Boundary element algorithms for Somigliana's identity

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Introduction

In this paper we present a method for the evaluation of the finite part or Cauchy principal value of singular surface integrals occurring in boundary element methods. In section 1 we briefly discuss Somigliana's identity. This identity is the starting point for the calculation of elastic deformation and stress in elastostatics by use of boundary element methods. For details we refer e.g. to the book of C.A. Brebbia and J. Dominguez (1989).In section 2 we describe the boundary element approximation of the geometry of the elastic solid under consideration. If the source point of the fundamental solution of the Navier-Cauchy equilibrium equation is a vertex of a triangular element of the boundary element mesh, then the integration over that triangle must be performed analytically. For the evaluation of the finite part or Cauchy principal value of singular integrals we use a two-dimensional version of the theorem of Gauss. This idea for the evaluation of the finite part of singular integrals in boundary element algorithms may be useful in other applications also. In section 3 we present the formulas for these integrals for implementation in boundary element algorithms in elastostatics based on Somigliana's identity.

1. Somigliana's identity

Consider a homogeneous isotropic elastic solid. The solid body in the reference (undeformed) configuration occupies a bounded region Ω in space. Coordinates of points and components of tensors are relative to a fixed right-handed cartesian coordinate system. We use the referential or lagrangian description of the material particles of the elastic solid, i.e. the material particles of the solid are identified by their coordinates $x = (x_1, x_2, x_3)$ in the reference configuration. The coordinates of particle x in the current (deformed) configuration are denoted by $x + u$, where $u = (u_1, u_2, u_3)$ is the displacement vector of particle x in the current configuration. The boundary surface of Ω is denoted by Γ and n is the unit outward normal vector on Γ. The elastic solid is loaded by external forces. We distinguish volume forces in the interior particles and surface traction forces in the particles on the boundary surface. The external volume force per unit volume in an interior point of Ω is denoted by b. The external traction force per unit area in a point on Γ is written as f. We consider the case that the elastic solid is in static equilibrium. Let p be an interior point of Ω. We refer to p as the source point. By use of the indicial notation Somigliana's identity can be written as

$$(1.1) \qquad u_k(p) = \int_\Omega u^*_{ki}[p] b_i d\Omega + \int_\Gamma u^*_{ki}[p] f_i d\Gamma - \int_\Gamma t^*_{ki}[p] u_i d\Gamma,$$

where

E = modulus of elasticity of the elastic material;

ν = Poisson's ratio of the elastic material;

$\mu := \dfrac{E}{2(1+\nu)}$ = shear modulus of the elastic material;

δ_{ij} = Kronecker delta $:= \begin{cases} 1, & i = j\ , \\ 0, & i \neq j\ ; \end{cases}$

ρ = distance to the source point p;

and the components of the fundamental solution of the Navier-Cauchy equilibrium equation corresponding to the source point p are put in the form

$$u^*_{ki}[p] = \frac{1}{16\pi\mu(1-\nu)\rho}((3-4\nu)\delta_{ki} + \rho_{,k}\rho_{,i}),$$

$$\rho_{,k} := \frac{\partial \rho}{\partial x_k};$$

$$t^*_{ki}[p] = \frac{-1}{8\pi(1-\nu)\rho^2}(3\rho_{,k}\rho_{,i}\rho_{,j}n_j + (1-2\nu)(\delta_{ki}\rho_{,j}n_j + n_k\rho_{,i} - n_i\rho_{,k})).$$

For details the reader is referred to the book of C.A. Brebbia and J. Dominguez (1989).
We now take the source point p on the boundary surface Γ. Let $\varepsilon > 0$ and the sphere with radius ε and centered at $p \in \Gamma$ is denoted by S^ε. Introduce

$$\Omega^\varepsilon := \Omega \cup S^\varepsilon, \quad B^\varepsilon = \Omega^\varepsilon \backslash \Omega.$$

The boundary surface of Ω^ε is written as Γ^ε, and

$$\Gamma^\varepsilon_1 := \Gamma^\varepsilon \cap \Gamma, \quad \Gamma^\varepsilon_2 := \Gamma^\varepsilon \backslash \Gamma^\varepsilon_1.$$

The source point p is an interior point of Ω^ε. So we can apply (1.1) to obtain

$$(1.2) \qquad u_k(p) = \int_\Omega u^*_{ki}[p]b_i d\Omega + \int_{B^\varepsilon} u^*_{ki}[p]b_i dB^\varepsilon + \int_{\Gamma^\varepsilon_1} u^*_{ki}[p]f_i d\Gamma^\varepsilon_1 + \int_{\Gamma^\varepsilon_2} u^*_{ki}[p]f_i d\Gamma^\varepsilon_2 + \\ - \int_{\Gamma^\varepsilon_1} t^*_{ki}[p]u_i d\Gamma^\varepsilon_1 - \int_{\Gamma^\varepsilon_2} t^*_{ki}[p]u_i d\Gamma^\varepsilon_2.$$

For $\varepsilon \downarrow 0$ we get

$$(1.3) \qquad (\delta_{ki} + c_{ki}(p))u_i(p) = \int_\Omega u^*_{ki}[p]b_i d\Omega + \int_\Gamma u^*_{ki}[p]f_i d\Gamma - F.P. \int_\Gamma t^*_{ki}[p]u_i d\Gamma,$$

where

$$c_{ki}(p) := \lim_{\varepsilon \downarrow 0} \int_{\Gamma^\varepsilon_2} t^*_{ki}[p] d\Gamma^\varepsilon_2,$$

$$F.P. \int_\Gamma t^*_{ki}[p]u_i d\Gamma := \lim_{\varepsilon \downarrow 0} \int_{\Gamma^\varepsilon_1} t^*_{ki}[p]u_i d\Gamma^\varepsilon_1.$$

If the boundary surface Γ is differentiable at $p \in \Gamma$, then

$$c_{ki}(p) = -\frac{1}{2}\delta_{ki}. \tag{1.4}$$

The integral $\int_\Gamma t^*_{ki}[p]u_i d\Gamma$ does not exist. Therefore in (1.3) we write $F.P. \int_\Gamma t^*_{ki}[p]u_i d\Gamma$, i.e. the finite part or Cauchy principal value of the integral under consideration.
Now suppose that there are no external volume forces, i.e. $b = 0$, then Somigliana's identity for the source point $p \in \Gamma$ can be put in the form

$$(\delta_{ki} + c_{ki}(p))u_i(p) = \int_\Gamma u^*_{ki}[p]f_i d\Gamma - F.P. \int_\Gamma t^*_{ki}[p]u_i d\Gamma. \tag{1.5}$$

If we translate the elastic solid over a vector, say a, i.e. replace u by $u + a$, then (1.5) is still satisfied. So

$$\delta_{ki} + c_{ki}(p) = -F.P. \int_\Gamma t^*_{ki}[p]d\Gamma. \tag{1.6}$$

From (1.5) and (1.6) we conclude that Somigliana's identity for the source point $p \in \Gamma$ can be written as

$$\int_\Gamma u^*_{ki}[p]f_i d\Gamma = F.P. \int_\Gamma t^*_{ki}[p](u_i - u_i(p))d\Gamma. \tag{1.7}$$

The integral equation (1.7) is the starting point for boundary element methods in elastostatics.

2. Boundary element approximation

In the boundary element approximation of Somigliana's identity (1.7) we triangulate the boundary surface Γ. The vertices of the triangles are points on Γ and are said to be the nodes of the so-called boundary element mesh M. Let $\Delta \in M$ be a triangle with vertices N_1, N_2, N_3. The position vectors of the nodes N_1, N_2, N_3 are denoted by x^1, x^2, x^3 respectively. Let p be a point of Δ with position vector

$$x = \xi_1 x^1 + \xi_2 x^2 + \xi_3 x^3 =: \xi_j x^j, \tag{2.1}$$

such that

$$\xi_1 + \xi_2 + \xi_3 = 1, \quad \xi_i \geq 0, \ \ i = 1, 2, 3.$$

The triangle $N_1 N_2 P$ is denoted by Δ_3. For the area $|\Delta_3|$ of Δ_3 we have $|\Delta_3| = \xi_3|\Delta|$.

Hence,

$$\xi_i = |\Delta_i|/|\Delta|, \quad \text{i} = 1,2,3. \tag{2.2}$$

The displacement vector u and the external traction force f per unit area in the nodes N_i, i $= 1,2,3$, are denoted by u^i and f^i respectively. The displacement vector u and the external traction force f per unit area in the point $x = \xi_j x^j \in \Delta$ are approximated by

$$u = \xi_j u^j, \quad f = \xi_j f^j. \tag{2.3}$$

Fix the node $p \in \Gamma$ of the mesh M and let $k \in \{1,2,3\}$. The boundary element approximation of the integral equation (1.7) can be put in the form

$$\sum_{\Delta \in M} (C_{kij}(p,\Delta) f_i^j - Ekij(p,\Delta)(u_i^j - u_i(p))) = 0, \tag{2.4}$$

where

$$C_{kij}(p,\Delta) := \int_\Delta u_{ki}^*[p]\xi_j d\Delta$$

$$E_{kij}(p,\Delta) := F.P. \int_\Delta t_{ki}^*[p]\xi_j d\Delta.$$

If the node $p \in \Gamma$ is not a vertex of the triangle Δ, then the integrals can be approximated by standard numerical methods e.g. Gauss quadrature. However, if the node $p \in \Gamma$ is a vertex of Δ, then the integrals are evaluated analytically. The results are presented in the next section.

3. Boundary element integrals

Let $p \in \Gamma$ be a node and $\Delta \in M$ a triangle such that p is a vertex of Δ. Without loss of generality we may assume $p = N_1$, where N_1, N_2, N_3 are the vertices of Δ. The angles of Δ at N_1, N_2, N_3 are denoted by $\alpha_1, \alpha_2, \alpha_3$ and z_1, z_2, z_3 is the length of the sides N_2N_3, N_3N_1, N_1N_2 respectively. The position vector of the node N_i, $i = 1,2,3$, is written als x^i. The components of the unit outward normal vector n on Δ can be put in the form

$$\begin{aligned} n_1 &= \frac{1}{sin\, \alpha_1}\left(\frac{x_2^2 - x_2^1}{z_3}\frac{x_3^3 - x_3^1}{z_2} - \frac{x_3^2 - x_3^1}{z_3}\frac{x_2^3 - x_2^1}{z_2}\right), \\ n_2 &= \frac{1}{sin\, \alpha_1}\left(\frac{x_3^2 - x_3^1}{z_3}\frac{x_1^3 - x_1^1}{z_2} - \frac{x_1^2 - x_1^1}{z_3}\frac{x_3^3 - x_3^1}{z_2}\right), \\ n_3 &= \frac{1}{sin\, \alpha_1}\left(\frac{x_1^2 - x_1^1}{z_3}\frac{x_2^3 - x_2^1}{z_2} - \frac{x_2^2 - x_2^1}{z_3}\frac{x_1^3 - x_1^1}{z_2}\right). \end{aligned} \tag{3.1}$$

To write down the formula for $C_{kij}(p,\Delta) = C_{kij}(N_1,\Delta)$ we introduce some notations.

$$(3.2) \qquad \beta := ln(cotan(\frac{\alpha_2}{2})cotan(\frac{\alpha_3}{2})).$$

$$v_1 := \beta,$$

$$(3.3) \qquad v_2 := \frac{z_2}{z_1}(\beta \cos\alpha_3 + \frac{z_3}{z_2} - 1),$$

$$v_3 := \frac{z_3}{z_1}(\beta \cos\alpha_2 + \frac{z_2}{z_3} - 1).$$

$$p_{ki} := \frac{1}{\sin^2\alpha_1}\frac{x_k^2 - x_k^1}{z_3}\frac{x_i^2 - x_i^1}{z_3},$$

$$(3.4) \qquad q_{ki} := \frac{1}{\sin^2\alpha_1}(\frac{x_k^2 - x_k^1}{z_3}\frac{x_i^3 - x_i^1}{z_2} + \frac{x_i^2 - x_i^1}{z_3}\frac{x_k^3 - x_k^1}{z_2}),$$

$$r_{ki} := \frac{1}{\sin^2\alpha_1}\frac{x_k^3 - x_k^1}{z_2}\frac{x_i^3 - x_i^1}{z_2}.$$

$$(3.5) \qquad A_{ki} := p_{ki}\,(\cos\alpha_3 - \cos(\alpha_3 - \alpha_1) + \beta\sin^2\alpha_3) + q_{ki}\,(\cos\alpha_2 + \cos\alpha_3 - \beta\sin\alpha_2\sin\alpha_3) +$$

$$+ r_{ki}\,(\cos\alpha_2 - \cos(\alpha_2 - \alpha_1) + \beta\sin^2\alpha_2).$$

$$(3.6) \qquad B_{ki} := p_{ki}\,(\frac{\sin^2\alpha_1}{\sin\alpha_2} + \beta\sin(2\alpha_3) - 2(\sin\alpha_3 + \sin(2\alpha_3 + \alpha_2))) +$$

$$+ q_{ki}\,(\beta\sin(\alpha_3 - \alpha_2) - 2(\sin\alpha_3 - \sin\alpha_2)) +$$

$$+ r_{ki}\,(-\frac{\sin^2\alpha_1}{\sin\alpha_3} - \beta\sin(2\alpha_2) + 2(\sin\alpha_2 + \sin(2\alpha_2 + \alpha_3))).$$

$$\overline{C}_{ki1} := A_{ki},$$

$$(3.7) \qquad \overline{C}_{ki2} := \frac{z_2}{z_1}(A_{ki}\cos\alpha_3 + B_{ki}\sin\alpha_3),$$

$$\overline{C}_{ki3} := \frac{z_3}{z_1}(A_{ki}\cos\alpha_2 - B_{ki}\sin\alpha_2).$$

As a result of tedious and elementary calculations we get

$$(3.8) \qquad C_{kij}(p,\Delta) = C_{kij}(N_1,\Delta) = \frac{|\Delta|}{16\pi z_1\mu(1-\nu)}((3-4\nu)\delta_{ki}v_j + \overline{C}_{kij}).$$

We now come to the formula for $E_{kij}(p,\Delta) = E_{kij}(N_1,\Delta)$.
Use (1.1), (2.4) and $\rho_{,j}n_j = 0$ to write

$$(3.9)\qquad E_{kij}(p,\Delta)=E_{kij}(N_1,\Delta)=\frac{-(1-2\nu)}{8\pi(1-\nu)}F.P.\int\limits_{\Delta}\frac{n_k\rho_{,i}-n_i\rho_{,k}}{\rho^2}\xi_j d\Delta.$$

If $j=2$ or $j=3$, then the integral on the right-hand side in (3.9) exists and can be evaluated analytically.
For $j=1$ we get

$$(3.10)\qquad E_{ki1}(N_1,\Delta)=\frac{-(1-2\nu)}{8\pi(1-\nu)}F.P.\int\limits_{\Delta}\frac{n_k\rho_{,i}-n_i\rho_{,k}}{\rho^2}d\Delta-E_{ki2}(N_1,\Delta)-E_{ki3}(N_1,\Delta).$$

The finite part of the integral in (3.10) is evaluated as follows.
Let $\varepsilon>0$ and define $\Delta_\varepsilon\subset\Delta$ as depicted in Figure 3.1.

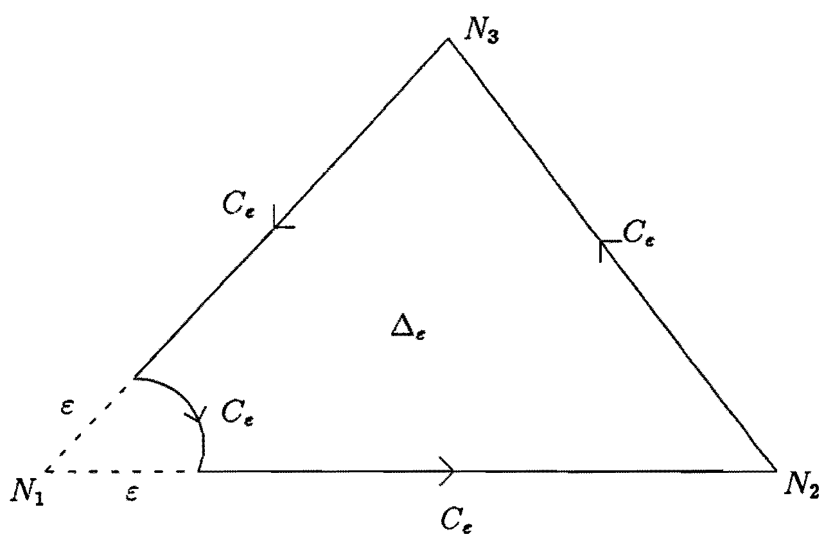

Figure 3.1.

The directions of the unit tangent vector t of the boundary curve C_ε of Δ_ε and the unit outward normal vector n on Δ_ε correspond in the sense of the right-hand screw. Integration by parts yields (use $n.grad\frac{1}{\rho}=0$ on Δ_ε)

$$(3.11)\qquad \int\limits_{\Delta_\varepsilon} n\times grad\frac{1}{\rho}d\Delta_\varepsilon=\int\limits_{C_\varepsilon}\frac{1}{\rho}tdC_\varepsilon.$$

From an elementary calculation we obtain

$$(3.12)\qquad \int\limits_{\Delta_\varepsilon} n\times grad\frac{1}{\rho}d\Delta_\varepsilon=\int\limits_{C_\varepsilon}\frac{1}{\rho}tdC_\varepsilon=\frac{x^3-x^2}{z_1}\beta+\frac{x^2-x^1}{z_3}(1+ln\frac{z_3}{\varepsilon})+\frac{x^1-x^3}{z_2}(1+ln\frac{z_2}{\varepsilon}).$$

Since the last two terms on the right-hand side in (3.12) cancel in the integration over de triangles with p as vertex, we conclude from (3.12)

$$
(3.13) \qquad
\begin{aligned}
D_{23} &:= -F.P.\int_{\Delta} \frac{n_2\rho_{,3} - n_3\rho_{,2}}{\rho^2} d\Delta = \frac{x_1^3 - x_1^2}{z_1}\beta =: -D_{32},\\
D_{31} &:= -F.P.\int_{\Delta} \frac{n_3\rho_{,1} - n_1\rho_{,3}}{\rho^2} d\Delta = \frac{x_2^3 - x_1^2}{z_1}\beta =: -D_{13},\\
D_{12} &:= -F.P.\int_{\Delta} \frac{n_1\rho_{,2} - n_{2},\rho_{,1}}{\rho^2} d\Delta = \frac{x_3^3 - x_3^2}{z_1}\beta =: -D_{21},\\
&D_{11} = D_{22} = D_{33} := 0.
\end{aligned}
$$

From (3.10) and (3.13) we conclude

$$
(3.14) \qquad E_{ki1}(N_1,\Delta) + E_{ki2}(N_1,\Delta) + E_{ki3}(N_1,\Delta) = \frac{1-2\nu}{8\pi(1-\nu)} D_{ki}.
$$

To write down the formula for $E_{kij}(p,\Delta) = E_{kij}(N_1,\Delta)$ we introduce the notation

$$
(3.15) \qquad
\begin{aligned}
S_{i1} &:= \frac{1}{sin\,\alpha_3}\left(\frac{x_i^3 - x_i^1}{z_2}(cos\,\alpha_2 + cos\,\alpha_3) - \frac{x_i^3 - x_i^2}{z_1}(1 - cos\,\alpha_1)\right),\\
S_{i2} &:= \frac{1}{sin\,\alpha_3}\frac{z_2}{z_1}\left(-\frac{x_i^3 - x_i^1}{z_2}(1 - cos\,\alpha_1) + \frac{x_i^3 - x_i^2}{z_1}(cos\,\alpha_3 - cos(\alpha_3 - \alpha_1) + \beta sin^2\alpha_3)\right),\\
S_{i3} &:= \frac{1}{sin\,\alpha_2}\frac{z_2}{z_1}\left(-\frac{x_i^3 - x_i^1}{z_2}(1 - cos\,\alpha_1) + \frac{x_i^3 - x_i^2}{z_1}(cos\,\alpha_3 + cos\,\alpha_2 - \beta\, sin\,\alpha_2 sin\,\alpha_3)\right).
\end{aligned}
$$

After elementary calculations we obtain

$$
(3.16) \qquad E_{kij}(p,\Delta) = E_{kij}(N_1,\Delta) = \frac{1-2\nu}{8\pi(1-\nu)}\,(\delta_{1j}D_{ki} + n_k S_{ij} - n_i S_{kj}).
$$

References

C.A. Brebbia and J. Dominguez: Boundary elements: an introductory course, MacGraw-Hill, 1989.