

The r-rank of the groups of exceptional Lie type

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The r -Rank of the groups of exceptional Lie type

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ABSTRACT

In this note, we prove the following result, settling a question raised at the end of [Borel & Serre, 1953], cf. [Borel, 1983 pp. 228 and 708]. A related result for Lie groups of type E_8 was recently proved by J.F. Adams.

THEOREM. *Let r be a prime and G a simple algebraic group of exceptional type over an algebraically closed field of characteristic $\neq r$. Let E be an elementary abelian r -subgroup of G of maximal rank. Then $\text{rank}(E) = \text{Lie rank}(G)$ with the exception of $r=2$ and $G = G_2, F_4$, the adjoint E_7 , and E_8 , in which cases $\text{rank}(E) = \text{Lie rank}(G) + 1$. Moreover, E is unique up to conjugacy.*

1. THE PRIME 2

In this section we prove the following

THEOREM. *Let G be an algebraic group of type G_2, F_4, E_6 , adjoint E_7 , simply connected E_7 , or E_8 over an algebraically closed field of characteristic $\neq 2$, and let E be an elementary abelian 2-group in G of maximal order. Then $|E| = 2^3, 2^5, 2^6, 2^8, 2^7, 2^9$ in the respective cases. Moreover, in each case any two such elementary abelian subgroups are conjugate.*

PROOF. By a theorem of [Springer & Steinberg, 1970], due to [Borel & Serre, 1953] in the Lie group case, E is a subgroup of $N_G(T)$ for some maximal torus T of G . In particular, $|E| \leq 2^l \cdot |W|$, where l is the Lie rank of G and $W = N_G(T)/T$, so E is finite. We shall deal with each case separately, although

the arguments are similar. The idea is to produce a certain subgroup containing the preimage in N of a Sylow 2-subgroup of W .

G_2 . Let J_1, J_2 be commuting (nonconjugate) fundamental SL_2 's. We may take $T \leq D = J_1 J_2$. Moreover $N_D(T)/T$ contains a Sylow 2-subgroup of $W = N_G(T)/T$, so we may assume $E \leq D$. Let $Z(D) = \langle e \rangle$. Maximality of E then implies $E = \langle e, x_1 x_2, y_1 y_2 \rangle$, where $x_1, y_1 \in J_1, x_2, y_2 \in J_2, x_1^2 = x_2^2 = y_1^2 = y_2^2 = [x_1, y_1] = [x_2, y_2] = e$. It is clear that any two such groups are conjugate in D .

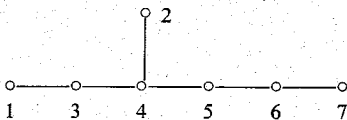
F_4 . There is an involution in F_4 with centralizer D , the simply connected group of type B_4 . We may take $T \leq D$ and check that $N_D(T)/T$ contains a Sylow 2-subgroup of $N_G(T)/T$. Hence, we may take $E \leq D$. An involution in SO_9 lifts to an involution in D if and only if the eigenspace for eigenvalue -1 has dimension a multiple of 4. A direct check then shows that D has 2-rank 5 and all elementary abelian subgroups of D of order 2^5 are conjugate.

E_6 . Set $V = \Omega_2(T) = \{t \in T \mid t^2 = 1\}$. Then V is the natural module for $O^-(6, 2) \cong W$. By § 8 of [Aschbacher & Seitz, 1976] W has 4 classes of involutions, represented by a_2, c_2, b_1 , and b_3 . Here the subscript is the dimension of the commutator space of the involution. The involutions in $\Omega^-(6, 2)$ are conjugates of a_2 and c_2 . Long root subgroups of W are generated by conjugates of a_2 , the commutator space $[V, a_2]$ is totally singular, and $C_V(a_2) = [V, a_2]^\perp$. Finally, by (19.9)(ii) of [Aschbacher & Seitz, 1976], applied to $O^-(6, 2) \cong U_4(2) \cdot 2$, we have $C_W(b_3) \leq C_W(b_1)$ for suitable choice of b_1 .

To prove the theorem for E_6 we may and shall assume that E is not contained in a maximal torus of G , $|E| \geq 2^6$, and $E \leq N_G(T)$. Set $\bar{E} = ET/T$, and for $x \in E$, write $\bar{x} = xT$. If \bar{E} centralizes b_1 , then there is a fundamental SL_2 normalized by T , containing a preimage in $N(T)$ of b_1 , and such that $E \leq SL_2 \circ SL_6$. A direct check then shows that E is necessarily contained in a maximal torus, a contradiction. Hence \bar{E} does not centralize a b_1 involution. In particular, $\bar{E} \leq \Omega^-(6, 2)$. Moreover, if $C_V(\bar{E})$ contains a nonsingular vector v , then \bar{E} centralizes the unique involution b_1 of W satisfying $[V, b_1] = \langle v \rangle$, a contradiction. Therefore, $C_V(\bar{E})$ is totally singular, $|E \cap T| \leq 4$, and $|\bar{E}| \geq 2^4$.

Let $\bar{E} \leq P$ be the stabilizer in $O^-(6, 2)$ of a singular 1-space of V . Then $P = O_2(P)L$, where $L \cong \Omega^-(4, 2)$ and $O_2(P)$ is the natural module for L . Since $|\bar{E}| \geq 2^4$, an easy argument shows that $\bar{E} = O_2(P)$ and so $|\bar{E}| = 2^4$. Hence, \bar{E} contains distinct a_2 involutions \bar{x}, \bar{y} . Then $C_V(\bar{x})$ and $C_V(\bar{y})$ have distinct radicals, so the singular points of $C_V(\bar{x}) \cap C_V(\bar{y})$ span a subspace of dimension ≤ 1 . Consequently, $|E| \leq |\bar{E}| \cdot |E \cap T| \leq 2^4 \cdot 2 < 2^6$. This contradiction finishes the proof of the E_6 case.

E_7 . Fix a maximal torus T and corresponding system of root groups. Let Σ denote a maximal set of pairwise commuting fundamental SL_2 's from this system. If we label the diagram as follows



then we can take $\Sigma = \{J_1, \dots, J_7\}$, where $J_i = \langle U_{\pm\beta_i} \rangle$ and the β_i are as follows:

$$\beta_1 = 2234321, \quad \beta_2 = 0112221, \quad \beta_3 = 0000001, \quad \beta_4 = 0112100,$$

$$\beta_5 = 0000100, \quad \beta_6 = 0100000, \quad \beta_7 = 0010000.$$

Set $Z(J_i) = \langle e_i \rangle$ and $J = J_1, \dots, J_7$. Then $\{e_1, \dots, e_7\}$ is a set of commuting involutions which span $Z = Z(J)$.

LEMMA 1 (E_7).

- (i) $N_G(J)/J \cong L_3(2)$ and $N_G(J)$ is 2-transitive on Σ , hence on $\{e_1, \dots, e_7\}$.
- (ii) If G is simply connected, the relations on $\{e_1, \dots, e_7\}$ are spanned by $\{e_4e_5e_6e_7, e_2e_3e_6e_7, e_1e_2e_5e_6\}$. So $|Z| = 2^4$.
- (iii) If G is adjoint, the relations on $\{e_1, \dots, e_7\}$ are spanned by $\{e_4e_5e_6e_7, e_2e_3e_6e_7, e_1e_2e_5e_6, e_1e_2e_3\}$. So $|Z| = 2^3$.

PROOF. For each i , the centralizer $C_G(J_i)$ is of type D_6 . Within D_6 a maximal commuting product of fundamental SL_2 's corresponds to a decomposition of the usual orthogonal module into three perpendicular 4-spaces. One checks that S_4 is induced on such a commuting product, transitive on the 6 copies of SL_2 . Hence, $N_G(J)$ is 2-transitive on $\{J_1, \dots, J_7\}$, $N_G(J)/J$ has order 168, and (i) follows.

For (ii) and (iii) first check that $e_4e_5e_6e_7, e_2e_3e_6e_7, e_1e_2e_5e_6, e_1e_2e_3$ are each in $Z(G)$ (show that they centralize each root group corresponding to a fundamental root). Hence, in the simple group $|Z| \leq 2^3$. Equality must hold since $L_3(2)$ acts nontrivially on Z . This gives (iii). For (ii), view $E_7 \leq E_8$ and note that $e_4e_5e_6e_7, e_2e_3e_6e_7, e_1e_2e_5e_6$ are in $Z(E_8) = 1$, while $e_1e_2e_3$ is not.

One can now list explicitly all relations on the e_i 's, listing tuples of integers to indicate corresponding products of e_i 's which are trivial.

G simply connected: 4567, 2367, 1256, 1247, 2345, 1357, 1346.

(*)

G adjoint: 4567, 2367, 1256, 1247, 2345, 1357, 1346,
123, 145, 347, 356, 167, 246, 257,
1234567

LEMMA 2 (E_7). Let $E \leq J$ be an elementary abelian 2-group.

- (i) There exist subgroups Q_i of J_i ($1 \leq i \leq 7$) such that $Q_i = \langle x_i, y_i \rangle$ is quaternion of order 8, $N_{J_i}(Q_i)$ induces S_3 on Q_i , and $E \leq Q = Q_1 \dots Q_7$.
- (ii) $|E| \leq 2^8, 2^7$ according to whether G is adjoint or simply connected.
- (iii) If G is adjoint, there is a unique J -class of elementary abelian groups of order 2^8 , represented by $\langle Z, x_4x_5x_6x_7, x_2x_3x_6x_7, x_1x_2x_5x_6, x_1x_2x_3, y_1 \dots y_7 \rangle$.
- (iv) If G is simply connected, there is a unique J -class of elementary abelian groups of order 2^7 , represented by $\langle Z, x_4x_5x_6x_7, x_2x_3x_6x_7, x_1x_2x_5x_6 \rangle$.
- (v) Any 2-group in G is conjugate to a subgroup of $N_G(J)$.

PROOF. Consider $EZ/Z \leq J/Z$ and project to each of the simple summands. Each projection of E is contained in the Klein 4-subgroup of a group isomorphic to S_4 . The preimages of the S_4 's are the normalizers of the Q_i 's. This gives (i).

For the other parts take E of maximal order. Then $Z \leq E$. Suppose $e \in E - Z$. Conjugating by a suitable element in the product of the normalizers of the Q_i 's we may assume e is a product of certain of the elements x_1, \dots, x_7 . Since e is an involution the relations force $e = x_i x_j x_k x_l$, $x_i x_j x_k$, or $x_1 \dots x_7$, where $ijkl$ or ijk is one of the tuples in (*).

For each i , $[x_i, y_i] = e_i$. Moreover, inspection of the above tuples shows: $|\{i, j, k, l\} \cap \{r, s, t\}| = 0$ or 2 and $|\{i, j, k, l\} \cap \{r, s, t, v\}| = 2$ if $\{i, j, k, l\} \neq \{r, s, t, v\}$. The proof of (ii), (iii), and (iv) is completed using these facts and an easy check of cases. Finally, (v) follows since $E \leq N_G(T)$ and the orders of $N_G(T)/T$ and $N_J(T)/T$ have the same 2-part (2^{10}).

LEMMA 3 (E_7). Assume G is adjoint and $E \leq N_G(J)$ is an elementary abelian 2-group. Then $|E| \leq 2^8$, equality possible only if E is G -conjugate to a subgroup of J .

PROOF. Suppose $|E| \geq 2^8$, $E \leq N_G(J)$, but $E \not\leq J$. Let $X = EJ/J$, regarded as a subgroup of $L_3(2)$. Hence, $X \cong \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. The permutation action of $N_G(J)/J$ on X is the same as that on Z^* . Let $Y = E \cap J$, with $Y \leq Q = Q_1 \dots Q_7$ as in Lemma 2, and E normalizing Q (use the fact that $N_G(J_i) = J_i C_G(J_i)$ for each i). Set $a_i = x_i Z$ and $b_i = y_i Z$.

CASE 1. $C_Z(E) \cong \mathbb{Z}_2$. By transitivity we may assume $C_Z(E) = \langle e_1 \rangle$. Since involutions in $L_3(2)$ have a 2-dimensional fixed space on the usual module, $X \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. So $Y = E \cap J$ is elementary abelian of order at least 2^6 and $|YZ/Z| \geq 2^5$.

$R = C_{J/Z}(X)$ is the product of groups of type PSL_2 , one for each orbit of X on Σ . Now, X has orbits of size 1, 2, 2, 2. Write $R = R_1 \dots R_4$, each $R_i \cong PSL_2$ and $R_1 = J_1 Z/Z$. If $\{J_i, J_j\}$ is an orbit, then $e_i e_j$ is fixed by E , hence $e_1 = e_i e_j$. Thus $1ij$ is one of the triples above. So the orbits are $\{J_2, J_3\}$, $\{J_4, J_5\}$, $\{J_6, J_7\}$, with corresponding PSL_2 's R_2, R_3, R_4 . $YZ/Z \cap R_1 = 1$ (since $Y \cap J_1 = \langle e_1 \rangle$). So conjugating by an appropriate element of $N(Q)$ we may assume that the image of YZ/Z under projection to $R_2 R_3 R_4$ contains a hyperplane of $\langle a_2 a_3, b_2 b_3, a_4 a_5, b_4 b_5, a_6 a_7, b_6 b_7 \rangle$. Intersecting the projection with $\langle a_2 a_3, b_2 b_3 \rangle$, $\langle a_4 a_5, b_4 b_5 \rangle$, and $\langle a_6 a_7, b_6 b_7 \rangle$, we may assume Y contains elements projecting to $a_2 a_3$, $a_4 a_5$, and $a_6 a_7$. Hence, we may assume Y contains $x_1 x_2 x_3$, $x_1 x_4 x_5$, and $x_1 x_6 x_7$. But also, Y contains an element projecting to an involution in $\langle b_2 b_3, b_4 b_5 \rangle$, forcing Y to be nonabelian. Contradiction.

CASE 2. $C_Z(E) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then E fixes 3 J_i 's, but does not centralize Z . So we may assume E normalizes J_1, J_2 , and J_3 . No element of $L_3(2)$ fixes more than 3 elements of the usual module, so X is semiregular on $\{J_4, J_5, J_6, J_7\}$.

First assume $X \cong \mathbb{Z}_2$. Then YZ/Z has order at least 2^5 and without loss of

generality we may assume the nontrivial orbits of E on Σ to be $\{J_4, J_5\}$ and $\{J_6, J_7\}$. Now $Y \cap J_1 J_2 J_3$ is not contained in Z , so we may assume $x_1 x_2 x_3 \in Y$. If $Y \cap J_1 J_2 J_3 = \langle e_1, e_2, x_1 x_2 x_3 \rangle$, then the image of Y under projection to $J_4 J_5 J_6 J_7 Z/Z$ coincides with $\langle a_4 a_5, b_4 b_5, a_6 a_7, b_6 b_7 \rangle$ and this forces Y to be nonabelian. So assume $x_1 x_2 x_3, y_1 y_2 y_3$ are both in Y . As above, we may assume Y contains an element projecting to $a_4 a_5$, which again forces Y to be nonabelian. Thus $X \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Hence E has a unique nontrivial orbit on Σ of size 4 and $|YZ/Z| \geq 2^4$. It follows that $|Y \cap J_1 J_2 J_3| \geq 2^4$, so we may assume $x_1 x_2 x_3, y_1 y_2 y_3 \in Y$. Now $N_G(J_1) \cap N_G(J_2) \cap N_G(J_3) = J_1 J_2 J_3 D$, where $D = D^0$ is simply connected of type D_4 (indeed, $Z(D) = \langle e_4 e_5, e_5 e_7 \rangle$). Take $h \in E - \langle Z, x_1 x_2 x_3, y_1 y_2 y_3 \rangle$. Since h commutes with $x_1 x_2 x_3$ and $y_1 y_2 y_3$, we may take $h \in D$. Now D has just 1 class of involutions in $D - Z(D)$, represented by e_4 (corresponding to involutions in SO_8 of type $(1)^4(-1)^4$). Hence $C_D(h)$ is D -conjugate to $C_D(e_4) = J_4 J_5 J_6 J_7$. Thus, $E \leq J_1 J_2 J_3 C_D(h)$, a D -conjugate of J . This completes the proof of Lemma 3.

LEMMA 4 (E_7). Assume G is simply connected and $E \leq N_G(J)$ is an elementary abelian 2-group. Then $|E| \leq 2^7$, equality possible only if E is G -conjugate to a subgroup of J .

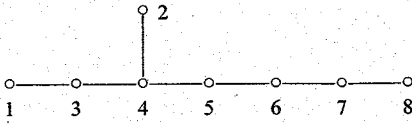
PROOF. Assume $|E| \geq 2^7$ and $E \not\leq J$. Then, up to conjugacy in $N_G(J)/J \cong L_3(2)$, we have that $X = EJ/J$ is one of the groups listed in the table below, where a, b, c are elements of $N_G(J)/J$ inducing the permutations $(2, 3)(6, 7)$, $(4, 5)(6, 7)$, $(4, 6)(5, 7)$, respectively, on Σ . A direct check shows that, in each case, $C_Z(X)$ is as indicated in the table. Thus, the rank of $E \cap Z$ (a subgroup of $C_Z(X)$) is at most 3, 2, and 3, so that $|(E \cap J)Z/Z| \geq 2^3, 2^3$, and 2^2 , in the respective cases.

On the other hand, if $q = q_1 \dots q_7$, where $q_i \in Q_i Z/Z$, is an involution then the tuple of indices i with $q_i \neq 1$ is a 4-tuple of (*). Moreover, if q is centralized by X , this tuple must be invariant under the permutation action of X on Σ . In the table, under $\text{inv}(X)$, those tuples from (*) are listed which are X -invariant. It readily follows from the structure of $\text{inv}(X)$ that $(E \cap Q)Z/Z$ has size at most 2^2 in all three cases. Therefore, we must have $X = \langle b, c \rangle$, $E \geq \langle e_1, e_2, e_3 \rangle$, and, without loss of generality, $(E \cap J)Z/Z = \langle x_4 x_5 x_6 x_7, y_4 y_5 y_6 y_7 \rangle Z/Z$. In particular $E \leq N_G(J_1)N_G(J_2)N_G(J_3)$, and we can finish as in the previous lemma.

X	$\langle c \rangle$	$\langle a, b \rangle$	$\langle b, c \rangle$
$C_Z(X)$	$\langle e_1, e_2, e_3 \rangle$	$\langle e_1, e_2, e_3 \rangle$	$\langle e_1, e_2, e_3 \rangle$
$\text{inv}(X)$	4567, 1357, 1346	4567, 2367, 2345	4567

The E_7 case of the theorem follows from Lemmas 2, 3, and 4.

E_8 . We proceed as for E_7 . Again T is a maximal torus, and Σ a maximal set of pairwise commuting fundamental SL_2 's. We label the diagram



and take $\Sigma = \{J_1, \dots, J_8\}$, where $(J_i)_{1 \leq i \leq 7}$ as for E_7 and $J_8 = \langle U_{\pm\beta_8} \rangle$, with $\beta_8 = 23465432$. Set $Z(J_i) = \langle e_i \rangle$ and $J = J_1 \dots J_8$. Then $\{e_1, \dots, e_8\}$ is a set of commuting involutions spanning $Z = Z(J)$.

LEMMA 5 (E_8).

- (i) $N_G(J)/J \cong \mathbb{Z}_2^3 L_3(2)$ and $N_G(J)$ is 3-transitive on Σ , hence on $\{e_1, \dots, e_8\}$.
- (ii) The relations on $\{e_1, \dots, e_8\}$ are given by the tuples of even length in (*) and the tuples obtained by joining 8 to the tuples of odd length in (*).

PROOF. For each $i \in \{1, \dots, 8\}$, the group $C_G(J_i)$ is of type E_7 , so the lemma is easily derived from Lemma 1.

LEMMA 6 (E_8). Let $E \leq J$ be an elementary abelian 2-group.

- (i) There exist subgroups Q_i of J_i such that $Q_i = \langle x_i, y_i \rangle$ is quaternion of order 8, $N_{J_i}(Q_i)$ induces S_3 on Q_i , and $E \leq Q = Q_1 \dots Q_8$.
- (ii) $|E| \leq 2^9$.
- (iii) There is a unique J -class of elementary abelian subgroups of order 2^9 , represented by $\langle Z, x_4 x_5 x_6 x_7, x_2 x_3 x_6 x_7, x_1 x_2 x_5 x_6, x_1 x_2 x_3 x_8, y_1 \dots y_8 \rangle$.
- (iv) Any 2-group in G is conjugate to a subgroup of $N_G(J)$.

PROOF. Similar to Lemma 2.

LEMMA 7 (E_8).

- (i) Let $K = \langle e_j e_k \mid 1 \leq j, k \leq 8 \rangle$ and $R/J = O_2(N(J)/J)$. Then K is a hyperplane in Z and $R = N(J) \cap C(K)$.
- (ii) $R - J$ contains a conjugate d of e_1 such that each involution in $R - J$ is $N(J)$ -conjugate to an involution in dK .
- (iii) If $ijkl$ is a 4-tuple as in Lemma 5(ii) and if x_i, x_j, x_k, x_l are elements of order 4 in J_i, J_j, J_k, J_l , respectively, then $x_i x_j x_k x_l \in e_1^G$.

PROOF. $N(J)$ acts on K since it permutes Σ , and clearly K is a hyperplane in Z . So $N_G(J)$ induces $L_3(2)$ on K and (i) follows. Observe that R/J acts regularly on Σ .

Let $z \in K^*$. Then $J \leq D = C_G(z) = D_8$ (half-spin). Consider SO_{16} (an image of the covering group of D) and its subgroup $\bar{D} = SO_{16} \cap (O_4)^4$. Set $(\bar{D})^0 = \bar{J}$, a group corresponding to J . Choose reflections t_1, t_2, t_3, t_4 , one from each O_4 . The product of any two of these is in SO_{16} , and these products generate an elementary abelian group \bar{S} of order 8 which acts faithfully on the set $\bar{\Sigma}$ of simple factors of \bar{J} . Let \bar{R} denote the subgroup corresponding to R . Then $\bar{S} \cap \bar{R}$ is not contained in \bar{J} . Since $t = t_1 t_2 t_3 t_4$ is the unique element in \bar{S} acting semi-regularly on $\bar{\Sigma}$, we have $t \in \bar{R} - \bar{J}$. In SO_{16} , t is conjugate to an involution in a

fundamental SL_2 . Translating this to D we conclude that there must exist an element $d \in (D \cap R) - J$, with d a conjugate of e_1 .

To prove (ii) let t be any involution in $R - J$. Since $(R/J)^\#$ is fused in $N_G(J)$, we may assume $tJ = dJ$. Hence, J/Z is the direct product of simple groups permuted semiregularly by t . Therefore, all involutions in dJ are conjugates of those in dZ . Hence, we may assume $t \in dZ$. Also, $C_Z(d) = K$ (since $Z - K = \{e_1, \dots, e_8\}$). So the only involutions in dZ are in fact in dK . This proves (ii).

For (iii) again consider D and choose $X \circ Y \leq D$ with X, Y of type D_4 . Then X and Y are simply connected and we may take $J \leq X \circ Y$, where $J \cap X$ and $J \cap Y$ are each a product of 4 of the fundamental SL_2 's. Say $J \cap X = J_r J_s J_u J_v$. One checks that $e_r e_s e_u e_v = 1$ so $rsuv$ is one of the 4-tuples of Lemma 5(ii). From 3-transitivity of $N(J)$ on Σ we may assume $\{r, s, u, v\} = \{i, j, k, l\}$. Set $x = x_i x_j x_k x_l$. The image of x in a quotient of X isomorphic to SO_8 is necessarily conjugate to the images of e_i, e_j, e_k , and e_l (by consideration of the action of this image on the orthogonal module). Without loss we may assume the kernel to the map is $\langle e_i e_j \rangle$. Hence, $x \sim e_i$ or $e_i (e_i e_j) = e_j$, proving (iii).

LEMMA 8 (E_8). *Let $E \leq N_G(J)$ be an elementary abelian 2-group. Then $|E| \leq 2^9$, equality possible only if E is G -conjugate to a subgroup of J .*

PROOF. Assume $|E| \geq 2^9$ and let $X = EJ/J$. If X has a fixed point, say J_8 , on Σ , then $E \leq N(J_8) = J_8 E_7$ and we are done by reduction to E_7 . Similarly, we may assume E centralizes no conjugate of e_1 .

Assume $X \cap (R/J) = 1$, so $|X| \leq 4$. Involutions in $N(J)/J$ fixing a point in Σ fix exactly 4 points, so from the above paragraph we conclude X contains a regular involution, say x . Then $C_Z(x) \leq K$ (as $Z - K = \{e_1, \dots, e_8\}$) and x is non-trivial on K (as $x \notin R$). Thus, $|E \cap Z| \leq |C_Z(E)| \leq 4$. But $|E \cap J| \geq 2^7$, whence $(E \cap J)Z$ is an elementary abelian group of order at least 2^9 .

Apply Lemma 6. Replacing E by a J -conjugate, if necessary, we may assume $(E \cap J)Z/Z = \langle Z, x_4 x_5 x_6 x_7, x_2 x_3 x_6 x_7, x_1 x_2 x_5 x_6, x_1 x_2 x_3 x_8, y_1 \dots y_8 \rangle$. However, x must centralize $(E \cap J)Z/Z$ and have no fixed points on Σ . Checking possible orbits of x we see this to be impossible.

We may now assume $X \cap (R/J) \neq 1$ and let $s \in (E \cap R) - J$. Lemma 7(ii) implies $sK = aK$ for some $a \in e_1^G$. From the first paragraph it follows that E does not centralize K . In particular, E is not contained in R . Let $f \in E - R$. Then $|E \cap Z| \leq |C_Z(E)| \leq |C_Z(s) \cap C_Z(f)| = |C_K(f)| = 4$.

It follows that $E \cap J$ must contain an element of the form $d = x_i x_j x_k x_l z$, where $ijkl$ is a tuple as in Lemma 5, x_r is of order 4 in J_r for $r \in \{i, j, k, l\}$, and $z \in Z$. Note that $\{i, j, k, l\}$ is necessarily a union of two orbits of $\langle s \rangle$. Also X must act on $\{i, j, k, l\}$ and also on its complement (as E centralizes s).

If $|X| \leq 4$, then as above $|E \cap J| \geq 2^9$ and we again obtain (recall that $|E \cap Z| \leq 4$) a contradiction using Lemma 6. Hence, $|X| \geq 8$. Restricting the abelian group X to $\{i, j, k, l\}$ we obtain an element $1 \neq x \in X$ fixing i, j, k , and l . Also X is transitive on either $\{i, j, k, l\}$ or its complement $\{i', j', k', l'\}$.

Order considerations imply $E \cap J$ must also contain an element of the form $x_i x_j x_k x_l z'$, so we may assume X is transitive on $\{i, j, k, l\}$. Let $e \in E$ satisfy $eJ = x$. Then $d = d^e = (x_i x_j x_k x_l)^e z^e$. But e normalizes each of J_i, J_j, J_k, J_l , so $x_i^e = x_i, x_j^e = x_j, x_k^e = x_k, x_l^e = x_l$ for $t = 0, 1$. Transitivity forces $x_r^e = x_r, e_r^t$ for each $r \in \{i, j, k, l\}$, and so $(x_i x_j x_k x_l)^e = x_i x_j x_k x_l$. Thus, $z = z^e$ and so $z \in C_Z(e) = \langle e_i, e_j, e_k, e_l \rangle$. Hence $d \sim x_i x_j x_k x_l$ (use an element of $\langle y_i, y_j, y_k, y_l \rangle$) and so by Lemma 7(iii), $d \in e_1^G$, contradicting the first paragraph.

The E_8 case of the theorem is now immediate from Lemmas 6 and 8.

COROLLARY. *Let q be an odd prime power. Then the 2-rank of ${}^2G_2(q), G_2(q), F_4(q), E_6(q), {}^2E_6(q), E_7(q), \hat{E}_7(q), E_8(q)$ is 3, 3, 5, 6, 6, 8, 7, 9 in the respective cases.*

PROOF. Let G be the algebraic group and let q be a power of the prime p . If σ is a field endomorphism, then it is immediate from the description given that the elementary abelian 2-groups of maximal rank can be taken in $O^p(G_\sigma)$. Suppose G is of type E_6 and that $\sigma = q\tau$, where τ is a graph automorphism. Set $E = \Omega_2(T)$, where T is a σ -stable torus contained in a σ -stable Borel subgroup. Let $w_0 \in N(T)$ represent the long word $w_0 \in W = N(T)/T$. Since τw_0 acts on T by inversion it fixes E elementwise; hence $\sigma w_0 = q\tau w_0$ fixes E elementwise. The result follows since Lang's Theorem implies that σ and σw_0 are G -conjugate. Finally, consideration of the centralizer of an involution shows that the 2-rank of ${}^2G_2(q)$ is 3.

2. ODD PRIMES

In this section r is an odd prime and G is an algebraic group of exceptional type over an algebraically closed field of characteristic distinct from r . We begin with a general lemma.

LEMMA 9. *Let J be an algebraic group over an algebraically closed field of characteristic $p \neq r$. Suppose $J = T_s \circ J_1 \circ \dots \circ J_k$, a central product of an s -dimensional torus and k groups isomorphic to SL_r . Then the r -rank of J is $s + k(r - 1)$ and all elementary abelian r -subgroups of maximal rank are contained in a maximal torus of J .*

PROOF. Assume $2 < r \neq p$. J contains a maximal torus of rank $s + k(r - 1)$, so the r -rank of J is at least $s + k(r - 1)$. Since the r -rank of both SL_r and PSL_r is easily checked to be $r - 1$, the first assertion follows by induction, factoring out T_s and all but one of the SL_r 's. These remarks also show that J/J_i has r -rank $s + (k - 1)(r - 1)$, for each $1 \leq i \leq k$. Let E be an elementary abelian r -subgroup of J having maximal rank and fix i ($1 \leq i \leq k$). By the above, $E \cap J_i$ has rank $r - 1$, and since $E \cap J_i$ is abelian it is contained in a maximal torus T'_i of $J_i \cong SL_r$. Moreover, $C_{J_i}(E \cap J_i) = T'_i$. Hence, $E \leq \bigcap_i C_{J_i}(E \cap J_i) = T_s T'_1, \dots, T'_k$, a maximal torus of J . The lemma follows.

By the results of [Springer & Steinberg, 1970] every elementary abelian r -group in G can be embedded in a torus if $r > 3$ for G_2 and F_4 , $r > 5$ for E_6 and E_7 , and $r > 7$ for E_8 . Thus, we only consider the remaining odd primes. Let T be a maximal torus of G . The following subgroups D of G contain T and are such that $N_D(T)/T$ contains a Sylow r -group of $N_G(T)/T$.

G	$r=3$	$r=5$	$r=7$
G_2	A_2		
F_4	A_2A_2		
E_6	$(A_2A_2A_2)3$	T_2A_4	
E_7	$T_1(A_2A_2A_2)3$	T_3A_4	
E_8	$A_2(A_2A_2A_2)3$	A_4A_4	T_2A_6

Here, T_i stands for a torus of rank i , and A_{r-1} for a fundamental subgroup isomorphic to SL_r .

PROPOSITION. *If r is an odd prime and G is an algebraic group of exceptional Lie type over an algebraically closed field of characteristic $\neq r$, then the r -rank of G is the Lie rank of G . Moreover, all elementary abelian r -subgroups of maximal rank are conjugate and contained in a maximal torus of G .*

PROOF. Let E be an elementary abelian r -subgroup of G of rank at least the Lie rank of G . In view of the previous comments we may take r to be one of the primes in the table above and assume that $E \leq D$, where D is also given in the table. A dimension check shows that D^0 contains a maximal torus of G , so the result follows from Lemma 9 provided $E \leq D^0$.

Suppose there is $e \in E - D^0$. Then $r=3$, $G=E_6, E_7$, or E_8 . Here D contains a normal subgroup S with $S \cong 1, T_1$, or A_2 , respectively, $D/SZ(D)$ the wreath product of PSL_3 with \mathbb{Z}_3 , and $C_{D/SZ(D)}(e) \cong PSL_3 \times \mathbb{Z}_3$. So the 3-rank of E is at most 3 plus the 3-rank of $C_{SZ(D)}(e)$. From the action of e it is clear that the latter is at most 2, 3, 4, respectively, so this is a contradiction.

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