + \bar{N})C_1) \\ & \times (zI - A - KC_1)^{-1}(E_Q + KD_Q) \end{aligned}$$

and

$$\begin{aligned} T_q = & [(C_P + D_P F)(zI - A - BF)^{-1}B + D_P]Q_s(z) \\ & \times [C_1(zI - A - KC_1)^{-1}(E_Q + KD_Q) + D_Q] \end{aligned}$$

where $Q_s := Q - \bar{N}$. Obviously in order for the closed-loop system transfer matrix to be equal to 0, \bar{N} must be such that $D_P \bar{N} D_Q = 0$. Also, from (Trentelman & Stoorvogel 1995), it is clear that whenever an H_2 optimal controller for Σ exists, the following conditions must be true:

- (i) $\text{im}[E_Q + B(N + \bar{N})D_Q] \subseteq \mathcal{V}^\circ(\Sigma_{2P})$,
- (ii) $\ker[C_P + D_P(N + \bar{N})C_1] \supseteq \mathcal{S}^\circ(\Sigma_{1Q})$,
- (iii) $\mathcal{S}^\circ(\Sigma_{1Q}) \subseteq \mathcal{V}^\circ(\Sigma_{2P})$,
- (iv) $(A + B(N + \bar{N})C_1)\mathcal{S}^\circ(\Sigma_{1Q}) \subseteq \mathcal{V}^\circ(\Sigma_{2P})$.

Thus, in view of the above conditions, it follows that $T_0 \equiv 0$. Hence, the condition $T_{z_{PQ} w_{PQ}}(\Sigma_{PQ} \times \Sigma_C) = 0$ is equivalent to $T_q = 0$ or $Q_s \in \mathbf{Q}_s$. This leads to the results of Theorem 7. \square

B Proof of Lemma 9

It can be checked that

$$\begin{aligned}
& T_{z_{\text{PQ}}w_{\text{PQ}}}(\Sigma_{\text{PQ}} \times \Sigma_{\text{C}}) \\
&= (C_{\text{P}} + D_{\text{P}}F)(zI - A - BF)^{-1}E_{\text{Q}} \\
&\quad + [C_{\text{P}} - (C_{\text{P}} + D_{\text{P}}F)(zI - A - BF)^{-1}(zI - A)] \\
&\quad\quad \times (zI - A - K_{\text{p}}C_1)^{-1}(E_{\text{Q}} + K_{\text{p}}D_{\text{Q}}) \\
&= (C_{\text{P}} + D_{\text{P}}F)(zI - A - BF)^{-1}E_{\text{Q}} \\
&\quad - [(C_{\text{P}} + D_{\text{P}}F)(zI - A - BF)^{-1}BF + D_{\text{P}}F] \\
&\quad\quad \times (zI - A - K_{\text{p}}C_1)^{-1}(E_{\text{Q}} + K_{\text{p}}D_{\text{Q}}) \\
&= T_{s, z_{\text{PQ}}w_{\text{PQ}}}(z) - [(C_{\text{P}} + D_{\text{P}}F)(zI - A - BF)^{-1}B + D_{\text{P}}] \\
&\quad \times F(zI - A - K_{\text{p}}C_1)^{-1}(E_{\text{Q}} + K_{\text{p}}D_{\text{Q}}) \\
&= T_{s, z_{\text{PQ}}w_{\text{PQ}}}(z) - T_{s, z_{\text{PQ}}u_{\text{PQ}}}(z)\Phi(z).
\end{aligned}$$

In the above equations, $T_{s, z_{\text{PQ}}w_{\text{PQ}}}(z)$ denotes the transfer function from w_{PQ} to z_{PQ} when the static state feedback gain F is used to control Σ_{PQ} , and it is given by

$$T_{s, z_{\text{PQ}}w_{\text{PQ}}}(z) = (C_{\text{P}} + D_{\text{P}}F)(zI - A - BF)^{-1}E_{\text{Q}}. \quad (\text{B.1})$$

Also, $T_{s, z_{\text{PQ}}u_{\text{PQ}}}(z)$ is the transfer function from u_{PQ} to z_{PQ} when the static state feedback gain F is used to control Σ_{PQ} , and it is given by

$$T_{s, z_{\text{PQ}}u_{\text{PQ}}}(z) = (C_{\text{P}} + D_{\text{P}}F)(zI - A - BF)^{-1}B + D_{\text{P}}. \quad (\text{B.2})$$

Next, we observe that, since $F \in \mathbf{F}_s^*(A, B, E_{\text{Q}}, C_{\text{P}}, D_{\text{P}})$, we have $T_{s, z_{\text{PQ}}w_{\text{PQ}}}(z) = 0$. Also, we observe that Σ_2 being left invertible ensures the left invertibility of $\Sigma_{2\text{PQ}}$, and consequently the left invertibility of $T_{s, z_{\text{PQ}}u_{\text{PQ}}}(z)$ as left invertibility is invariant under state feedback. These observations lead to the result of Lemma 9. \square

C Proof of Lemma 11

Let F be in $\mathbf{F}_s^*(A, B, E_{\text{Q}}, C_{\text{P}}, D_{\text{P}})$. Utilizing the Youla parameterization, the general class of stabilizing controllers for Σ_{PQ} can be parameterized as

$$\Sigma_{\text{C}} : \begin{cases} \sigma\xi = (A + BF + K_{\text{p}}C_1)\xi - K_{\text{p}}y + By_1 \\ u = F\xi + y_1 \\ y_1 = \mathcal{Q}(y - C_1\xi), \end{cases} \quad (\text{C.1})$$

where \mathcal{Q} is the input-output operator associated with a transfer matrix $Q \in \mathcal{RH}_{\infty}$. The latter is a free parameter, where \mathcal{RH}_{∞} denotes the set of proper and stable rational matrices. Also, F and K_{p} are any fixed matrices such that

$A + BF$ and $A + K_p C_1$ are stable matrices while in this case F is chosen to be the one specified in the set $\mathbf{F}_s^*(A, B, E_Q, C_p, D_p)$. Next, under a controller of the type (C.1), it is straightforward to compute $T_{z_{PQ}w_{PQ}}(z)$ which is the transfer function from w_{PQ} to z_{PQ} of Σ_{PQ} ,

$$T_{z_{PQ}w_{PQ}}(z) = T_{s, z_{PQ}w_{PQ}}(z) \quad (\text{C.2})$$

$$\begin{aligned} & -T_{s, z_{PQ}u_{PQ}}(z) \left[F(zI - A - K_p C_1)^{-1} (E_Q + K_p D_Q) \right. \\ & \left. + Q(z) [C_1(zI - A - K_p C_1)^{-1} (E_Q + K_p D_Q) + D_Q] \right], \quad (\text{C.3}) \end{aligned}$$

where $T_{s, z_{PQ}w_{PQ}}$, as given in (B.1), is the transfer function from w_{PQ} to z_{PQ} , and where $T_{s, z_{PQ}u_{PQ}}(z)$, as given in (B.2), is the transfer function from u_{PQ} to z_{PQ} , both when the static state feedback controller characterized by F is used to control Σ_{PQ} . Since $F \in \mathbf{F}_s^*(A, B, E_Q, C_p, D_p)$, we note that $T_{s, z_{PQ}w_{PQ}}(z) = 0$, and hence (C.2) can be rewritten as

$$\begin{aligned} T_{z_{PQ}w_{PQ}}(z) &= -T_{s, z_{PQ}u_{PQ}}(z) \left[F(zI - A - K_p C_1)^{-1} (E_Q + K_p D_Q) \right. \\ & \left. + Q(z) [C_1(zI - A - K_p C_1)^{-1} (E_Q + K_p D_Q) + D_Q] \right]. \quad (\text{C.4}) \end{aligned}$$

Now, assume that there exists a strictly proper controller that solves the DDPMS for Σ_{PQ} . Since $T_{z_{PQ}w_{PQ}}(z) = 0$ for such a controller, one can conclude that there exists a $Q \in \mathcal{RH}_\infty$ such that

$$\begin{aligned} T_{s, z_{PQ}u_{PQ}}(z) & \left[F(zI - A - K_p C_1)^{-1} (E_Q + K_p D_Q) \right. \\ & \left. + Q(z) [C_1(zI - A - K_p C_1)^{-1} (E_Q + K_p D_Q) + D_Q] \right] = 0. \quad (\text{C.5}) \end{aligned}$$

Next, we observe that the left invertibility of Σ_{PQ} implies the left invertibility of $T_{s, z_{PQ}u_{PQ}}(z)$, and hence (C.5) can be rewritten as

$$\begin{aligned} F(zI - A - K_p C_1)^{-1} (E_Q + K_p D_Q) \\ + Q(z) [C_1(zI - A - K_p C_1)^{-1} (E_Q + K_p D_Q) + D_Q] = 0. \quad (\text{C.6}) \end{aligned}$$

We next have an important observation, namely that the problem of existence of a Q that satisfies (C.6) is equivalent to the solvability of the DDPMS for the following auxiliary system,

$$\Sigma^{aux} : \begin{cases} \dot{x} &= (A + K_p C_1)' x + C_1' u + F' w \\ y &= + I w \\ z_{PQ} &= (E + K_p D_Q)' x + D_Q' u. \end{cases}$$

It is simple to verify, by using the control law $u = -Q'y$ (where Q is the input-output operator associated with Q') with the system Σ^{aux} , that the closed-loop

transfer function from w to z is equal to the right hand side of (C.6). Also, we note that the stability of Q is necessary for the internal stability of the closed-loop system consisting of $u = -Q'y$ and Σ^{aux} . Now to examine the solvability condition for the DDPMS for Σ^{aux} , we first look at the subsystem Σ_1^{aux} of Σ^{aux} which is characterized by $((A + K_p C_1)', F', 0, I)$, and then note that Σ_1^{aux} is square invertible and of minimum phase with no infinite zeros of order greater than or equal to one (note that $\mathcal{S}^\odot(\Sigma_1^{aux}) = 0$). Thus, in view of this and the results of (Stoorvogel & van der Woude 1991), the solvability condition for the DDPMS of Σ^{aux} is given by

$$\text{im } F' \subseteq \mathcal{V}^\odot(\Sigma_2^{aux}) \quad (\text{C.7})$$

where Σ_2^{aux} is the second subsystem of Σ^{aux} and is characterized by $((A + K_p C_1)', C_1', (E_Q + K_p D_Q)', D_Q')$. Now, it is straightforward to verify that the condition (C.7) is equivalent to

$$\mathcal{S}^\odot(\Sigma_{\text{IPQ}}) \subseteq \ker F. \quad (\text{C.8})$$

Hence it follows that, for all F in $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$, the fact that there exists a strictly proper controller that solves the DDPMS for Σ_{PQ} implies (C.8) and hence $\mathbf{K}_p^*(F)$ is non-empty. \square

D Proof of Theorem 14

Let $F \in \mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$. Then, from Lemma 13, it follows that $F \in \mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$. With this choice of F , we observe that

$$T_{s, z_{\text{PQ}} u_{\text{PQ}}}(z) = (C_P + D_P F)(zI - A - BF)^{-1} B + D_P = \begin{pmatrix} T(z) & 0 \end{pmatrix}, \quad (\text{D.1})$$

where $T_{s, z_{\text{PQ}} u_{\text{PQ}}}$, as given in (B.2), is the transfer function from u_{PQ} to z_{PQ} when the static state feedback controller characterized by F is used to control Σ_{PQ} . Also, $T(z) = T_{s, z_{\text{PQ}} u_{\text{PQ}}}(z)\Gamma'$, and, moreover, T is left invertible. Next, consider a prediction estimator based controller that solves the DDPMS for Σ_{PQ} , and uses the given state feedback gain F . Then from the proof of Lemma 9 (see Appendix B) and in view of (D.1), it follows that $A + K_p C_1$ must be stable, and that

$$\Gamma F(zI - A - K_p C_1)^{-1} (E_Q + K_p D_Q) = 0.$$

This leads to the results of Theorem 14. \square

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