Recurrence and realization of 2-D systems

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Recurrence and Realization of 2-D systems

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Introduction

In this note a class of transfer matrices is described which can be realized by a method described in [1]. This is done in such a way that the resulting recursive equations can be evaluated in a straightforward manner.

1. Problem description

Let P/Q denote a proper transfer matrix in two variables.
We have therefor the following $P \in \mathbb{R}^{m\times n}[s][z]$, $Q \in \mathbb{R}[s][z]$, (for definitions see [1]) and properness is characterized by:

(1.1) 1) $deg_z(Q) \leq deg_z(P)$ *(*$deg_z(Q)$ *denotes degree in* z *of* Q).
2) *the degree in* s *of the coefficient of the highest power in* z *of* Q *is not less than the degree of all other coefficients of* Q *and* P.

In [1] it is shown that a proper transfer matrix can be realized in subsequently a first level and a second level realization.
It is easily seen that the impulse response of a proper transfer matrix has its support in the first quadrant of $\mathbb{Z} \times \mathbb{Z}$.
We will now generalize the realization procedure to a class of non symmetric halfplane filters (NSHP filters).
A NSHP is a subset of $\mathbb{Z} \times \mathbb{Z}$ of the following kind

(1.2) $\{(k,h) \mid k > 0 \text{ or } k = 0 \text{ and } h \geq 0\}$.

For more on NSHP filters see [2], [3].

In the next we will consider transfer matrices which have their support in a subset H_q of a NSHP. This subset will be of the following kind:

(1.3) $H_q = \{(k,h) \mid k \geq 0,\ h \geq -qk \text{ for some positive integer } q\}$.

Remark. The case $q \leq 0$ can directly been solved by the method of [1].
For q = 2 H_q is thus the following set

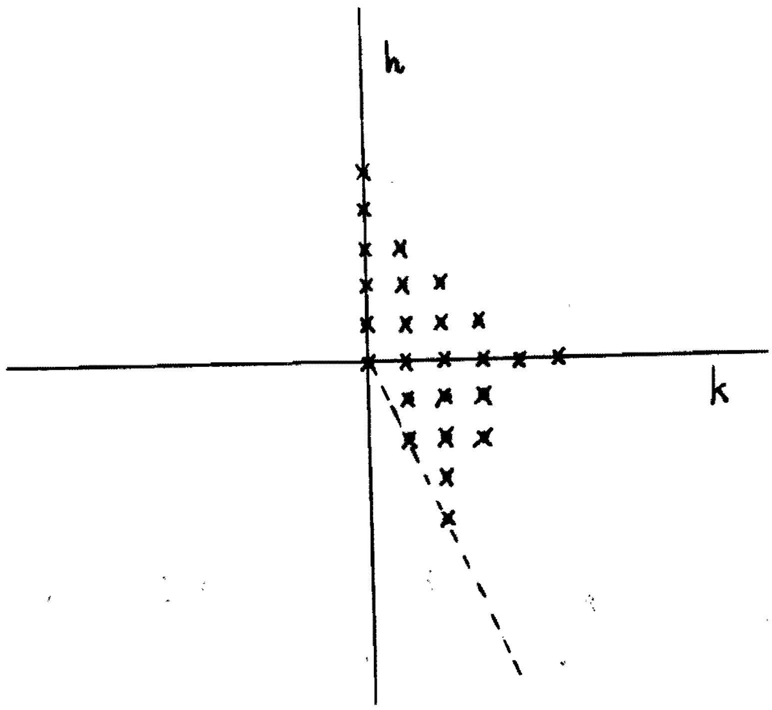

fig. 1.

2. The realization method

For the ease of notation we will consider only transfer functions. The case of transfer matrices is completely analogous.

Now consider a transfer function $T(z,s)$ with formal power series expansion:

$$(2.1) \qquad T(z,s) = \sum_{k\in\mathbb{Z},h\in\mathbb{Z}} f_{kh} z^{-k} s^{-h} .$$

Suppose the impulse response $\{f_{kh}\}$ has its support in H_q.

As usual z and s denote the so called horizontal and vertical shift, thus:

$$z(x)_{kh} = x_{k+1,h}, \quad s(x)_{kh} = x_{k,h+1} .$$

We will now introduce two new shifts α and β by the following spectral transformation (see also [4])

$$(2.2) \qquad \begin{array}{ll} z = \alpha\beta^{q} & s = \beta \text{ or} \\ \alpha = zs^{-q} & \beta = s . \end{array}$$

Substituting (2.2) in (2.1) we obtain

$$\bar{T}(\alpha,\beta) = \sum_{k\in\mathbb{Z},h\in\mathbb{Z}} \bar{f}_{kh} \alpha^{-k} \beta^{-h} .$$

It is now clear that $\bar{f}_{kh} = 0$ for $k < 0$ or $h < 0$ $\bar{T}(\alpha,\beta)$ is now a proper transfer function as is easily verified.

Now we can apply the realization procedure of [1] to obtain a second level realization of $\bar{T}(\alpha,\beta)$.

We will write down only the input equations in Roessers form (see [1])

$$(2.3) \qquad \begin{bmatrix} \alpha(x)_{kh} \\ \beta(a)_{kh} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{kh}$$

or because $\alpha = zs^{-q}$, $\beta = s$

$$(2.4) \qquad \begin{bmatrix} z(x)_{kh} \\ s(a)_{kh} \end{bmatrix} = \begin{bmatrix} A_1 s^{q} & A_2 s^{q} \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_{kh} & + & B_1 s^{q} \\ a_{kh} & & B_2 \end{bmatrix} u_{kh}, \quad \begin{array}{l} k = 0,1,\ldots \\ h = -qk,-qk+1,\ldots \end{array}$$

where s^q is defined by: $s^q(s)_{kh} = x_{k,h+q}$.

In this way we obtained a generalized second level realization of T(z,s) where the dependence in the state space equations can be grafically shown as follows:

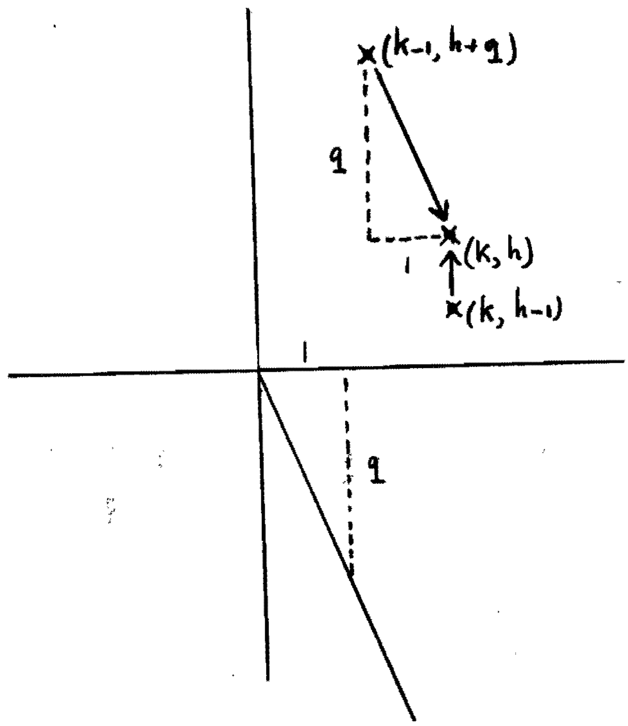

fig. 2.

The initial conditions must be specified for

$$x_{0,h} \text{ and } a_{k,-qk}\ , \qquad h = 0,1,2,\ldots;\ k = 0,1,2,\ldots\ .$$

All the initial conditions will be zero.
The possibility of realizing T(z,s) by (2.4) can be seen directly from T(z,s) itself without considering the support of the impulse response.

Theorem (2.5). *If* T(z,s) P/Q *has the following two properties:*

1) $deg_z(Q) \geq deg_z(P)$.
2) *If* $deg_z(Q) = deg_z(P)$, *then the degree of the coefficient of the highest power in* z *of* Q *is not less then the degree of the corresponding coefficient of* P. *If* $deg_z(Q) > deg_z(P)$ *then the only condition is* 1)

then T(z,s) *can be realized by state equations of the form* (2.4).

Proof. Consider a spectral transform $z = \alpha\beta^q$, $s = \beta$ with q sufficiently large then T(z,s) is transformed into a proper transfer function $\bar{T}(\alpha,\beta)$ which can be realized by the method of [1].

Example. $(z,s) = \dfrac{z^2s^2 + z + s^4}{z^2s^2 + zs^3 + s^6}$

$$z = \alpha\beta^2,\ s = \beta$$

$$T(\alpha,\beta) = \frac{\alpha^2\beta^6 + \alpha\beta^2 + \beta^4}{\alpha^2\beta^6 + \alpha\beta^5 + \beta^6} \text{ is proper .}$$

From (2.4) and fig. 2 it is seen that the state $\begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix}$ can be computed recursively at each point (k,h) using only states and inputs that have already been computed.

Now consider spectral transformations:

$$z = \alpha^p\beta^q,\ s = \alpha^r\beta^t \tag{2.6}$$

with p,q,r,t nonnegative integers satisfying $qr - pt = -1$.

We then have $\alpha = z^ts^{-q}$, $\beta = z^{-r}s^p$.

Suppose we have a transfer function with impulse response having its support in the shaded sector of fig. 3.

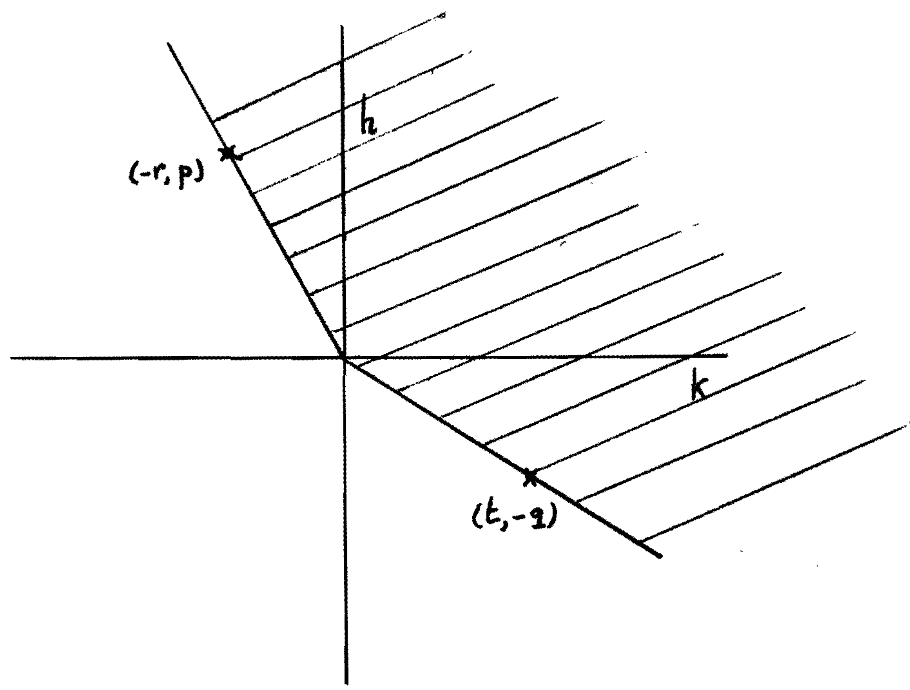

fig. 3.

Then it is directly seen that transforming $T(z,s)$ by (2.6) into $\bar{T}(\alpha,\beta)$ gives us a proper transfer function thus having the support of the impulse response in the first quadrant.
The analogous equations of (1.7) become

$$\begin{bmatrix} z^t(x)_{kh} \\ s^p(a)_{kh} \end{bmatrix} = \begin{bmatrix} A_1 s^q & A_2 s^q \\ A_3 z^r & A_4 z^r \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} B_1 s^q \\ B_2 z^r \end{bmatrix} u_{kh}, \quad \begin{matrix} k = 0,1,2,\dots \\ h = 0,1,2,\dots \end{matrix} \tag{2.7}$$

Initial conditions must be specified for:

$$x_{-hr,ph},\ a_{tk,-qk},\ k = 0,1,2,\dots;\ h = 0,1,2,\dots\ .$$

In fig. 4 initial conditions are thus specified on γ and δ.
The dependence in these equations (2.7) can be shown as follows:

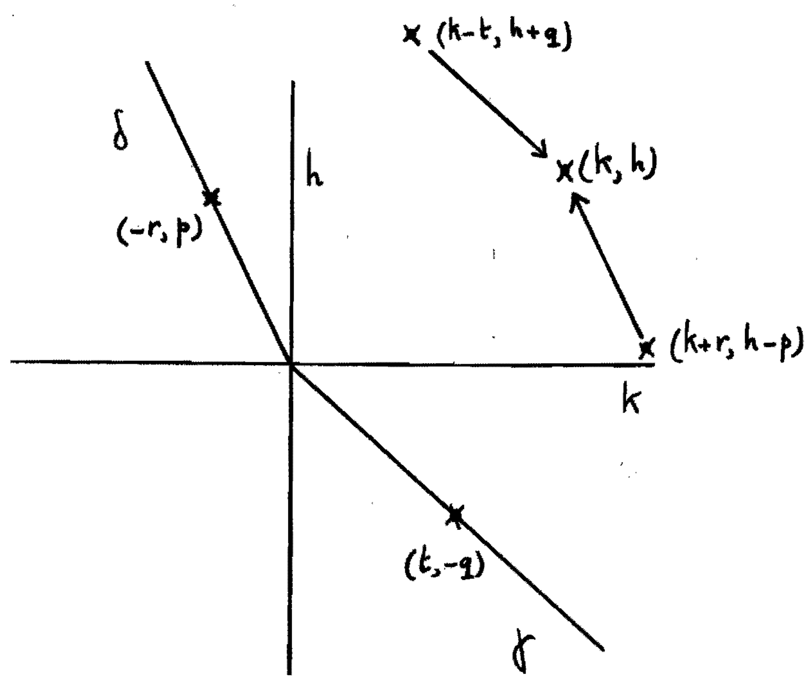

fig. 4.

Again the state $\begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix}$ at each (k,h) can be computed using only states that have already been computed. This is possible because $qr - pt$ is negative. When $qr - pt \geq 0$ it is no longer possible to compute the states recursively using only states that have already been computed.

Remark 1. The condition $qr - pt < 0$ acts like a causality property.

Remark 2. The spectral transformations in this construction are of course not unique. See also (2.9).

Remark 3. The fact that states can be computed using only states that have already been computed gives rist to an order on $\mathbf{Z} \in \mathbf{Z}$. See also [5], [6].

Now consider a transfer function $T(z,s)$ with impulse response having support in a sector like in fig. 3 with p,q,r,t nonnegative and $qr - pt < 0$. $qr - pt < 0$ ensures that the sector angle is less than π.

We will now construct a sector characterized by nonnegative integers p',q',r',t' such that

(2.8) 1) $q'r' - p't' = -1$.

2) The sector characterized by p',q',r',t' contains the sector characterized by p,q,r,t,

Herein "a sector S characterized by p,q,r,t" is the following

$$S = \{(k,h) \mid hr \geq -pk\} \cap \{(k,h) \mid ht \geq -qk\} .$$

Suppose now $p \neq 0$, $r \neq 0$ we may assume p and r are relatively prime thus there exist q_1 and t_1 such that:

$$q_1 r - pt_1 = -1$$

and thus:

$$(q_1 + np)r - p(t_1 + nr) = -1 \quad \text{for all } n \in \mathbf{Z}$$

because $q/t < p/r$ ($t = 0$ is excluded by $qr - pt < 0$) we have for sufficiently large n

$$q/t < \frac{q_1 + n_0 p}{t_1 + n_0 r} ,$$

(2.9) Now take $p' = p$, $q' = q_1 + n_0 p$, $r' = r$, $t' = t_1 + n_0 r$.

The sector characterized by p',q',r',t' satisfies the requirements (2.8). The case $r = 0$ can be taken care of by theorem (2.5). The case $p = 0$ is excluded because $qr - pt < 0$. Thus it is shown that the construction (1.11) is always possible.

Remark 4. The construction via (1.11) generally leads to dynamics of relatively high order.

Remark 5. By allowing transformations like

$$z = \alpha^{\pm 1}, \quad s = \beta^{\pm 1}$$

transfer functions with impulse response having support in other quadrants can be included. This is easily verified.

3. Conclusions

The method of [1] can be generalized to apply to transfer functions of the type considered in this note. This gives rise to a generalized form of state space equations as for example in (2.7).
The restriction for applying this method is the condition $qr - pt < 0$, $qr - pt < 0$ can be interpreted as a causality condition. The condition $qr - pt = -1$, which can always be assumed to be true by the construction (2.8), excludes the possibility of rational exponents in expressions like $\alpha = z^{t}s^{-q}$. Of course the roles of z and s can be interchanged in the foregoing and also cases, where s or z is replaced by s^{-1} or z^{-1} respectively, can be included.

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