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RANA 89-09
June 1989

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by

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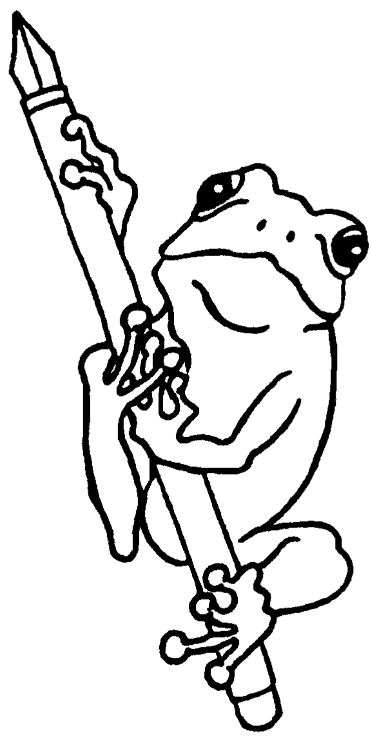

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Abstract

Hilbert spaces of harmonic functions are presented wherein differentiation operators are continuous or even compact

1. Introduction

We all say that differentiation operators are unbounded or equivalently not continuous in a Hilbert or Banach space of functions. In fact, differentiation operators are a standard illustrative example in the undergraduate text books on functional analysis, which is used to show that there are unbounded operators in Banach spaces. In the present note, however, we are going to give some naturally arising Hilbert spaces of harmonic functions wherein the differentiation operators are continuous or even compact.

Let us fix a few notations. $\mathbb{R}^q$ is the Euclidean space of dimension $q \geq 2$ with inner product $x \cdot y = \sum_{i=1}^{q} x_i y_i$ and corresponding norm $|x| = (x \cdot x)^{1/2}$ for $x,y \in \mathbb{R}^q$. $B^q(R) = \{x \mid x \in \mathbb{R}^q, |x| < R\}$; $B^q \equiv B^q(1)$. $S^{q-1} = \{x \in \mathbb{R}^q \mid |x| = 1\}$ is the unit sphere in $\mathbb{R}^q$. The points on S^{q-1} are denoted ζ,η et al. As usual $L^2(S^{q-1})$ stands for the L^2 space of functions defined on S^{q-1} with inner product

$$(u, \mathrm{v}) = \int_{S^{q-1}} u(\zeta)\, \overline{\mathrm{v}(\zeta)}\, d\sigma_{q-1}(\zeta)$$

and corresponding norm $\|u\| = [(u,u)]^{1/2}$. Here $d\sigma_{q-1}$ represents the usual Lebesque-measure element and $\sigma_{q-1} = \int_{S^{q-1}} 1 \cdot d\sigma_{q-1}(\zeta)$ is the total measure of S^{q-1}.

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2. Identifications of Weighted Hilbert Spaces of Harmonic Functions on $\mathbb{R}^q$ with Domains of Positive Self-adjoint Operators in $L^2(S^{q-1})$

Let $\mu : (0,\infty) \to (0,\infty)$ be a Lebesque measurable function which is nonnegative almost everywhere. It is easy to see that

$$HA^q(\mu) = \{u \mid u(x) \text{ harmonic on } \mathbb{R}^q \text{ and } \|u\|_\mu = [\int_{\mathbb{R}^q} |u(x)|^2 \mu(|x|)\, dx]^{1/2} < \infty\} \quad (1)$$

is a pre-Hilbert space with corresponding inner product

$$(u, v)_\mu = \int_{\mathbb{R}^q} u(x)\, \overline{v(x)}\, \mu(|x|)\, dx. \quad (2)$$

On the other hand let $\{\lambda_m \mid m \in \mathbb{N}_0\}$ be a sequence of nonnegative numbers. Recall that we have the standard identity decomposition $L^2(S^{q-1}) = \bigoplus_{m=0}^{\infty} \mathrm{H}_m^q$ (cf. e.g., [7]). So, associated with the given sequence $\{\lambda_m \mid m \in \mathbb{N}_0\}$ a positive self-adjoint operator Λ in $L^2(S^{q-1})$ is well defined as follows:

$$u = \sum_{m=0}^{\infty} P_m u$$

$$\Lambda u = \sum_{m=0}^{\infty} \lambda_m P_m u \quad (4)$$

and

$$D(\Lambda) = \{u \in L^2(S^{q-1}) \mid \sum_{m=0}^{\infty} \lambda_m^2 \|P_m u\|^2 < \infty\} \quad (5)$$

where P_m is the projection of $L^2(S^{q-1})$ onto H_m^q. The domain $D(\Lambda)$, equipped with the graph inner product

$$(u, v)_\Lambda = \sum_{m=0}^{\infty} \lambda_m^2 (P_m u\, ,\, P_m v) \quad (6)$$

and corresponding norm

$$\|u\|_\Lambda = (\sum_{m=0}^{\infty} \lambda_m^2 \|P_m u\|^2)^{1/2} \quad (7)$$

is a Hilbert space.

Our main aim in this section is to identify a space $HA^q(\mu)$ with a space $D(\Lambda)$ for an admissible pair $(\mu, \{\lambda_m\})$ of weight function and nonnegative sequence. In order to formulate and prove the result we need an auxiliary lemma.

Lemma 1. (cf. [2] and [3] Theorem IV.1.10)

(i) $u(\zeta) \in L^2(S^{q-1})$ can be extended to a harmonic function $u(x)$ on $B^q(R)$ $(R > 1)$ iff

$$\sum_{m=0}^{\infty} r^{2m} \|P_m u\|^2 < \infty \text{ for any } r \in (0,R). \tag{8}$$

(ii) If $u(x)$ is a harmonic function on $B^q(R)$ $(R > 1)$ then

$$\sum_{m=0}^{\infty} r^m (P_m u)(\zeta) = \sum_{m=0}^{\infty} (P_m u)(x) = u(x) \tag{9}$$

where the series converges uniformly on each ball $B^q(r)$ with $r \in (0,R)$. ▯

Now we can give

Theorem 2. Assume that a weight function $\mu : (0,\infty) \to (0,\infty)$ and a sequence $\{\lambda_m \mid m \in \mathbb{N}_0\}$ satisfy the conditions below

i) $\mu_m \equiv (\int_0^{\infty} r^{2m+q-1} \mu(r)\, dr)^{1/2} < \infty, \ \forall m \in \mathbb{N}_0$

ii) $R^m / \lambda_m = O(1), \ \forall R > 0$

iii) $\{\lambda_m\} \sim \{\mu_m\}$, i.e. $0 < \liminf \dfrac{\lambda_m}{\mu_m} \le \limsup \dfrac{\lambda_m}{\mu_m} < \infty$.

Then the space $HA^q(\mu)$ is isomorphic to the space $D(\Lambda)$ as normed spaces; the isomorphy is exactly the restriction-extension correspondence as in Lemma 1 above. Moreover, if, instead of iii) above, $\lambda_m = \mu_m$ for all $m \in \mathbb{N}_0$, then the isomorphy is actually an isometry.

Proof. Let $u(\zeta) \in D(\Lambda)$. Because of condition ii) Lemma 1 ensures that $u(\zeta)$ extends uniformly to a harmonic function $u(x)$ on $\mathbb{R}^q$, namely

$$u(x) = u(r\,\zeta) = \sum_{m=0}^{\infty} r^m (P_m u)(\zeta). \tag{10}$$

It is then clear that

$$\int_{S^{q-1}} |u(r\zeta)|^2 \, d\sigma_{q-1}(\zeta) = \sum_{m=0}^{\infty} r^{2m} \|P_m u\|^2. \tag{11}$$

Therefore

$$\|u\|_\mu^2 = \int_{\mathbb{R}^q} \mu(|x|) \, |u(x)|^2 \, dx$$

$$= \int_0^\infty r^{q-1} \mu(r) \, dr \int_{S^{q-1}} |u(r\zeta)|^2 \, d\sigma_{q-1}(\zeta)$$

$$= \int_0^\infty r^{q-1} \mu(r) \, dr \sum_{m=0}^{\infty} r^{2m} \|P_m u\|^2$$

$$= \sum_{m=0}^{\infty} \mu_m^2 \|P_m u\|^2 \tag{12}$$

which implies that $u(x) \in HA^q(\mu)$.

Conversely, if $u(x) \in HA^q(\mu)$, then Lemma 1 (ii) implies that equality (10) is valid, so are (11) and (12).

Thus, by condition iii) and equality (12) we conclude that the spaces $H^q(\mu)$ and $D(\Lambda)$ are isomorphic as normed spaces via the natural restriction-extension correspondence. In particular, if $\lambda_m = \mu_m$ for all $m \in \mathbb{N}_0$, then they are isometric. $\square$

Corollary 3. Under the conditions in Theorem 2 the space $HA^q(\mu)$ is complete. So it is a Hilbert space. $\square$

Corollary 4. Under the conditions in Theorem 2 we have $HA^q(\mu) = \bigoplus_{m=0}^{\infty} H_m^q$. Moreover, if $\{e_{m,j}^q(\zeta) \mid 1 \le j \le N(q,m)\}$ is an orthonormal basis in H_m^q, then $\{\frac{1}{\mu_m} e_{m,j}^q(x) \mid 1 \le j \le N(q,m)\}$ is an orthonormal basis in H_m^q considered a subspace of $HA^q(\mu)$.

Proof. For $p_m \in H_q^q$ and $q_n \in H_n^q$ we have

$$(p_m, q_n)_\mu = \int_{\mathbb{R}^q} \mu(|x|) \, p_m(x) \, \overline{q_n(x)} \, dx = \left(\int_0^\infty r^{m+n+q-1} \mu(r) \, dr\right) (p_m, q_n). \tag{13}$$

Hence $H_m^q \perp H_n^q$ in $HA^q(\mu)$ if $m \neq n$, for $\mathrm{H}_m^q \perp \mathrm{H}_n^q$. Equality (13) implies in particular that

$$\|p_m\|_\mu^2 = \mu_m^2 \, \|p_m\|^2. \tag{14}$$

So equality (12) becomes

$$\|u\|_\mu^2 = \sum_{m=0}^{\infty} \|P_m u\|_\mu^2. \tag{15}$$

Thus $HA^q(\mu) = \bigoplus_{m=0}^{\infty} HA_m^q$. The last assertion in the present corollary follows from (13) and (14) directly. □

Corollary 5. Under the conditions in Theorem 2 the Hilbert space $HA^q(\mu)$ has a reproducing kernel $K_\mu^q(x,y)$. Explicitly $K_\mu^q(x,y) = \sum_{m=0}^{\infty} K_{\mu,m}^q(x,y)$, where, for each m, $K_{\mu,m}^q(x,y)$ is the reproducing kernel of the subspace H_m^q, namely

$$K_{m,k}^q(x,y) = \frac{N(q,m)}{\sigma_{q-1}\,\mu_m^2} \, |x|^m \, |y|^m \, P_m^q \left[\frac{x}{|x|} \cdot \frac{y}{|y|} \right]. \tag{16}$$

Here P_m^q are the Legendre polynomials (cf. [6] and [7]).

Proof. Of the basic theory of reproducing kernel we refer to [1], [8] and [5].

Let $u \in HA^q(\mu)$. Then $u(x) = \sum_{m=0}^{\infty} r^m \, (P_m u)\,(\zeta)$. In view of the estimate $N(q,m) \le K_q \, m^{q-2}$ (cf. [Mü]) and condition ii) in Theorem 2 we have

$$\begin{aligned} |u(x)| &\le \sum_{m=0}^{\infty} r^m \, |(P_m u)\,(\zeta)| \\ &\le \sum_{m=0}^{\infty} r^m \left[\frac{N(q,m)}{\sigma_{q-1}} \right]^{1/2} \|P_m u\| \\ &\le \left[\sum_{m=0}^{\infty} \frac{N(q,m)\, r^{2m}}{\sigma_{q-1}\,\lambda_m^2} \right]^{1/2} \Big(\sum_{m=0}^{\infty} \lambda_m^2 \, \|P_m u\|^2 \Big)^{1/2} \\ &\le C(r)\, \|u\|_\mu \end{aligned} \tag{17}$$

where $C(r)$ is a constant depending only on q, μ and $\{\lambda_m\}$. The above inequality shows that for each fixed $x \in \mathbb{R}^q$, the functional $u \mapsto u(x)$ is continuous on $HA^q(\mu)$. Therefore the Hilbert space $HA^q(\mu)$ possesses a reproducing kernel K_μ^q.

Let $\{e^q_{m,j} \mid 1 \le j \le N(q,m)\}$ be an orthonormal basis in $\mathbf{H}^q_m$. Corollary 4 above says that $\{\frac{1}{\mu_m} e^q_{m,j}(x) \mid 1 \le j \le N(q,m)\}$ is an orthonormal basis in H^q_m. So its reproducing kernel $K^q_{\mu,m}$ is given by

$$K^q_{\mu,m}(x,y) = \sum_{j=1}^{N(q,m)} \frac{1}{\mu_m} e^q_{m,j}(x) \cdot \overline{\frac{1}{\mu_m} e^q_{m,j}(y)}$$

$$= \frac{|x|^m \, |y|^m}{\mu_m^2} \sum_{j=1}^{N(q,m)} e^q_{m,j}\left[\frac{x}{|x|}\right] \overline{e^q_{m,j}\left[\frac{y}{|y|}\right]}$$

$$= \frac{N(q,m)}{\sigma_{q-1}\,\mu_m^2} \, |x|^m \, |y|^2 \, P^q_m\left[\frac{x}{|x|} \cdot \frac{y}{|y|}\right]. \tag{18}$$

Since $HA^q(\mu) = \bigoplus_{m=0}^{\infty} H^q_m$, the reproducing kernel $K^q_\mu(x,y)$ for $HA^q(\mu)$ equals $\sum_{m=0}^{\infty} K^q_{\mu,m}(x,y)$. ▯

Corollary 6.

(i) For $u \in H^q_m$ holds the estimate

$$|u(x)| \le \left[\frac{N(q,m)}{\sigma_{q-1}}\right]^{1/2} \frac{|x|^m}{\mu_m} \|u\|_\mu . \tag{19}$$

(ii) For $u \in HA^q(\mu)$ holds the estimate

$$|u(x)| \le \left[\sum_{m=0}^{\infty} \frac{N(q,m)}{\sigma_{q-1}} \frac{|x|^{2m}}{\mu_m^2}\right]^{1/2} \|u\|_\mu . \tag{20}$$

(iii) For $u \in HA^q(\mu)$ and $m \in \mathbb{N}_0$

$$(P_m u)(x) = \frac{N(q,m)\,|x|^m}{\sigma_{q-1}\,\mu_m^2} \int_{\mathbb{R}^q} |y|^m \, P^q_m\left[\frac{x}{|x|} \cdot \frac{y}{|y|}\right] u(y)\,dy. \tag{21}$$

Proof. Assertion (i) and (ii) follow directly from Corollary 5. On multiplying both sides of (18) by $u(y)$ and integating over $\mathbb{R}^q$ we obtain equality (21) immediately. ▯

3. Conditions for Differential Operators to be Continuous or Compact in $HA^q(\mu)$

In this section Theorem 2 in the previous section is applied to give conditions such that differentiation operators are continuous or compact in the Hilbert space $HA^q(\mu)$.

Theorem 7. Suppose the conditoins in Theorem 2 are satisfied.

(i) If $\lambda_m^2 / \lambda_{m+1}^2 = O(m^{-q})$, then the differentiation operator $\partial_k (1 \le k \le q)$ is continuous on $HA^q(\mu)$.

(ii) If $\lambda_m^2 / \lambda_{m+1}^2 = o(m^{-q})$, then the differentiation operator $\partial_k (1 \le k \le q)$ is compact on $HA^q(\mu)$.□

In the proof of the theorem we need an estimate of spherical harmonics given in [4]:

Lemma 8. For $h \in H_m^q$ holds the estimate

$$|(\partial_k h)(\zeta)| \le m \left[\frac{N(q,m)}{\sigma_{q-1}} \right]^{1/2} \|h\|.$$

□

Proof of Theorem 7. Let $u(x)$ be a harmonic function on $\mathbb{R}^q$. The $u(x) = \sum_{m=0}^{\infty} (P_m u)(x)$ where the convergence is uniform on compacta of $\mathbb{R}^q$ and $\sum_{m=0}^{\infty} R^{2m} \|P_m u\|^2 < \infty$ for any $R > 0$ (Lemma 1). By Lemma 8 we have

$$\begin{aligned} |\partial_k [(P_m u)(x)]| &\le r^{m-1} \, |[\partial_k (P_m u)](\zeta)| \\ &\le r^{m-1} \, m \left[\frac{N(q,m)}{\sigma_{q-1}} \right]^{1/2} \|P_m u\| \\ &\le \left[\frac{K_q}{\sigma_{q-1}} \right]^{1/2} R^{-1} \, m^{q/2} \left[\frac{r}{R} \right]^{m-1} \cdot R^m \|P_m u\| \quad (0 < r < R). \end{aligned} \tag{22}$$

Therefore the series $\sum_{m=0}^{\infty} \partial_k [(P_m u)(x)]$ converges uniformly on compacta of $\mathbb{R}^q$. Hence

$$(\partial_k u)(x) = \sum_{m=0}^{\infty} \partial_k \, [(P_m u)(x)]. \tag{23}$$

In particular, for $u \in HA^q(\mu)$ we have

$$
\begin{aligned}
\|\partial_k u\|_\Lambda^2 &= \sum_{m=0}^{\infty} \lambda_m^2 \, \|P_m(\partial_k u)\|^2 \\
&= \sum_{m=0}^{\infty} \lambda_m^2 \, \|\partial_k(P_{m+1} u)\|^2 \\
&\le \sum_{m=0}^{\infty} \lambda_m^2 \int_{S^{q-1}} |[\partial_k(P_{m+1} u)]\,(\zeta)|^2 \, d\sigma_{q-1}(\zeta) \\
&\le \sum_{m=0}^{\infty} \lambda_m^2 \, (m+1)^2 \, \frac{N(q,m)}{\sigma_{q-1}} \, \|P_{m+1} u\|^2 \, \sigma_{q-1} \quad \text{(Lemma 8)} \\
&\le \sum_{m=0}^{\infty} K_q (m+1)^q \, \frac{\lambda_m^2}{\lambda_{m+1}^2} \cdot \lambda_{m+1}^2 \, \|P_{m+1} u\|^2 .
\end{aligned} \tag{24}
$$

(i) If $\dfrac{\lambda_m^2}{\lambda_{m+1}^2} = O(m^{-q})$, then the above inequality (24) shows that $(\partial_k u)\,(\zeta) \in D(\Lambda)$, and $\|\partial_k u\|_\Lambda \le c\|u\|_\Lambda$ for some constant c independent on u. By Theorem 2 we then conclude that $(\partial_k u)\,(x) \in HA^q(\mu)$ and $\|\partial_k u\|_\mu \le c'\|u\|_\mu$ with c' another constant. Thus the differentiation operator ∂_k is continuous on $HA^q(\mu)$.

(ii) If $\dfrac{\lambda_m^2}{\lambda_{m+1}^2} = o(m^{-q})$, then the inequality (24) implies that for a given bounded set $\mathbf{B}$ in $D(\Lambda)$ the set $\{\partial_k u \mid u \in \mathbf{B}\}$ is bounded in $D(\Lambda)$ and for any $\varepsilon > 0$ there exists M such that $\sum_{m \ge M}^{\infty} \lambda_m^2 \, \|P_m \partial_k u\|^2 < \varepsilon$ for all $u \in \mathbf{B}$. Thus $\{\partial_k u \mid u \in \mathbf{B}\}$ is a relatively compact set in $D(\Lambda)$. Therefore ∂_k is a compact operator on $D(\Lambda)$, so is it on $HA^q(\mu)$ by Theorem 2. $\square$

4. Examples

Let $\mu(r) = e^{-ar^b}$ $(a,b > 0)$. Then the differentiation operators $\partial_k (1 \le k \le q)$ are compact on $HA^q(\mu)$ for $b < \frac{2}{q}$ and continuous on $HA^q(\mu)$ for $b = \frac{2}{q}$. This assertion follows from Theorem 2 and Theorem 7 above and the following two lemmas.

Lemma 9.

$$u_m^2 = \int_0^\infty e^{-ar^b} r^{2m+q-1}\, dr \simeq (2\pi)^{1/2} b^{-1/2} (eab)^{-\frac{2m+q}{b}} (2m+q)^{\frac{2m+q}{b}-\frac{1}{2}}. \tag{25}$$

Proof. Substituting s for ar^b we have

$$\begin{aligned} u_m^2 &= \int_0^\infty e^{-ar^b} r^{2m+q-1}\, dr \\ &= \int_0^\infty e^{-s} a^{-\frac{2m+q}{b}} b^{-1} s^{\frac{2m+q}{b}-1}\, ds \\ &= a^{-\frac{2m+q}{b}} b^{-1} \Gamma\left(\frac{2m+q}{b}\right). \end{aligned}$$

Then an invocation to Stirrings's formula

$$\Gamma(\theta) \simeq (2\pi)^{1/2} \theta^{\theta-1/2} e^{-\theta} \quad (\theta \to \infty)$$

yields relation (25). □

Lemma 10. Set $\lambda_m^2 = \left(\frac{2m}{eab}\right)^{\frac{2}{b}m} m^{\frac{q}{b}-\frac{1}{2}}$ $(m \in \mathbb{N}_0)$. Then $\frac{R^{2m}}{\lambda_m^2} = O(1)$ for all $R > 0$ and $\{\lambda_m\} \sim \{\mu_m\}$.

Moreover

$$\frac{\lambda_m^2}{\lambda_{m+1}^2} \sim m^{-\frac{2}{b}}.$$

□

Further examples of Hilbert spaces of type $HA^q(\mu)$ wherein differentiation operators are continuous or compact are given in [3].

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Acknowledgement:

The author would like to express his gratitude to Prof. De Graaf for his inspiring advices.

Key Words: Differential operators; spherical harmonics; Hilbert spaces of harmonic functions; reproducing kernels; Stirring's formula.

1980 AMS Classification: 46E20, 47F05, 47B38, 31B05, 33A45, 33A15