

Uniform asymptotics of a Bessel-function series occurring in a transmission-line problem

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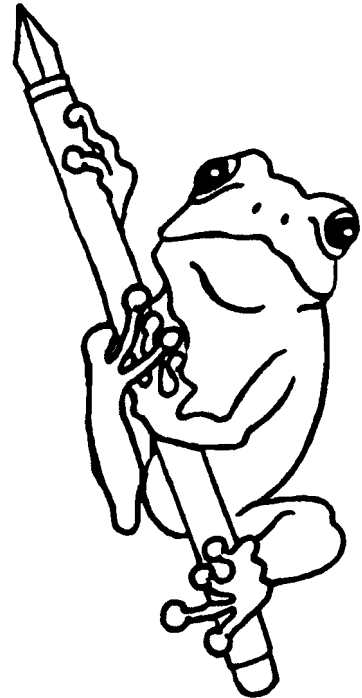
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UNIFORM ASYMPTOTICS OF
A BESSEL-FUNCTION SERIES
OCCURRING IN A
TRANSMISSION-LINE PROBLEM

by

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Uniform asymptotics of a Bessel-function series occurring in a transmission-line problem

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Abstract: Consider the Bessel-function series $S(\theta) = J_k(k) + 2 \sum_{n=1}^{\infty} J_{n+k}(k) \cos(2n\theta)$, where $0 \leq \theta \leq \pi$ and $k = i\lambda$ with real $\lambda \rightarrow \infty$. We determine the complete asymptotic expansion of $S(\theta)$, uniformly valid in θ . The expansion is obtained by the method of steepest descent applied to a contour integral representation for $S(\theta)$. The result is used to establish the high-frequency asymptotics of the linear current density on a thin conducting strip that is part of a transmission line.

Keywords: Uniform asymptotic expansion, Bessel function, method of steepest descent, linear current density, transmission line.

Abbreviated title: Asymptotics of a Bessel-function series

1. Introduction

In a recent paper [3], Belevitch and Boersma determined the transmission parameters (capacitance and impedance per unit length) of a line consisting of a thin strip of finite width, and of a thin return wire situated above the centre of the strip. In terms of Cartesian coordinates x, y, z , the transmission line is infinite in the z -direction, the thin strip has a centre line $|x| \leq a, y = 0$, and the return wire is a cylinder of small radius r centred at $x = 0, y = c$. The strip cross-section is a very thin ellipse of semi-axes a and b , with $b \ll a$. The cylinder is perfectly conducting, whereas the strip has a finite conductivity σ and permeability μ . In the mathematical analysis the elliptic shape of the strip is replaced by the infinitely thin strip $|x| \leq a, y = 0$, with a surface conductivity $2\sigma h$ where $h = b(1 - x^2/a^2)^{1/2}$. The go-conductor (i.e. the strip) carries an alternating current $I e^{i\omega t}$, which returns by the return wire. The time dependence by $e^{i\omega t}$ is suppressed throughout. Since the thickness of the strip is supposed to be small with respect to the skin depth $\delta = (2/\omega\mu\sigma)^{1/2}$, the skin effect in the strip is purely lateral. Then the current density ($A m^{-2}$) is uniform along the thickness coordinate and it suffices to determine the lateral distribution of the linear current density $j(x)$ ($A m^{-1}$) on the strip. In [3] the case of an elliptic cross-section of the strip has been chosen on purpose, because this case is amenable to analytical treatment leading to analytical expressions for the linear current density $j(x)$. The high-frequency asymptotics (as $\omega \rightarrow \infty$) of $j(x)$ is discussed in [3, Sec. 9], however, the asymptotic analysis is formal and not rigorous from a mathematical view-point. Consequently, a certain anomaly or nonuniformity arises in the behaviour of $j(x)$ at the edges $x = \pm a$ of the strip. It is the aim of this paper to remove the anomaly and to correct the high-frequency asymptotics by a rigorous analysis. Thus we shall derive a uniform asymptotic expansion for the linear current density $j(x)$, that is uniformly valid on the strip $-a \leq x \leq a$ including the edges.

For simplicity the asymptotic analysis is restricted to the case of a return at large distance from the strip, corresponding to $c = \infty$. Alternatively, one may consider the strip of elliptic cross-section surrounded by a cylindrical return conductor of large radius, concentric with the go-conductor. The latter transmission-line problem has been solved before by Belevitch [2]. In both problems the proximity effect of the return conductor is neglected, whereas the lateral skin effect in the strip remains unaffected. From [2, form. (20)] or [3, form. (7.45)] we quote the following result for the linear current density $j(x)$ at $x = a \cos \theta$:

$$j(a \cos \theta) = \frac{2I}{\pi a} \frac{\sin \theta}{J_k'(k)} [J_k(k) + 2 \sum_{n=1}^{\infty} J_{n+k}(k) \cos(2n\theta)], \quad 0 \leq \theta \leq \pi. \quad (1.1)$$

Here, $J_{n+k}(k)$ and $J_{n+k}'(k)$ stand for the Bessel function of the first kind and its derivative, of order $n + k$ and argument k ; the parameter k is a normalized frequency given by

$$k = \frac{1}{4} i \omega \mu \sigma a b, \quad (1.2)$$

where μ is the permeability of the medium. The total current in the strip should be equal to I ; indeed, by integration of (1.1) and use of a standard recurrence relation for Bessel functions we

obtain

$$\int_{-a}^a j(x) dx = a \int_0^\pi j(a \cos \theta) \sin \theta d\theta = I \frac{J_k(k) - J_{k+1}(k)}{J_k'(k)} = I. \quad (1.3)$$

Of particular interest is the asymptotic behaviour of the lateral skin effect, as it is reached at high frequencies such that the skin depth δ is small with respect to \sqrt{ab} , or more precisely, such that $b \ll \delta \ll \sqrt{ab}$; see [2, p. 222], [3, Sec. 4]. In that case the electromagnetic problem for the strip becomes frequency-independent and can be solved in a direct manner. The resulting asymptotic linear current density is found from [2, form. (7)] or [3, form. (4.9)] as

$$j_\infty(x) = \frac{I}{\pi(a^2 - x^2)^{1/2}}, \quad (1.4)$$

generally valid for a thin strip of width $2a$ and arbitrary cross-section. Now one would expect that $j(x)$ tends to $j_\infty(x)$ as $\omega \rightarrow \infty$, corresponding to $|k| \rightarrow \infty$. However, a comparison of (1.1) and (1.4) immediately shows the anomaly already mentioned: at the edges $x = \pm a$ of the strip $j(x)$ vanishes for all finite k , whereas $j_\infty(x)$ is infinite there. Moreover, it is desirable to determine the high-frequency asymptotic behaviour of $j(x)$ in more detail, going beyond the expected leading term $j_\infty(x)$. For these reasons we shall derive a uniform asymptotic expansion (as $|k| \rightarrow \infty$) for the Bessel-function series

$$S(\theta) = J_k(k) + 2 \sum_{n=1}^{\infty} J_{n+k}(k) \cos(2n\theta), \quad 0 \leq \theta \leq \pi \quad (1.5)$$

occurring in (1.1). The expansion obtained is uniformly valid for $0 \leq \theta \leq \pi$, and is next used to construct high-frequency asymptotic expansions for the linear current density $j(x)$. As a result it is found that $j(x) \rightarrow j_\infty(x)$ as $\omega \rightarrow \infty$, only on the open interval $-a < x < a$ and not uniformly.

This paper is organized as follows. In Section 2 we deduce an integral representation for $S(\theta)$, which is amenable to asymptotic analysis by the method of steepest descent. The integrand has saddle points of second order and simple-pole singularities which may be close to or even coalesce with the saddle points. By means of an appropriate conformal transformation of the integration variable, the integral representation is reduced to a canonical form in the sense of [4, Sec. 9.3]. The uniform asymptotics of the canonical integral representation is treated in Section 3. By splitting off the contributions of the pole singularities we are led to a uniform asymptotic expansion for $S(\theta)$, in terms of the special function $G(z)$ and an asymptotic series in powers of $k^{-2/3}$. Various properties of the function $G(z)$ are established in Appendix A; while the coefficients in the asymptotic series are evaluated in Appendix B. Finally, in Section 4 we present both uniform and non-uniform high-frequency asymptotic expansions for the linear current density $j(x)$.

2. Integral representation for $S(\theta)$

For convenience we introduce the real parameter λ given by

$$\lambda = \frac{1}{4} \omega \mu \sigma a b , \quad (2.1)$$

so that $k = i\lambda$. Then $J_{n+k}(k)$ changes into the modified Bessel function $I_{n+i\lambda}(\lambda)$, and by use of [11, form. 6.22(3)], in which w is replaced by $w + \pi i/2$, we arrive at the integral representation

$$J_{n+k}(k) = e^{n\pi i/2} e^{-\pi\lambda/2} I_{n+i\lambda}(\lambda) = \frac{1}{2\pi i} \int_{\infty-3\pi i/2}^{\infty+\pi i/2} \exp[\lambda(i \sinh w - iw)] e^{-nw} dw . \quad (2.2)$$

Insert this representation into the series (1.5) for $S(\theta)$ and interchange the summation and integration. In the resulting integrand there appears the summable series

$$1 + 2 \sum_{n=1}^{\infty} e^{-nw} \cos(2n\theta) = \frac{\sinh w}{\cosh w - \cos(2\theta)} , \quad (2.3)$$

valid for $\text{Re } w > 0$. Thus we obtain the following integral representation for $S(\theta)$:

$$S(\theta) = \frac{1}{2\pi i} \int_{\infty-3\pi i/2}^{\infty+\pi i/2} \exp[\lambda \Phi(w)] \frac{\sinh w}{\cosh w - \cos(2\theta)} dw , \quad 0 \leq \theta \leq \pi \quad (2.4)$$

where

$$\Phi(w) = i \sinh w - iw . \quad (2.5)$$

In (2.4) the integrand is analytic in the complex w -plane except for pole singularities at $w = \pm 2i\theta + 2n\pi i$, $n \in \mathbb{Z}$. The integration contour runs from $\infty - 3\pi i/2$ to $\infty + \pi i/2$, such that all poles are to the left of the contour.

The asymptotics of the integral (2.4) as $\lambda \rightarrow \infty$ will be treated by the method of steepest descent, also called the saddle point method; cf. [11, Chapter VIII], [7, Chapter 5], [4, Chapter 7]. Starting from the equation

$$\Phi'(w) = i(\cosh w - 1) = 0 , \quad (2.6)$$

it is found that the saddle points of $\Phi(w)$ are located at $w = 2n\pi i$, $n \in \mathbb{Z}$. Furthermore, since $\Phi''(2n\pi i) = 0$, $\Phi'''(2n\pi i) = i \neq 0$, the saddle points are of order 2. From each saddle point there emanate three steepest-descent paths (SDP) and three steepest-ascent paths (SAP). For the saddle point $w = 0$, the SDPs are described by

$$\text{Im } \Phi(w) = \text{Im } \Phi(0) = 0 , \quad \text{Re } \Phi(w) \leq \text{Re } \Phi(0) = 0 , \quad (2.7)$$

which by setting $w = u + iv$, is reduced to

$$\sinh u \cos v - u = 0, \quad -\cosh u \sin v + v \leq 0. \quad (2.8)$$

Clearly, one SDP coincides with the negative imaginary axis $u = 0, v \leq 0$, while the other two SDPs are described by the equation

$$\cos v = \frac{u}{\sinh u}, \quad u \geq 0 \text{ or } u \leq 0, \quad 0 \leq v < \pi/2. \quad (2.9)$$

These SDPs are symmetric with respect to the imaginary axis, and have the line $v = \pi/2$ as their asymptote when $u \rightarrow \pm\infty$. At $w = 0$ the SDPs make an angle $\pi/6$ with the positive or negative real axis. A sketch of the SDPs and SAPs through $w = 0$ is shown in Fig. 1. Here the SAPs are described by

$$\text{Im } \Phi(w) = \text{Im } \Phi(0) = 0, \quad \text{Re } \Phi(w) \geq \text{Re } \Phi(0) = 0, \quad (2.10)$$

and it is easily seen that the SAPs are just the mirror images of the SDPs with respect to the real axis. The SDPs and SAPs emanating from the saddle point $w = 2n\pi i$ are found by a translation over $2n\pi i$.

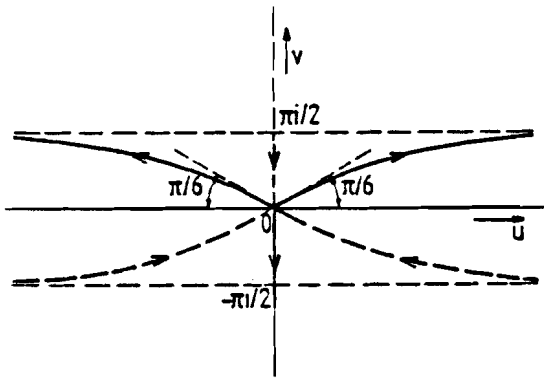


Fig. 1. SDPs (solid curves) and SAPs (dashed curves) emanating from the saddle point $w = 0$; the arrows indicate the direction of decreasing $\text{Re } \Phi(w)$.

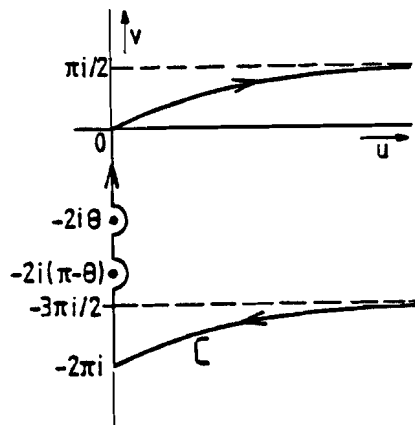


Fig. 2. Integration contour C in the case $0 < \theta < \pi$.

The integration contour in (2.4), which runs from $\infty - 3\pi i/2$ to $\infty + \pi i/2$, is now deformed into the contour C which is a concatenation of SDPs; the deformation should be such that no poles of the integrand are intercepted. In the case $0 < \theta < \pi$, the contour C consists of the SDPs from the saddle points $w = 0, w = -2\pi i$ to $w = \infty + \pi i/2, w = \infty - 3\pi i/2$, respectively, and of the line segment connecting $w = 0$ and $w = -2\pi i$, with semi-circular indentations to the right of the poles at $w = -2i\theta$ and $w = -2i(\pi - \theta)$; see Fig. 2. In the cases $\theta = 0, \theta = \pi$, the contour C is to be modified in an obvious manner. With the contour C thus determined, we have the basic integral representation

$$S(\theta) = \frac{1}{2\pi i} \int_C \exp[\lambda \Phi(w)] \frac{\sinh w}{\cosh w - \cos(2\theta)} dw, \quad 0 \leq \theta \leq \pi. \quad (2.11)$$

Along the contour C the exponent $\Phi(w)$ is real and attains its maximum value 0 at the saddle point $w = 0$. Then the asymptotic expansion (as $\lambda \rightarrow \infty$) of the integral (2.11) can be determined by a standard application of Laplace's method. In the derivation, the two cases $0 < \theta < \pi$ and $\theta = 0$ or π have to be distinguished and treated separately, which is a disadvantage. Moreover, in the case $0 < \theta < \pi$ the asymptotic analysis presupposes $\lambda \sin^3 \theta$ (rather than λ) to be large, and as a consequence the asymptotic expansion obtained is not uniformly valid for $0 < \theta < \pi$. To overcome these defects, we will derive a uniform asymptotic expansion for $S(\theta)$, uniformly valid for $0 \leq \theta \leq \pi$, along the lines of [4, Chapter 9]. As a preliminary, we first reduce the integral representation (2.11) to a proper canonical form in the sense of [4, Sec. 9.3].

In (2.11), we apply the transformation of integration variable defined by

$$\Phi(w) = i \sinh w - iw = -\frac{1}{6} t^3. \quad (2.12)$$

Here, the introduction of t^3 relates to the fact that $w = 0$ is a second-order saddle point of $\Phi(w)$. To determine t as a single-valued function of w , we choose the branch of the cube root as

$$t = iw \left[\frac{6}{w^3} (\sinh w - w) \right]^{1/3} = iw \left[1 + \frac{1}{20} w^2 + \frac{1}{840} w^4 + \dots \right]^{1/3}, \quad (2.13)$$

in which $[\]^{1/3} = 1$ at $w = 0$. As a result, t is an analytic function of w in a vicinity of $w = 0$. The relations (2.12) and (2.13) define a mapping, $t = t(w)$, of the complex w -plane into the complex t -plane. We now examine where this mapping is conformal, and therefore invertible. Consider in the w -plane the domain \mathcal{D} that is bounded below by the SDPs from $w = -2\pi i$ to $w = \pm \infty - 3\pi i/2$, and bounded above by the SAPs from $w = 2\pi i$ to $w = \pm \infty + 3\pi i/2$; see Fig. 3. The domain \mathcal{D} is symmetric with respect to the real axis, and we will determine the image of \mathcal{D} under the mapping $t = t(w)$. From (2.12) and (2.13) it is readily found that the real and imaginary axes of the w -plane transform according to

$$w = u \quad (u \in \mathbb{R}) \rightarrow t = i \left[6(\sinh u - u) \right]^{1/3} \operatorname{sgn}(u), \quad (2.14)$$

$$w = iv \quad (v \in \mathbb{R}) \rightarrow t = - \left[6(v - \sin v) \right]^{1/3} \operatorname{sgn}(v).$$

Hence, the real axis of the w -plane is mapped one-to-one onto the imaginary axis of the t -plane; the segment $[-2\pi i, 2\pi i]$ along the v -axis is mapped one-to-one onto the segment $[-(12\pi)^{1/3}, (12\pi)^{1/3}]$ along the Re t -axis.

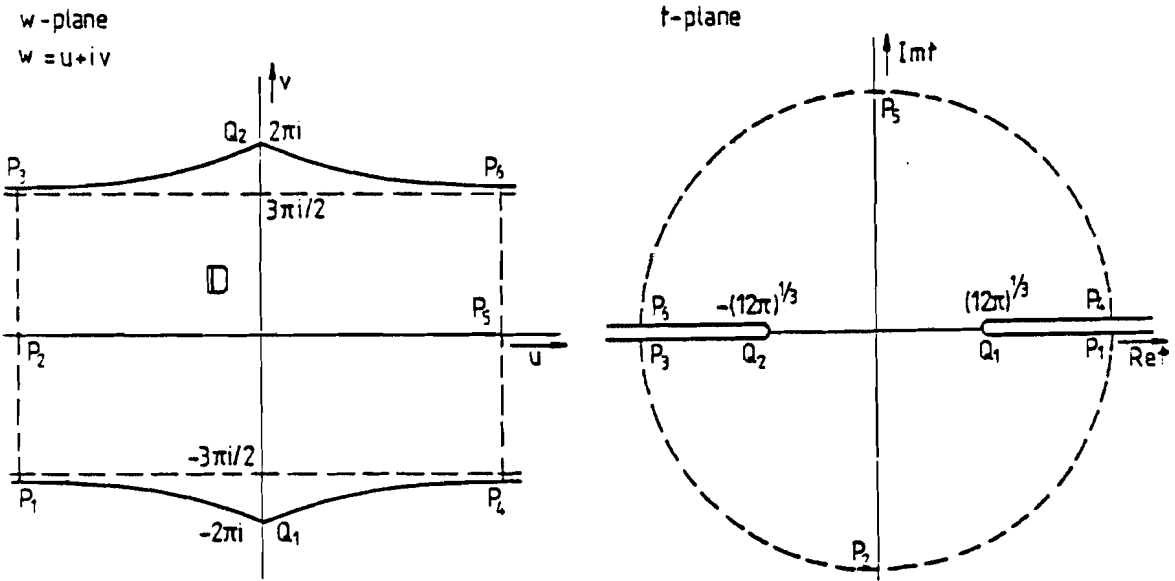


Fig. 3. Conformal mapping of the domain D onto the t -plane with branch cuts $[(12\pi)^{1/3}, \infty)$ and $(-\infty, -(12\pi)^{1/3}]$ along the real axis. Corresponding points are marked by the same letters.

Next, we investigate the mapping of the boundary of the domain D in Fig. 3. Along the SDPs $Q_1 P_4$ and $Q_1 P_1$ emanating from $w = -2\pi i$, one has $\Phi(w) = \Phi(-2\pi i) - \tau = -2\pi - \tau$, where $\tau \geq 0$ increases monotonically with distance to $w = -2\pi i$. More precisely, one has $\Phi(w) = -2\pi - \tau - i0$ along $Q_1 P_4$ and $\Phi(w) = -2\pi - \tau + i0$ along $Q_1 P_1$. By taking the proper branch of the cube root of $\Phi(w)$, the resulting t -images are found as $t = (12\pi + 6\tau)^{1/3} \pm i0$, where the upper (lower) sign applies to $Q_1 P_4$ ($Q_1 P_1$). Thus the two SDPs are mapped one-to-one onto the upper and lower sides of the half-line $\text{Re } t \geq (12\pi)^{1/3}$, $\text{Im } t = 0$. Likewise, the two SAPs emanating from $w = 2\pi i$ are mapped one-to-one onto the upper and lower sides of the half-line $\text{Re } t \leq -(12\pi)^{1/3}$, $\text{Im } t = 0$. Furthermore, we observe that for $\text{Re } w = u \rightarrow \pm\infty$ the mapping $t = t(w)$ can be approximated by

$$t^3 = -6\Phi(w) = \mp 3i e^{\pm w}, \quad t = \pm 3^{1/3} i e^{\pm w/3}, \quad (\text{Re } w \rightarrow \pm\infty) \quad (2.15)$$

where the proper branch of the cube root has been taken. Hence, for large positive u_0 the segments $w = u_0 + iv$, $-3\pi/2 \leq v \leq 3\pi/2$, and $w = -u_0 + iv$, $-3\pi/2 \leq v \leq 3\pi/2$, transform into the semi-circles $|t| = (3e^{u_0})^{1/3}$, $0 \leq \arg t \leq \pi$, and $|t| = (3e^{-u_0})^{1/3}$, $-\pi \leq \arg t \leq 0$, respectively, as represented by the dashed lines in Fig. 3. From the preceding results we conclude that the mapping function $t = t(w)$ is univalent or one-to-one on the boundary of the domain D . Then by Darboux's theorem [5, p. 115], the mapping is also univalent in D . Thus the function $t = t(w)$, defined by (2.12) and (2.13), determines a conformal mapping of the domain D in the w -plane onto the t -plane with branch cuts $[(12\pi)^{1/3}, \infty)$ and $(-\infty, -(12\pi)^{1/3}]$ along the real axis. The latter result has also been proved by Meijer [9, Teil IV, pp. 11-15] in a somewhat different manner. The function $t = t(w)$, univalent in D , has the inverse function $w = w(t)$, say, which is analytic in the cut t -plane. For later use we establish the Taylor series of $w(t)$ around $t = 0$, viz.

$$w = w(t) = -it - \frac{i}{60} t^3 - \frac{i}{1400} t^5 + O(t^7), \quad (2.16)$$

convergent for $|t| < (12\pi)^{1/3}$.

We now return to the integral representation (2.11) for $S(\theta)$, in which we introduce the change of integration variable $w \rightarrow t = t(w)$, defined by (2.12) and (2.13). Notice that the original integration contour C lies in the domain \mathcal{D} of Fig. 3. Under the mapping $t = t(w)$, C transforms into the contour L in the complex t -plane; L consists of the positive real axis and the half-line $\arg t = 2\pi/3$, and runs from ∞ to $\infty e^{2\pi i/3}$. In addition, the contour L has semi-circular indentations above $t = \gamma$ and $t = \gamma_1$, which are the images of the poles at $w = -2i\theta$ and $w = -2i(\pi - \theta)$; hence,

$$\gamma = [6(2\theta - \sin(2\theta))]^{1/3}, \quad \gamma_1 = [6(2\pi - 2\theta + \sin(2\theta))]^{1/3}. \quad (2.17)$$

In this manner we arrive at the integral representation in canonical form:

$$S(\theta) = \frac{1}{2\pi i} \int_L \exp[-\frac{1}{6} \lambda t^3] \frac{\sinh w}{\cosh w - \cos(2\theta)} \frac{dw}{dt} dt, \quad 0 \leq \theta \leq \pi. \quad (2.18)$$

Here the integrand is an analytic function of t in the cut t -plane (with branch cuts $[(12\pi)^{1/3}, \infty)$ and $(-\infty, -(12\pi)^{1/3}]$ along the real axis), except for pole singularities at $t = \pm\gamma$ and $t = \pm\gamma_1$.

3. Uniform asymptotic expansion of $S(\theta)$

Since $S(\theta) = S(\pi - \theta)$, it suffices to establish the asymptotics of $S(\theta)$ for $0 \leq \theta \leq \pi/2$. In that case the integral (2.18) has the following features: the poles $t = \pm\gamma_1$ of the integrand are bounded away from the second-order saddle point $t = 0$, whereas the poles $t = \pm\gamma$ may be close to or even coalesce with the saddle point. Coalescence occurs when $\theta = 0$ and this explains the nonuniformity of the asymptotic expansion discussed below (2.11). At the simple poles $t = \pm\gamma$ the residues of the integrand in (2.18) are given by

$$\text{Res}_{t=\pm\gamma} \frac{\sinh w}{\cosh w - \cos(2\theta)} \frac{dw}{dt} = \text{Res}_{w=\mp 2i\theta} \frac{\sinh w}{\cosh w - \cos(2\theta)} = 1. \quad (3.1)$$

Following Van der Waerden [10], we now subtract the contributions of the poles $t = \pm\gamma$ by setting

$$\frac{\sinh w}{\cosh w - \cos(2\theta)} \frac{dw}{dt} = \frac{1}{t - \gamma} + \frac{1}{t + \gamma} + H(t; \theta). \quad (3.2)$$

Then the function $H(t; \theta)$ has poles at $t = \pm\gamma_1$ only, and $H(t; \theta)$ is analytic in t for $|t| < \min_{0 \leq \theta \leq \pi/2} \gamma_1 = (6\pi)^{1/3}$; note that the latter bound does not depend on θ . Furthermore, we introduce the special function $G(z)$ defined by

$$G(z) = \frac{3}{2\pi i} \int_{-\infty \exp(-\pi i/3)}^{\infty \exp(\pi i/3)} \frac{e^{t^3}}{t-z} dt, \quad z \in \mathcal{C} \quad (3.3)$$

where the integration contour runs from $\infty e^{-\pi i/3}$ to $\infty e^{\pi i/3}$, such that the pole at $t = z$ lies to the left of the contour. It is easily recognized that $G(z)$ is an entire function. Further properties of $G(z)$ are established in Appendix A. On inserting (3.2) into (2.18), the first and second resulting integrals are expressible in terms of the function $G(z)$, and we obtain

$$S(\theta) = \frac{1}{3} G((\lambda/6)^{1/3} \gamma e^{-\pi i/3}) + \frac{1}{3} G(-(\lambda/6)^{1/3} \gamma e^{-\pi i/3}) \\ + \frac{1}{2\pi i} \int_L \exp[-\frac{1}{6} \lambda t^3] H(t; \theta) dt, \quad 0 \leq \theta \leq \pi/2. \quad (3.4)$$

The final integral in (3.4) is a Laplace integral and its asymptotic expansion (as $\lambda \rightarrow \infty$) is found by replacing $H(t; \theta)$ by its power-series expansion around $t = 0$, followed by a term-by-term integration. Since $H(t; \theta)$ is an odd function of t , its series-expansion takes the form

$$H(t; \theta) = \sum_{n=0}^{\infty} A_n(\theta) t^{2n+1}, \quad |t| < (6\pi)^{1/3}, \quad 0 \leq \theta \leq \pi/2. \quad (3.5)$$

Here, the coefficients $A_n(\theta)$ which are yet to be determined, are bounded and continuous functions of θ for $0 \leq \theta \leq \pi/2$. In the term-by-term integration we need the auxiliary integral

$$\frac{1}{2\pi i} \int_L \exp[-\frac{1}{6} \lambda t^3] t^{2n+1} dt = \frac{e^{(2n+2)\pi i/3}}{\pi} \sin((2n+2)\pi/3) \int_0^{\infty} \exp[-\frac{1}{6} \lambda t^3] t^{2n+1} dt \\ = \frac{e^{(2n+2)\pi i/3}}{3\pi} \sin((2n+2)\pi/3) \Gamma\left[\frac{2n+2}{3}\right] \left[\frac{\lambda}{6}\right]^{-(2n+2)/3}. \quad (3.6)$$

In this manner we obtain the complete uniform asymptotic expansion of $S(\theta)$, viz.

$$S(\theta) \sim \frac{1}{3} G(-i(k/6)^{1/3} \gamma) + \frac{1}{3} G(i(k/6)^{1/3} \gamma) \\ - \frac{1}{3\pi} \sum_{n=0}^{\infty} (-1)^n 6^{(2n+2)/3} \sin((2n+2)\pi/3) \Gamma\left[\frac{2n+2}{3}\right] A_n(\theta) k^{-(2n+2)/3}, \quad (3.7)$$

$$(|k| \rightarrow \infty, 0 \leq \theta \leq \pi/2)$$

where we have restored the original parameter k of (1.2) by setting $\lambda = -ik$. The present asymptotic expansion which is uniformly valid for $0 \leq \theta \leq \pi/2$, involves the sum of two functions $G(\mp i(k/6)^{1/3} \gamma)$ with γ given by (2.17), and an asymptotic series in powers of $k^{-2/3}$. The defining integral (3.3) for $G(z)$ acts as a "canonical integral" (in the terminology of [4, Sec. 9.3]) which has the same features as the original integral representation (2.18) for $S(\theta)$: the integrand in (3.3) has a simple pole at $t = z$, which may be close to or even coalesce with the second-order saddle point $t = 0$.

The coefficients $A_n(\theta)$ with $n = 0, 1, 2, \dots$, $0 \leq \theta \leq \pi/2$, are evaluated in Appendix B. From (B.6) and (B.9) we have for $A_0(\theta)$:

$$A_0(\theta) = \frac{2}{\gamma^2} - \frac{1}{2 \sin^2 \theta}, \quad 0 < \theta \leq \pi/2; \quad A_0(0) = -\frac{1}{10}, \quad (3.8)$$

where γ is given by (2.17). Results for $A_1(\theta)$ and $A_2(\theta)$ are presented in (B.12) and (B.13). For general $n = 0, 1, 2, \dots$, the coefficient $A_n(\theta)$ is found to be (cf. (B.21) and (B.25))

$$A_n(\theta) = \frac{2}{\gamma^{2n+2}} - \left[\frac{2^{-2n-1}}{(\sin \theta)^{2n+2}} + \frac{b_n^{(n)}}{(\sin \theta)^{2n}} + \dots + \frac{b_1^{(n)}}{\sin^2 \theta} \right], \quad 0 < \theta \leq \pi/2; \quad (3.9)$$

$$A_n(0) = c_0^{(n)}, \quad (3.10)$$

where the coefficients $b_n^{(n)}, \dots, b_1^{(n)}, c_0^{(n)}$ are determined by the Laurent-series expansion of $2\gamma^{-2n-2}$ in powers of $\sin \theta$:

$$\begin{aligned} \frac{2}{\gamma^{2n+2}} &= \frac{2^{-2n-1}}{(\sin \theta)^{2n+2}} + \frac{b_n^{(n)}}{(\sin \theta)^{2n}} + \dots + \frac{b_1^{(n)}}{\sin^2 \theta} \\ &+ \sum_{m=0}^{\infty} c_m^{(n)} (\sin \theta)^{2m}, \quad 0 < \theta \leq \pi/2. \end{aligned} \quad (3.11)$$

It is pointed out that the expression between square brackets in (3.9) is precisely the principal part of the Laurent series (3.11), whereas $c_0^{(n)}$ in (3.10) is the constant term in the same Laurent series. The coefficient $A_n(\theta)$ can be explicitly determined by evaluation of (3.11) using the expansions (B.3) or (B.4); the evaluation might be carried out by means of a computer algebra program package. From (3.9) and (3.11) it follows that $A_n(\theta) \rightarrow c_0^{(n)} = A_n(0)$ as $\theta \downarrow 0$. Hence, the coefficient $A_n(\theta)$ is bounded and continuous in θ for $0 \leq \theta \leq \pi/2$.

Finally, we determine the asymptotics of $S(\theta)$ in two special cases. Firstly, let $\theta = 0$, then $\gamma = 0$ and $G(0) = 1$, whereupon the asymptotic expansion (3.7) specializes to

$$S(0) \sim \frac{2}{3} - \frac{1}{3\pi} \sum_{n=0}^{\infty} (-1)^n 6^{(2n+2)/3} \sin((2n+2)\pi/3) \Gamma\left[\frac{2n+2}{3}\right] A_n(0) k^{-(2n+2)/3}, \quad (3.12)$$

$$(|k| \rightarrow \infty).$$

Secondly, consider the case $0 < \theta \leq \pi/2$. Suppose that $|k|^{1/3} \gamma$ or equivalently $|k| \sin^3 \theta$ is large, then the functions $G(\mp i(k/6)^{1/3} \gamma)$ in (3.7) can be replaced by their asymptotic expansions (A.5). As a result we obtain the complete asymptotic expansion

$$S(\theta) \sim \frac{1}{3\pi} \sum_{n=0}^{\infty} (-1)^n 6^{(2n+2)/3} \sin((2n+2)\pi/3) \Gamma\left[\frac{2n+2}{3}\right] \cdot \left[\frac{2}{\gamma^{2n+2}} - A_n(\theta) \right] k^{-(2n+2)/3}, \quad (|k| \sin^3 \theta \rightarrow \infty, \quad 0 < \theta \leq \pi/2). \quad (3.13)$$

From (3.9) it is recognized that the coefficient $2\gamma^{-2n-2} - A_n(\theta)$ in (3.13) is equal to the principal part of the Laurent series of $2\gamma^{-2n-2}$ in powers of $\sin \theta$. Hence, this coefficient has a singularity at $\theta = 0$. By symmetry the asymptotic expansion (3.13) is valid for $0 < \theta < \pi$, but not uniformly. The expansion fails in "boundary layers" around $\theta = 0$ and $\theta = \pi$, of width $O(|k|^{-1/3})$.

4. High-frequency asymptotics of the current density

It is recalled from (1.1) and (1.5) that the linear current density on the strip is given by

$$j(a \cos \theta) = \frac{2I}{\pi a} \frac{\sin \theta}{J'_k(k)} S(\theta), \quad 0 \leq \theta \leq \pi \quad (4.1)$$

where the parameter k is given by (1.2). This section deals with the asymptotics of the current density at high frequencies, corresponding to $\omega \rightarrow \infty$ or $|k| \rightarrow \infty$. The complete asymptotic expansion of $J'_k(k)$ has been derived by Meijer [9, Teil IV, p. 9], reading

$$J'_k(k) \sim \frac{1}{3\pi} \sum_{n=0}^{\infty} 6^{(2n+2)/3} \sin((2n+2)\pi/3) \Gamma\left[\frac{2n+2}{3}\right] b_n k^{-(2n+2)/3}, \quad (|k| \rightarrow \infty) \quad (4.2)$$

where b_n is the coefficient of u^n in the power-series expansion of

$$\left[1 + \frac{u}{3!} + \frac{u^2}{5!} + \dots \right] \left[1 + \frac{6}{5!} u + \frac{6}{7!} u^2 + \dots \right]^{- (2n+2)/3}.$$

By retaining only the first two terms in (4.2), we have

$$J'_k(k) = \frac{2^{-1/3} 3^{1/6}}{\pi} \Gamma\left(\frac{2}{3}\right) k^{-2/3} - \frac{2^{-2/3} 3^{-1/6}}{5\pi} \Gamma\left(\frac{1}{3}\right) k^{-4/3} + O(k^{-8/3}), \quad (|k| \rightarrow \infty) \quad (4.3)$$

in accordance with [1, form. 9.3.33]. In (4.1) we substitute for $S(\theta)$ the uniform asymptotic expansion (3.7). The asymptotic series in (3.7) is truncated after the third term with a remainder of order $O(k^{-8/3})$, and is next divided by the expansion (4.3) of $J'_k(k)$. The necessary values of the coefficients $A_0(\theta)$ and $A_1(\theta)$ are taken from (3.8) and (B.12). As a result we obtain the uniform asymptotic expansion for the current density j :

$$\begin{aligned}
 j(a \cos \theta) = \frac{2I}{\pi a} \sin \theta & \left[\frac{G(-i(k/6)^{1/3} \gamma) + G(i(k/6)^{1/3} \gamma)}{3J'_k(k)} \right. \\
 & - \left[\frac{2}{\gamma^2} - \frac{1}{2 \sin^2 \theta} \right] - \left[\frac{4}{\gamma^4} + \frac{2}{5\gamma^2} - \frac{1}{4 \sin^4 \theta} \right] q k^{-2/3} \\
 & \left. - \left[\frac{4}{5\gamma^4} + \frac{2}{25\gamma^2} - \frac{1}{20 \sin^4 \theta} \right] q^2 k^{-4/3} + O(k^{-2}) \right], \quad (4.4)
 \end{aligned}$$

($|k| \rightarrow \infty$, $0 \leq \theta \leq \pi/2$)

where $q = 6^{-1/3} \Gamma(\frac{1}{3}) / \Gamma(\frac{2}{3})$, γ is given by (2.17), and the function $G(z)$ is defined by (3.3). The present asymptotic expansion is uniformly valid for $0 \leq \theta \leq \pi/2$.

In case $0 < \theta < \pi$ and $|k| \sin^3 \theta$ is large, we may substitute the non-uniform asymptotic expansion (3.13) for $S(\theta)$ in (4.1). Only the first two terms in (3.13) are retained, and we next divide by the two-term expansion (4.3) for $J'_k(k)$. In this manner we arrive at the asymptotic expansion

$$j(a \cos \theta) = \frac{I}{\pi a \sin \theta} \left[1 + \frac{q}{2 \sin^2 \theta} k^{-2/3} + \frac{q^2}{10 \sin^2 \theta} k^{-4/3} + O((k \sin^3 \theta)^{-2}) \right], \quad (4.5)$$

($|k| \sin^3 \theta \rightarrow \infty$, $0 < \theta < \pi$)

which obviously is not uniformly valid for $0 < \theta < \pi$. The expansion fails in "boundary layers" around $\theta = 0$ and $\theta = \pi$, of width $O(|k|^{-1/3})$. The leading term in (4.5) is equal to the asymptotic linear current density $j_\infty(a \cos \theta)$ as given by (1.4). Thus we conclude that $j(x) \rightarrow j_\infty(x)$ as $\omega \rightarrow \infty$, only on the open interval $-a < x < a$ and not uniformly. At the edges $x = \pm a$ of the strip $j(x)$ vanishes, and near both edges there is a transition region of width $O(|k|^{-1/3})$, in which $j(x)$ increases steeply from 0 to its asymptotic value $j_\infty(x)$.

Appendix A. Properties of the function $G(z)$

In (3.3) the function $G(z)$ has been defined by

$$G(z) = \frac{3}{2\pi i} \int_{\infty \exp(-\pi i/3)}^{\infty \exp(\pi i/3)} \frac{e^{t^3}}{t-z} dt, \quad z \in \mathbb{C}. \quad (A.1)$$

Here the integration contour runs from $\infty e^{-\pi i/3}$ to $\infty e^{\pi i/3}$, such that the pole at $t = z$ lies to the left of the contour. Obviously, $G(z)$ is an entire function. Two alternative representations for $G(z)$ are derived now. Firstly, by integrating along the contour L_0 formed by the two half-lines

$\arg t = \pm \pi/3$, we obtain

$$G(z) = \begin{cases} 3e^{z^3} + \frac{3}{2\pi i} \int_{L_0} \frac{e^{t^3}}{t-z} dt, & -\pi/3 < \arg z < \pi/3, \\ \frac{3}{2\pi i} \int_{L_0} \frac{e^{t^3}}{t-z} dt, & \pi/3 < \arg z < 5\pi/3. \end{cases} \quad (\text{A.2})$$

For $z \in L_0$, the function $G(z)$ is defined by continuity. Secondly, by making the change of variable $t^3 \rightarrow t$ in (A.1), we are led to the representation

$$G(z) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} \frac{e^t t^{-2/3}}{t^{1/3} - z} dt. \quad (\text{A.3})$$

Here the integration contour is a loop which starts and ends at $-\infty$, and encircles the origin counter-clockwise, such that $-\pi < \arg t < \pi$ and $|t| > |z|^3$ along the contour. We will establish various properties of the function $G(z)$.

(i) For large $|z|$ the main contribution to the integral in (A.2) stems from a vicinity of the saddle point $t = 0$. In this vicinity we have the expansion

$$\frac{1}{t-z} = - \sum_{n=0}^{\infty} t^n z^{-n-1}, \quad |t| < |z|$$

which is inserted into (A.2). Then by a term-by-term integration, using the auxiliary integral

$$\frac{3}{2\pi i} \int_{L_0} e^{t^3} t^n dt = \frac{3}{\pi} \sin((n+1)\pi/3) \int_0^{\infty} e^{-t^3} t^n dt = \frac{1}{\pi} \sin((n+1)\pi/3) \Gamma\left[\frac{n+1}{3}\right], \quad (\text{A.4})$$

we obtain the complete asymptotic expansion of $G(z)$, viz.

$$G(z) \sim \begin{cases} 3e^{z^3} - \frac{1}{\pi} \sum_{n=0}^{\infty} \sin((n+1)\pi/3) \Gamma\left[\frac{n+1}{3}\right] z^{-n-1}, & (|z| \rightarrow \infty, |\arg z| \leq \pi/6) \\ -\frac{1}{\pi} \sum_{n=0}^{\infty} \sin((n+1)\pi/3) \Gamma\left[\frac{n+1}{3}\right] z^{-n-1}, & (|z| \rightarrow \infty, \pi/6 < \arg z < 5\pi/6). \end{cases} \quad (\text{A.5})$$

Notice that the term $3e^{z^3}$ is dominant over the asymptotic series in the sector $|\arg z| \leq \pi/6$. In the sectors $\pi/6 < |\arg z| < \pi/3$ the term $3e^{z^3}$ is exponentially small and has therefore been ignored.

(ii) We insert the expansion

$$\frac{1}{t^{1/3} - z} = \sum_{n=0}^{\infty} t^{-(n+1)/3} z^n, \quad |t| > |z|^3$$

into the integral (A.3) and integrate term-by-term. Then by means of Hankel's contour integral for the Γ -function, we are led to the series-expansion

$$G(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\frac{1}{3}n + 1)} \quad (\text{A.6})$$

which is convergent for all $z \in \mathcal{C}$. In particular, it is found that $G(0) = 1$.

(iii) By differentiation of (A.2) with respect to z and subsequent integration by parts in the resulting integral, we deduce the relation

$$G'(z) - 3z^2 G(z) = \frac{9}{2\pi i} \int_{L_0} e^{t^3} (t+z) dt = \frac{3^{3/2}}{2\pi} [\Gamma(\frac{2}{3}) + \Gamma(\frac{1}{3})z] \quad (\text{A.7})$$

where the latter integral was evaluated by means of (A.4). Differentiate (A.7) twice with respect to z , then the function $G(z)$ is found to satisfy the differential equation

$$y''' - 3z^2 y'' - 12z y' - 6y = 0. \quad (\text{A.8})$$

Further solutions of (A.8) are given by $y = G(z e^{\pm 2\pi i/3})$ and $y = e^{z^3}$, and these solutions are related by

$$G(z) + G(z e^{2\pi i/3}) + G(z e^{-2\pi i/3}) = 3 e^{z^3}, \quad (\text{A.9})$$

as it can easily be verified by means of (A.1).

(iv) Rewrite (A.7) as

$$\frac{d}{dz} [e^{-z^3} G(z)] = \frac{3^{3/2}}{2\pi} e^{-z^3} [\Gamma(\frac{2}{3}) + \Gamma(\frac{1}{3})z]$$

and integrate the latter relation from 0 to z . Recalling the initial value $G(0) = 1$, we arrive at the representation

$$G(z) = e^{z^3} + \frac{3^{3/2}}{2\pi} e^{z^3} \int_0^z e^{-t^3} [\Gamma(\frac{2}{3}) + \Gamma(\frac{1}{3})t] dt. \quad (\text{A.10})$$

Alternatively, the integral in (A.10) can be expressed in terms of incomplete gamma functions yielding

$$G(z) = e^{z^3} \left[1 + \frac{\gamma(\frac{1}{3}, z^3)}{\Gamma(\frac{1}{3})} + \frac{\gamma(\frac{2}{3}, z^3)}{\Gamma(\frac{2}{3})} \right]. \quad (\text{A.11})$$

(v) The function $G(z)$ is a special case of Mittag-Leffler's function $E_\alpha(z)$ defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}. \quad (\text{A.12})$$

Indeed, from (A.6) it is immediately recognized that $G(z) = E_{1/3}(z)$. Most of the above properties of $G(z)$ can be deduced from the corresponding properties of $E_\alpha(z)$ as listed in [8, Sec. 18.1]. Since $\alpha = \frac{1}{3}$ is rational, the function $G(z)$ is also a special case of a function studied by Boersma

[6], which in its turn is a special case of Meijer's G -function. Thus, by means of [6, form. (14)] the function $G(z)$ can be expressed in terms of Meijer's G -function by

$$G(z) = \frac{3}{4\pi^2} G_{2,3}^{3,2} \left[-z^3 \left| \begin{matrix} \frac{2}{3}, \frac{1}{3} \\ 0, \frac{1}{3}, \frac{2}{3} \end{matrix} \right. \right]. \quad (\text{A.13})$$

Appendix B. Evaluation of the coefficients $A_n(\theta)$

In (3.2) we introduced the function $H(t; \theta)$ given by

$$H(t; \theta) = \frac{\sinh w}{\cosh w - \cos(2\theta)} \frac{dw}{dt} - \frac{1}{t - \gamma} - \frac{1}{t + \gamma}, \quad (\text{B.1})$$

where $w = w(t)$ is the inverse of the function $t = t(w)$ defined by (2.12) and (2.13). This appendix deals with the evaluation of the coefficients $A_n(\theta)$ that appear in the power-series expansion of $H(t; \theta)$ around $t = 0$, viz.

$$H(t; \theta) = \sum_{n=0}^{\infty} A_n(\theta) t^{2n+1}, \quad |t| < (6\pi)^{1/3}, \quad 0 \leq \theta \leq \pi/2, \quad (\text{B.2})$$

as quoted from (3.5). As a preliminary we determine the series-expansion of γ given by (2.17), in powers of $\sin \theta$. By setting $\sin \theta = \sigma$ and by utilizing the known series-expansions of $\arcsin \sigma$ and $(1 - \sigma^2)^{1/2}$, we have

$$\begin{aligned} \gamma &= [6(2\theta - \sin(2\theta))]^{1/3} = [12(\arcsin \sigma - \sigma(1 - \sigma^2)^{1/2})]^{1/3} \\ &= 2\sigma \left[\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!} \frac{3\sigma^{2n}}{2n+3} \right]^{1/3}, \quad |\sigma| \leq 1. \end{aligned} \quad (\text{B.3})$$

Here the notation $(a)_n$ stands for Pochhammer's symbol defined by

$$(a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1), \quad n = 1, 2, 3, \dots$$

On replacing σ by $\sin \theta$ and evaluating the leading terms in (B.3), the required expansion of γ is found to be

$$\begin{aligned} \gamma &= [6(2\theta - \sin(2\theta))]^{1/3} \\ &= 2 \sin \theta \left[1 + \frac{3}{10} \sin^2 \theta + \frac{9}{56} \sin^4 \theta + \frac{5}{48} \sin^6 \theta + \dots \right]^{1/3}, \end{aligned} \quad (\text{B.4})$$

valid for $0 \leq \theta \leq \pi/2$.

We now turn to the evaluation of the coefficients $A_n(\theta)$ for $0 \leq \theta \leq \pi/2$. These coefficients appear in the Taylor series (B.2) of $H(t; \theta)$, therefore

$$A_n(\theta) = \frac{H^{(2n+1)}(0; \theta)}{(2n+1)!} \quad (\text{B.5})$$

where the superscript refers to differentiation with respect to t . Starting from (B.1), we first determine $A_0(\theta)$ for $0 < \theta \leq \pi/2$:

$$A_0(\theta) = H'(0; \theta) = \left\{ \frac{1 - \cosh w \cos(2\theta)}{(\cosh w - \cos(2\theta))^2} \left[\frac{dw}{dt} \right]^2 + \frac{\sinh w}{\cosh w - \cos(2\theta)} \frac{d^2 w}{dt^2} \right\} \Big|_{t=0} + \frac{2}{\gamma^2} = \frac{2}{\gamma^2} - \frac{1}{2 \sin^2 \theta}, \quad 0 < \theta \leq \pi/2. \quad (\text{B.6})$$

Here it has been used that $w = 0$, $dw/dt = -i$, $d^2 w/dt^2 = 0$ at $t = 0$, known from (2.16). Next, let $\theta = 0$, then also $\gamma = 0$ and the function $H(t; \theta)$ in (B.1) simplifies into

$$H(t; 0) = \frac{\sinh w}{\cosh w - 1} \frac{dw}{dt} - \frac{2}{t}. \quad (\text{B.7})$$

Expansion of the latter function in powers of w and subsequently in powers of t by means of (2.16), yields

$$H(t; 0) = \frac{2}{t} \left[1 - \frac{1}{20} t^2 + O(t^4) \right] - \frac{2}{t} = -\frac{1}{10} t + O(t^3). \quad (\text{B.8})$$

Then for $\theta = 0$, the coefficient A_0 is found to be

$$A_0(0) = H'(0; 0) = -\frac{1}{10}. \quad (\text{B.9})$$

To recognize the connection between (B.6) and (B.9), we establish the expansion

$$\frac{2}{\gamma^2} = \frac{1}{2 \sin^2 \theta} \left[1 - \frac{1}{5} \sin^2 \theta + O(\sin^4 \theta) \right] \quad (\text{B.10})$$

deduced from (B.4). When inserting (B.10) into (B.6), the singular terms $1/(2 \sin^2 \theta)$ cancel and we find

$$A_0(\theta) = -\frac{1}{10} + O(\sin^2 \theta). \quad (\text{B.11})$$

From the latter result it is obvious that $A_0(\theta) \rightarrow -1/10 = A_0(0)$ as $\theta \downarrow 0$. Thus we conclude that the coefficient $A_0(\theta)$ is bounded and continuous in θ for $0 \leq \theta \leq \pi/2$.

In the same manner we determine the coefficients $A_1(\theta)$, $A_2(\theta)$ by use of (B.5). The calculations are straightforward but tedious, therefore we only present the results:

$$A_1(\theta) = \frac{2}{\gamma^4} - \frac{1/8}{\sin^4 \theta} + \frac{1/20}{\sin^2 \theta}, \quad 0 < \theta \leq \pi/2; \quad A_1(0) = -\frac{13}{1400}, \quad (\text{B.12})$$

$$A_2(\theta) = \frac{2}{\gamma^6} - \frac{1/32}{\sin^6 \theta} + \frac{3/160}{\sin^4 \theta} + \frac{9/5600}{\sin^2 \theta}, \quad 0 < \theta \leq \pi/2; \quad A_2(0) = -\frac{71}{84000}. \quad (\text{B.13})$$

By expansion of $2/\gamma^4$ and $2/\gamma^6$ in powers of $\sin \theta$, similar to (B.10), it has been verified that all singular terms (involving negative powers of $\sin \theta$) in (B.12) and (B.13) cancel, and that $A_{1,2}(\theta) \rightarrow A_{1,2}(0)$ as $\theta \downarrow 0$. Thus we conclude that the coefficients $A_1(\theta)$, $A_2(\theta)$ are bounded and continuous in θ for $0 \leq \theta \leq \pi/2$.

Next, we come to the evaluation of the coefficient $A_n(\theta)$ for general $n = 0, 1, 2, \dots$. By using Cauchy's integral formula for $H^{(2n+1)}(0; \theta)$ in (B.5), we find that $A_n(\theta)$ is represented by the integral

$$\begin{aligned} A_n(\theta) &= \frac{1}{2\pi i} \oint H(t; \theta) \frac{dt}{t^{2n+2}} \\ &= \frac{1}{2\pi i} \oint \left[\frac{\sinh w}{\cosh w - \cos(2\theta)} \frac{dw}{dt} - \frac{1}{t-\gamma} - \frac{1}{t+\gamma} \right] \frac{dt}{t^{2n+2}}. \end{aligned} \quad (\text{B.14})$$

Here, the symbol \oint denotes that the integral is taken along a circle $|t| = \rho$, with $\rho < (6\pi)^{1/3}$. We first consider the case $0 < \theta \leq \pi/2$, and require that $\rho < \gamma$. Then by use of the residue theorem, the representation (B.14) is reduced to

$$A_n(\theta) = \frac{2}{\gamma^{2n+2}} + \frac{1}{2\pi i} \oint \frac{\sinh w}{\cosh w - \cos(2\theta)} \frac{dw}{t^{2n+2}}, \quad (\text{B.15})$$

in which $t = t(w)$ is defined by (2.12) and (2.13). In (B.15) the integral is taken along a circle $|w| = \rho_1$ with $\rho_1 < 2\theta$, so that the poles $w = \pm 2i\theta$ lie outside the integration contour. By making the change of integration variable $w \rightarrow -2iw$, the representation (B.15) transforms into

$$A_n(\theta) = \frac{2}{\gamma^{2n+2}} + \frac{1}{2\pi i} \oint \frac{2 \sin w \cos w}{\sin^2 w - \sin^2 \theta} \frac{dw}{t^{2n+2}}, \quad (\text{B.16})$$

where the integral is along the circle $|w| = \rho_1/2$ with $\rho_1/2 < \theta$, and

$$\begin{aligned} t &= [-6i(\sinh(-2iw) + 2iw)]^{1/3} = [6(2w - \sin(2w))]^{1/3} \\ &= 2w \left[1 - \frac{1}{5} w^2 + \frac{2}{105} w^4 + \dots \right]^{1/3} \end{aligned} \quad (\text{B.17})$$

by (2.12) and (2.13). In the integral (B.16) we set $\sin \theta = \sigma$ and we make the substitution $\sin w = s$, which defines a conformal mapping of a vicinity of $w = 0$ onto a vicinity of $s = 0$ in the complex s -plane. As a result we obtain

$$\begin{aligned}
 A_n(\theta) &= \frac{2}{\gamma^{2n+2}} + \frac{1}{2\pi i} \oint \left[\frac{1}{s-\sigma} + \frac{1}{s+\sigma} \right] \frac{ds}{t^{2n+2}} \\
 &= \frac{2}{\gamma^{2n+2}} + \frac{1}{2\pi i} \oint \frac{2}{t^{2n+2}} \frac{ds}{s-\sigma}, \tag{B.18}
 \end{aligned}$$

where it has been used that t^{2n+2} is an even function of s . In (B.18) the integral is taken along a circle $|s| = \rho_2$ with $\rho_2 < \sigma$, so that the pole $s = \sigma$ lies outside the integration contour. The variable t in (B.18) should be considered as a function of s , given by

$$t = [12(\arcsin s - s(1-s^2)^{1/2})]^{1/3} = 2s \left[\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!} \frac{3s^{2n}}{2n+3} \right]^{1/3}, \quad |s| < 1 \tag{B.19}$$

similar to (B.3). Clearly, t is an analytic function of s for $|s| < 1$. Consider now the factor $2t^{-2n-2}$ in the integrand of (B.18). From (B.19) it readily follows that t^{-2n-2} is an analytic function of s for $0 < |s| < 1$, with a pole of order $2n+2$ at $s = 0$. Then the factor $2t^{-2n-2}$ which is even in s , can be expanded in a Laurent series around $s = 0$, viz.

$$\begin{aligned}
 2t^{-2n-2} &= 2^{-2n-1} s^{-2n-2} + b_n^{(n)} s^{-2n} + \dots + b_1^{(n)} s^{-2} \\
 &\quad + \sum_{m=0}^{\infty} c_m^{(n)} s^{2m}, \quad 0 < |s| < 1. \tag{B.20}
 \end{aligned}$$

It is well known that the final integral in (B.18) is equal to minus the principal part of the Laurent series (B.20), evaluated at $s = \sigma = \sin \theta$. Thus we find for the coefficient $A_n(\theta)$:

$$A_n(\theta) = \frac{2}{\gamma^{2n+2}} - \left[\frac{2^{-2n-1}}{(\sin \theta)^{2n+2}} + \frac{b_n^{(n)}}{(\sin \theta)^{2n}} + \dots + \frac{b_1^{(n)}}{\sin^2 \theta} \right], \quad 0 < \theta \leq \pi/2. \tag{B.21}$$

Notice that the coefficients $b_m^{(n)}$, $m = 1, 2, \dots, n$, can be obtained directly from the Laurent-series expansion of $2\gamma^{-2n-2}$ in powers of $\sin \theta$. Indeed, by setting $s = \sigma = \sin \theta$ in (B.19), one has $t = \gamma$ by (B.3), and the Laurent series (B.20) becomes

$$\begin{aligned}
 \frac{2}{\gamma^{2n+2}} &= \frac{2^{-2n-1}}{(\sin \theta)^{2n+2}} + \frac{b_n^{(n)}}{(\sin \theta)^{2n}} + \dots + \frac{b_1^{(n)}}{\sin^2 \theta} \\
 &\quad + \sum_{m=0}^{\infty} c_m^{(n)} (\sin \theta)^{2m}, \quad 0 < \theta \leq \pi/2. \tag{B.22}
 \end{aligned}$$

Consider next the case $\theta = 0$. Then also $\gamma = 0$, and the representation (B.14) simplifies into

$$A_n(0) = \frac{1}{2\pi i} \oint \left[\frac{\sinh w}{\cosh w - 1} \frac{dw}{dt} - \frac{2}{t} \right] \frac{dt}{t^{2n+2}}, \tag{B.23}$$

where the integral is taken along a circle $|t| = \rho$, with $\rho < (6\pi)^{1/3}$. It is easily recognized that the second part of the integral (B.23) vanishes. In the remaining integral we proceed as in the case

$0 < \theta \leq \pi/2$. Thus we apply the change of integration variable $w \rightarrow -2iw$, and the substitution $\sin w = s$. As a result we obtain

$$A_n(0) = \frac{1}{2\pi i} \oint \frac{2}{t^{2n+2}} \frac{ds}{s}, \quad (\text{B.24})$$

where the integral is taken along a circle $|s| = \rho_2$ with $\rho_2 < 1$, and the variable t should be considered as a function of s , given by (B.19). The integral (B.24) is known to be equal to the constant term in the Laurent series (B.20). Thus we find for the coefficient A_n at $\theta = 0$:

$$A_n(0) = c_0^{(n)}. \quad (\text{B.25})$$

This completes the evaluation of the coefficients $A_n(\theta)$. From (B.21) and (B.22) it readily follows that $A_n(\theta) \rightarrow c_0^{(n)} = A_n(0)$ as $\theta \downarrow 0$. Thus we conclude that the coefficient $A_n(\theta)$ is bounded and continuous in θ for $0 \leq \theta \leq \pi/2$.

References

- [1] M. Abramowitz and I.A. Stegun (Eds.), *Handbook of Mathematical Functions* (Dover, New York, 1965).
- [2] V. Belevitch, The lateral skin effect in a flat conductor, *Philips Tech. Rev.* **32** (1971) 221-231.
- [3] V. Belevitch and J. Boersma, Theory of a thin conducting strip with an overhead return wire, *Philips J. Res.* **44** (1990) 541-582.
- [4] N. Bleistein and R.A. Handelsman, *Asymptotic Expansions of Integrals* (Dover, New York, 1986).
- [5] R.P. Boas, *Invitation to Complex Analysis* (Random House, New York, 1987).
- [6] J. Boersma, On a function, which is a special case of Meijer's G -function, *Compos. Math.* **15** (1962) 34-63.
- [7] N.G. de Bruijn, *Asymptotic Methods in Analysis* (Dover, New York, 1981).
- [8] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions, Vol. III* (McGraw-Hill, New York, 1955).
- [9] C.S. Meijer, *Asymptotische Entwicklungen Besselscher, Hankelscher und verwandter Funktionen* (Dissertation Universität Groningen, Groningen, 1933).
- [10] B.L. van der Waerden, On the method of saddle points, *Appl. Sci. Res.* **B2** (1952) 33-45.
- [11] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Second Ed. (Cambridge University Press, Cambridge, 1958).

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