# On two-graphs, and Shult's characterization of symplectic and orthogonal geometries over GF(2)

TECHNISCHE HOGESCHOOL EINDHOVEN
NEDERLAND
ONDERAFDELING DER WISKUNDE

TECHNOLOGICAL UNIVERSITY EINDHOVEN
THE NETHERLANDS
DEPARTMENT OF MATHEMATICS



# On two-graphs, and Shult's characterization of symplectic and orthogonal geometries over GF(2)

by

J.J. Seidel

T.H.-Report 73-WSK-02

Januari 1973

# On two-graphs, and Shult's characterization of symplectic and orthogonal geometries over GF(2)

by

J.J. Seidel

## 1. Introduction

A graph satisfies the triangle property whenever, for each pair of adjacent vertices u and v, there exists a vertex f(u,v) adjacent to u and to v, such that every further vertex is adjacent to one or three of the vertices u, v, f(u,v). Shult [9] proved that the only regular graphs having the triangle property are the void graphs, the graphs obtained from the symplectic and the orthogonal geometries over GF(2), and the complete graphs. The present paper gives a different proof of this result, together with a slight generalization. If regularity is replaced by the condition that no vertex is adjacent to all other vertices, then the same class of graphs is obtained, except for the complete graphs. For further generalizations we refer to a forthcoming paper by Buekenhout and Shult [2].

The present proof is based on matrix methods. It uses the notion of regular two-graphs. This notion has been introduced by G. Higman, and was investigated by Taylor [10]. Regular two-graphs correspond to switching classes of graphs whose (1,-1)-adjacency matrix has 2 eigenvalues, cf. [7]. The relation between these notions and Shult's theorem is illustrated by the coincidence of Shult's first induction step, and the author's determination [6] of all graphs with the eigenvalues -3 and 2s + 1.

The present paper is self-contained. Section 2 collects the definitions and some theorems on regular two-graphs and on switching of graphs, including another result by Shult [8]. In section 3, symplectic and orthogonal geometries over GF(2) and their two-graphs are reviewed, on the basis of [1] and [10]. Section 4 consists of the proof of Shult's theorem. In the final section 5 this theorem is applied to a problem originating from Lie algebras of characteristic 2, proposed by Hamelink [4].

## 2. Regular two-graphs

Let $\Omega$ denote a finite set of n elements. Let $\Omega^{(i)}$ denote the set of all i-subsets of $\Omega$. An ordinary graph $(\Omega,E)$ consists of a vertex set $\Omega$ and an edge set $E \subset \Omega^{(2)}$.

Definition 2.1. A two-graph $(\Omega,\Delta)$ is a pair of a vertex set $\Omega$ and a triple set $\Delta \subset \Omega^{(3)}$, such that each 4-subset of $\Omega$ contains an even number of triples of $\Delta$.

For any $\omega \in \Omega$, the triple set $\Delta$ of any two-graph $(\Omega,\Delta)$ is determined by its triples containing $\omega$. Indeed, $\{\omega_1,\omega_2,\omega_3\} \in \Delta$ whenever an odd number of the other 3-subsets of $\{\omega,\omega_1,\omega_2,\omega_3\}$ belongs to $\Delta$.

Given any graph $(\Omega,E)$, let $\Delta$ be the set of the triples from $\Omega$ which carry an odd number of edges of E. Then $(\Omega,\Delta)$ is a 2-graph. Indeed, it is easily checked that for any graph on 4 vertices the number of its subgraphs on 3 vertices having an odd number of edges is even.

Definition 2.2. The switching class of graphs belonging to the two-graph $(\Omega,\Delta)$ is the set of all graphs with vertex set $\Omega$, which have $\Delta$ as the set of triples of vertices carrying an odd number of edges.

Given any two-graph $(\Omega,\Delta)$, its switching class of graphs is obtained as follows. Select any $\omega \in \Omega$, and partition $\Omega \setminus \{\omega\}$ into any 2 disjoint sets $\Omega_1$ and $\Omega_2$. Let E consist of the following pairs:

$\{\omega,\omega_1\}$, for all $\omega_1 \in \Omega_1$ ;
$\{\omega_1,\omega_1'\}$, for all $\omega_1,\omega_1' \in \Omega_1$, with $\{\omega,\omega_1,\omega_1'\} \in \Delta$ ;
$\{\omega_2,\omega_2'\}$, for all $\omega_2,\omega_2' \in \Omega_2$, with $\{\omega,\omega_2,\omega_2'\} \in \Delta$ ;
$\{\omega_1,\omega_2\}$, for all $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$, with $\{\omega,\omega_1,\omega_2\} \notin \Delta$ .

Then $(\Omega,E)$ belongs to the switching class of $(\Omega,\Delta)$. Conversely, every graph of the switching class of $(\Omega,\Delta)$ is obtained in this way.

Definition 2.3. The derived graph, with respect to $\omega \in \Omega$, of the two-graph $(\Omega,\Delta)$ is the graph on $\Omega \setminus \{\omega\}$ in which 2 vertices are adjacent whenever, together with $\omega$, they form a triple of $\Delta$.

The switching class of $(\Omega,\Delta)$ contains each of its derived graphs extended by the isolated vertex $\omega$. Indeed, take $\Omega_1 = \emptyset$ in the above construction.

With respect to any labeling of $\Omega$ any graph $(\Omega,E)$ is described by its $(-1,1)$-adjacency matrix A as follows. The elements of A are $a_{ii} = 0$ for all $i \in \Omega$, $a_{xy} = -1$ for adjacent $x,y \in \Omega$, and $a_{u,v} = 1$ for non-adjacent $u,v \in \Omega$. Thus, A is a symmetric matrix with zero diagonal of the order n. If $(\Omega,E)$ has the adjacency matrix A, then any graph $(\Omega,E')$ in its switching class has the adjacency matrix

$$A' = DAD ,$$

for some diagonal matrix D of order n with diagonal elements $\pm 1$. Obviously, $A'$ and A have the same spectrum. We shall say that $(\Omega,E')$ is obtained from $(\Omega,E)$ by switching with respect to the vertices of $\Omega$ which correspond to the elements $-1$ of D.

Definition 2.4. A two-graph $(\Omega,\Delta)$ is regular with the parameter k, whenever each pair from $\Omega$ is contained in a constant number k of triples of $\Delta$.

Theorem 2.5. A two-graph is regular if and only if the adjacency matrix of any graph in its switching class has 2 eigenvalues.

Proof. Let $(\Omega,E)$, with adjacency matrix A, be in the switching class of $(\Omega,\Delta)$. For any adjacent $x,y \in \Omega$, let $p(x,y)$ denote the number of the vertices which are adjacent to x and non-adjacent to y. For any non-adjacent $u,v \in \Omega$, let $q(u,v)$ denote the number of the vertices which are adjacent to u and non-adjacent to v. The regularity condition of $(\Omega,\Delta)$ says that

$$k = q(u,v) + q(v,u) = n - 2 - p(x,y) - p(y,x) .$$

On the other hand, the elements of the matrix $(A - \rho_1 I)(A - \rho_2 I)$, for any real $\rho_1$ and $\rho_2$, with $\rho_1 \geq \rho_2$, are

$$\begin{array}{ll} \rho_1\rho_2 + n - 1 & \text{, on the diagonal ;} \\ \rho_1 + \rho_2 + n - 2 - 2(p(x,y) + p(y,x)) & \text{, for } a_{xy} = -1 \text{ ;} \\ -(\rho_1 + \rho_2) + n - 2 - 2(q(u,v) + q(v,u)) & \text{, for } a_{uv} = 1 \text{ .} \end{array}$$

We now relate $\rho_1$ and $\rho_2$ to k and n by

$$n = 1 - \rho_1\rho_2 \ , \quad k = -\tfrac{1}{2}(\rho_1 + 1)(\rho_2 + 1) \ .$$

It follows that $(\Omega,\Delta)$ is a regular two-graph with the parameter k, if and only if

$$(A - \rho_1 I)(A - \rho_2 I) = 0 \ .$$

We close this section with a criterion for 2-transitivity of two-graphs, due to Shult [8]. For any graph $(\Omega,E)$ with adjacency matrix A, let

$$\Omega = \{x\} \cup \Omega_x \cup \Omega'_x \ ; \quad A = \begin{bmatrix} 0 & -j^T & j^T \\ -j & B & C \\ j & C^T & D \end{bmatrix}$$

denote the partition of $(\Omega,E)$ into $x \in \Omega$, the subgraph on the set $\Omega_x$ of the vertices adjacent to x, and the subgraph on the set $\Omega'_x$ of the vertices non-adjacent to x.

Theorem 2.6. Let $(\Omega \cup \{\omega\},\Delta)$ be any two-graph, and let $(\Omega,E)$ be its derived graph with respect to $\omega$. Suppose $(\Omega,E)$ admits a transitive automorphism group. Suppose that, for any $x \in \Omega$, there exist automorphisms $\sigma$ of the subgraph on $\Omega_x$, and $\tau$ of the subgraph on $\Omega'_x$, such that

$$\begin{bmatrix} B & C \\ C^T & D \end{bmatrix} \rightarrow \begin{bmatrix} B & -C \\ -C^T & D \end{bmatrix} \ .$$

Then $(\Omega \cup \{\omega\},\Delta)$ is a regular two-graph admitting a doubly transitive automorphism group.

Proof. Extend $(\Omega,E)$ by the isolated vertex $\omega$, and switch the resulting graph with respect to the vertices of $\Omega_x$. The graph thus obtained has the same adjacency matrix as $(\Omega \cup \{\omega\},E)$ if $\omega$ and x are interchanged, and $\Omega_x$ and $\Omega'_x$ are taken in the order $\sigma^{-1}\Omega_x$ and $\tau^{-1}\Omega'_x$, respectively. Therefore, $(\Omega \cup \{\omega\},\Delta)$ admits an automorphism which interchanges $\omega$ and x. From the transitivity of $(\Omega,E)$ it follows that $(\Omega \cup \{\omega\},\Delta)$ admits an automorphism which fixes $\omega$ and maps any $y \in \Omega$ onto any $z \in \Omega$. This implies that the two-graph admits a 2-transitive automorphism group, and hence is regular.

Remark 2.7. The notion of two-graph may be defined in terms of 2-dimensional cocycles, cf. [10], [11]. D.G. Higman [11] showed that theorem 2.6 has an extension to cocycles of arbitrary dimension $\geq 2$.

Remark 2.8. Any two-graph may be interpreted as a dependent set of equiangular lines in Euclidean space of some finite dimension, and conversely, cf. [12].

## 3. Symplectic and orthogonal geometries over GF(2)

Let $V = V(2m,2)$ denote the vector space of dimension $2m$ over GF(2), the binary field. $V$ carries a symplectic geometry whenever it is provided with a non-degenerate, alternating, bilinear form, that is, a form $B: V \times V \to GF(2)$ such that, for all $x,y,z \in V$,

$$(\forall_{u \in V}\ B(u,x) = 0) \Rightarrow (x = 0)\ ;\quad B(x,x) = 0\ ;$$

$$B(x + y,\ z) = B(x,z) + B(y,z)\ .$$

Calling a plane in $V(2m,2)$ hyperbolic whenever it is spanned by vectors $u$ and $v$ with $B(u,v) = 1$, we observe thay $V(2m,2)$ is a direct product of hyperbolic planes:

$$V = H \perp H \perp \ldots \perp H\ .$$

The linear transformations of $V(2m,2)$, leaving $B$ invariant, constitute the symplectic group. This group acts transitively on the vectors of $V \setminus \{0\}$.

Definition 3.1. The symplectic two-graph $\Sigma = \Sigma(2m,2)$ consists of the set $V$ of the vectors of $V(2m,2)$, and the set $\Delta$ of the triples of distinct $u,v,w \in V$ satisfying

$$B(u,v) + B(v,w) + B(w,u) = 0\ .$$

Theorem 3.2. $\Sigma(2m,2)$ is a regular two-graph, with the parameters

$$n = 2^{2m}\ ,\quad k = 2^{2m-1} - 2\ ,\quad \rho_1 = 1 + 2^m\ ,\quad \rho_2 = 1 - 2^m\ .$$

Proof. For $x_1,x_2,x_3,x_4 \in V$ and $x_{123} = B(x_1,x_2) + B(x_2,x_3) + B(x_3,x_1)$, etc, we have

$$x_{123} + x_{124} + x_{134} + x_{234} = 0\ .$$

Hence $x_{hij} = 1$ for an even number of triples, and $(V,\Delta)$ is a two-graph. Let $G$ denote the symplectic group extended by the translations of $V(2m,2)$. Then $G$ acts 2-transitively on $(V,\Delta)$, hence $(V,\Delta)$ is regular. For any $u \neq 0$ the set $\{x \in V \mid B(u,x) = 0\}$ contains $2^{2m-1}$ elements. This implies $k = 2^{2m-1} - 2$. The eigenvalues $\rho_1$ and $\rho_2$ follow from theorem 2.5.

Definition 3.3. The symplectic graph $S(2m,2)$ is the derived graph of $\Sigma(2m,2)$ with respect to 0. It has the vertex set $V \setminus \{0\}$, and any u and v are adjacent whenever $B(u,v) = 0$.

Theorem 3.4. The order and the eigenvalues of $S(2m,2)$ are

$$n = 2^{2m} - 1 \ , \quad \rho_0 = 2 \ , \quad \rho_1 = 1 + 2^m \ , \quad \rho_2 = 1 - 2^m \ .$$

Proof. The matrix of the values of the bilinear form, taken on the vectors of $V(2m,2)$,

$$B = [B(u,v)]_{u \in V, v \in V}$$

is related to the adjacency matrix A of $S(2m,2)$ by

$$2B - J + I = \begin{bmatrix} 0 & -j^T \\ -j & A \end{bmatrix} .$$

Since this matrix has the eigenvalues $1 \pm 2^m$, it follows that

$$(A - I - 2^m I)(A - I + 2^m I) = -J \ , \quad Aj = 2j \ ,$$

which implies the theorem.

The vector space $V(2m,2)$ carries an orthogonal geometry, whenever it is provided with a quadratic form, that is, a form $Q : V \to GF(2)$ such that $Q(0) = 0$, and

$$Q(x + y) + Q(x) + Q(y)$$

is a non-degenerate, alternating, bilinear form.

Theorem 3.5. Essentially, there are 2 quadratic forms, viz.

$$Q^+(x) = \xi_1\xi_2 + \xi_3\xi_4 + \ldots + \xi_{2m-1}\xi_{2m} \ , \text{ with } 2^{2m+1} + 2^{m-1} \text{ zeros;}$$

$$Q^-(x) = \xi_1^2 + \xi_2^2 + \xi_1\xi_2 + \ldots + \xi_{2m-1}\xi_{2m}, \text{ with } 2^{2m-1} - 2^{m-1} \text{ zeros.}$$

Proof. Let $V(2m,2)$ carry an orthogonal geometry, and let $B$ be the associated bilinear form. Then, for all $u,v \in V$ we have

$$Q(u + v) + Q(u) + Q(v) = B(u,v) .$$

The hyperbolic planes of $V(2m,2)$ are of two types:

type D: $Q(u) = Q(v) = 0$, $Q(u + v) = 1$, with $Q(x) = \xi_1\xi_2$ ;

type E: $Q(u) = Q(v) = Q(u + v) = 1$, with $Q(x) = \xi_1^2 + \xi_2^2 + \xi_1\xi_2$ ;

for $x = \xi_1 u + \xi_2 v$. We may write either

$$V(2m,2) = D \perp D \perp \ldots \perp D, \text{ or } V(2m,2) = E \perp D \perp \ldots \perp D .$$

Indeed, let $V(4,2)$ be spanned by $u,v,w,t$, with $B(u,v) = B(w,t) = 1$, all other values of $B$ zero, and $Q(u) = Q(v) = Q(w) = Q(t) = 1$. Then $\{u+v+w,\ v + w\}$ and $\{u + w + t,\ u + t\}$ span planes of type D, and $V(4,2) = E \perp E = D \perp D$. This proves the first assertion. In order to count the number of zeros of $Q^+(x)$, and of $Q^-(x)$, we first observe that these numbers add up to $2^{2m}$, as a consequence of

$$Q^-(x + e) = 1 + Q^+(x) , \quad \text{for } e = (1,1,0,\ldots,0) ,$$

and for all $x \in V$. Now $D \perp D \perp \ldots \perp D$ consists of the vectors $x + y$, $x \in X$, $y \in Y$, where $X$ and $Y$ are m-dimensional subspaces of $V(2m,2)$ on which the form $Q^+$ vanishes. The number of vectors $x + y$ with $Q^+(x + y) = B(x,y) = 0$ equals

$$2^m + (2^m - 1)2^{m-1} ,$$

viz. $2^m$ for $x = 0$, and $2^{m-1}$ for each $x \neq 0$. This completes the proof.

Definition 3.6. The orthogonal two-graph $\Omega^\varepsilon(2m,2)$, $\varepsilon = +,-$, consists of the set $\Omega^\varepsilon := \{x \in V \mid Q^\varepsilon(x) = 0\}$, and the set of the triples of distinct $u,v,w \in \Omega^\varepsilon$ satisfying

$$B(u,v) + B(v,w) + B(w,u) = 0 .$$

Definition 3.7. The orthogonal graph $O^\varepsilon(2m,2)$, $\varepsilon = +,-$, is the derived graph of $\Omega^\varepsilon(2m,2)$ with respect to $0$. Is has the vertex set $\Omega^\varepsilon \setminus \{0\}$, and any $u$ and $v$ are adjacent whenever $B(u,v) = 0$.

Theorem 3.8. $\Omega^\varepsilon(2m,2)$ are regular two-graphs, with the parameters

$$n = 2^{m-1}(2^m + 1),\ \rho_1 = 1 + 2^{m-1},\ \rho_2 = 1 - 2^m, \text{ for } \Omega^+(2m,2);$$
$$n = 2^{m-1}(2^m - 1),\ \rho_1 = 1 + 2^m,\ \rho_2 = 1 - 2^{m-1}, \text{ for } \Omega^-(2m,2).$$

Proof. $\Omega^\varepsilon(2m,2)$ are sub-two-graphs of $\Sigma(2m,2)$. We now apply theorem 2.6. The graph $O^\varepsilon(2m,2)$ has a transitive automorphism group (in fact, the orthogonal group). For any $x \in \Omega^\varepsilon \setminus \{0\}$, let $\Omega_x$ and $\Omega'_x$ be as in theorem 2.6. The mappings $\sigma : y \to y + x$, $y \in \Omega_x$, and $\tau : z \to z$, $z \in \Omega'_x$ are automorphisms of the subgraphs on $\Omega_x$, and on $\Omega'_x$, respectively. The second condition of theorem 2.6 is satisfied, since

$$B(y + x, z) = B(y,z) + 1 .$$

It follows that $\Omega^\varepsilon(2m,2)$ admits a 2-transitive automorphism group (in fact, the symplectic group), and is regular. In order to determine k for $\Omega^+(2m,2)$ we take any $u \neq 0$, $Q^+(u) = 0$, and use the m-dimensional X and Y on which $Q^+$ vanishes. The number of vectors $x + y$ such that $x \in X$, $y \in Y$, $Q^+(x + y) = 0$, $B(u,x) = B(u,y)$ equals

$$2^{m-1} + (2^m - 2)2^{m-2} + 2^{m-1} ,$$

viz. for $x = 0$, for each $x \notin \{0,u\}$, and for $x = u$. This proves

$$k = 2^{2m-2} + 2^{m-1} - 2 ,$$

from which the eigenvalues of $\Omega^+(2m,2)$ follow. By taking complements in the hyperplane $B(u,x) = 0$ we find for $\Omega^-(2m,2)$

$$k = 2^{2m-2} - 2^{m-1} - 2 .$$

Theorem 3.9. The order and the eigenvalues of $O^\varepsilon(2m,2)$ are

$$n = 2^{2m-1} + 2^{m-1} - 1,\ \rho_0 = 2 - 2^{m-1},\ \rho_1 = 1 + 2^{m-1},\ \rho_2 = 1 - 2^m,$$

for $\varepsilon = +$;

$$n = 2^{2m-1} - 2^{m-1} - 1,\ \rho_0 = 2 + 2^{m-1},\ \rho_1 = 1 + 2^m,\ \rho_2 = 1 - 2^{m-1},$$

for $\varepsilon = -$.

Proof. As the proof of theorem 3.4.

Theorem 3.10. The switching class of the symplectic two-graph $\Sigma(2m,2)$ contains strongly regular graphs with the parameters

$$n = 2^{2m},\ \rho_1 = 1 + 2^m,\ \rho_0 = \rho_2 = 1 - 2^m\ ;$$

$$n = 2^{2m},\ \rho_0 = \rho_1 = 1 + 2^m,\ \rho_2 = 1 - 2^m\ .$$

Proof. Let A be the adjacency matrix of the symplectic graph $S(2m,2)$, extended by the all-adjacent vertex 0. We switch this graph with respect to the vertices of $V \setminus \Omega^{\varepsilon}$, so as to obtain a graph with the adjacency matrix A'. Partitioning according to

$$V = \{0\} \cup (\Omega^{\varepsilon} \setminus \{0\}) \cup (V \setminus \Omega^{\varepsilon})$$

we put A and A' in the following form:

$$A = \begin{bmatrix} 0 & -j^T & -j^T \\ -j & B & C \\ -j & C^T & D \end{bmatrix}, \quad A' = \begin{bmatrix} 0 & -j^T & j^T \\ -j & B & -C \\ j & -C^T & D \end{bmatrix}.$$

The adjacency matrix A' applies, since it has constant row sums

$$2^{2m-1} - \varepsilon 2^{m-1} - (2^{2m-1} + \varepsilon 2^{m-1} - 1) = 1 - \varepsilon 2^m\ .$$

Theorem 3.11. The subgraph of the symplectic graph $S(2m,2)$ on the set $V \setminus \Omega^{\varepsilon}$ is strongly regular with the parameters

$$n = 2^{2m-1} - 2^{m-1},\ \rho_1 = 1 + 2^m,\ \rho_0 = \rho_2 = 1 - 2^{m-1},\ \text{for } \varepsilon = +\ ;$$

$$n = 2^{2m-1} + 2^{m-1},\ \rho_0 = \rho_1 = 1 + 2^{m-1},\ \rho_2 = 1 - 2^m,\ \text{for } \varepsilon = -\ .$$

The graph belongs to a regular two-graph isomorphic to $\Omega^{-\varepsilon}(2m,2)$.

Proof. Comparing the matrices A and A' mentioned in theorem 3.10 we observe that the subgraph of $S(2m,2)$ on $V \setminus \Omega^{\varepsilon}$, which has the adjacency matrix D, is regular with $\rho_0 = 1 - \varepsilon 2^{m-1}$. In addition, $|\Omega^{-\varepsilon}| = |V \setminus \Omega^{\varepsilon}|$, and any tripel $\{u,v,w\} \subset V \setminus \Omega^{\varepsilon}$ has an odd number of edges if and only if

$$B(u,v) + B(v,w) + B(w,u) = 0 .$$

This implies the proof of the theorem.

Definition 3.12. A graph $(\Omega,E)$ satisfies the triangle property, whenever for each adjacent $u,v \in \Omega$ there exists a vertex $f(u,v) \in \Omega$, adjacent to u and to v, such that every further $x \in \Omega$ is adjacent to one or three of u, v, $f(u,v)$.

Theorem 3.13. The symplectic graph $S(2m,2)$ and the orthogonal graphs $O^{\varepsilon}(2m,2)$ satisfy the triangle property.

Proof. For any distinct vertices u,v with $B(u,v) = 0$ the vertex $u + v$ serves as $f(u,v)$. Indeed, $u + v$ is adjacent to u and to v, and any vertex $x \notin \{u,v,u + w\}$ satisfies

$$B(u,x) + B(v,x) + B(u + v, x) = 0 .$$

## 4. Characterization of the symplectic and orthogonal graphs

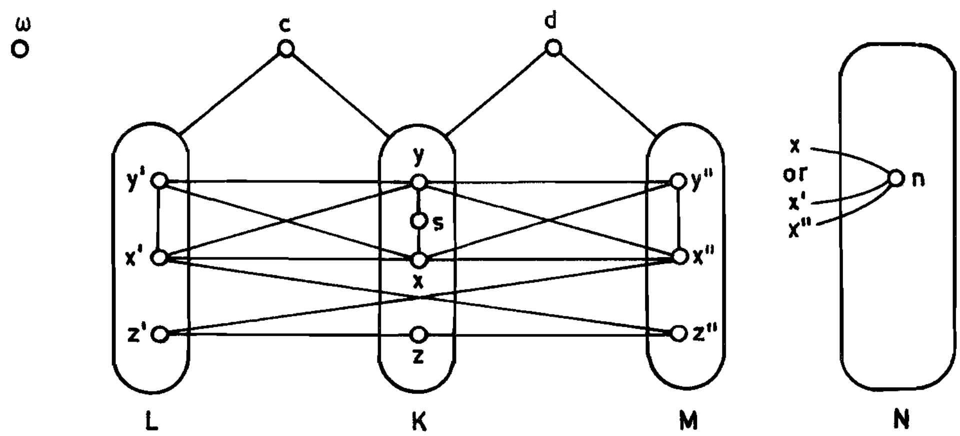
Shult's theorem states that, essentially, the symplectic and orthogonal graphs are characterized by the triangle property. This will be proved in theorem 4.15, after a series of lemma's.

Hypothesis 4.1. The graph $(\Omega,E)$ satisfies the triangle property 3.12. The graph $(\Omega,E)$ is not a void graph, and no vertex is adjacent to all other vertices.

Lemma 4.2. Given any adjacent $u,v \in \Omega$, there is exactly one $f(u,v) \in \Omega$, adjacent to u and to v, such that every further $x \in \Omega$ is adjacent to one or three of u, v, f(u,v).

Proof. Suppose that two distinct f(u,v) and f'(u,v) apply. Then the triangle property implies that f and f' are adjacent. Let f" be a vertex adjacent to f and to f' as required by the triangle property. We shall show that f" is adjacent to all other vertices, contrary to hypothesis 4.1. Indeed, arrange the vertices $\notin \{u,v,f\}$ into the sets P (adjacent to u, to v, to f), Q (only adjacent to u), R (only adjacent to v), S (only adjacent to f). Because of the triangle property, the vertices of each of these sets are adjacent to f", and so do the three remaining vertices. This conflicts to hypothesis 4.1, so the lemma is proved.

The following figure F is used frequently in the sequel. We shall apply the triangle property without further mentioning. The integer k will denote twice the cardinality of the set K.



Let any $c,d \in \Omega$ be non-adjacent. Let K, L, M, N be the sets of the vertices which are adjacent to c and to d, adjacent to c and not to d, adjacent to d and not to c, non-adjacent to c and to d, respectively. Then $K \neq \emptyset$ since $(\Omega,E)$ is not a void graph. For any $x \in K$ there are unique $x' := f(x,c) \in L$ and $x'' := f(x,d) \in M$, which are adjacent to x but mutually non-adjacent. For any adjacent $x,y \in K$ we have adjacent $x',y' \in L$, and $x'',y'' \in M$. For any non-adjacent $x,z \in K$ we have non-adjacent $x',z' \in L$, and $x'',z'' \in M$. Furthermore, we have adjacent $x,y'$, and $x',z''$, and non-adjacent $x,z'$, and $x',y''$. Finally, any $n \in N$ is adjacent to x, and non-adjacent to $x'$ and to $x''$, or conversely.

Lemma 4.3. Let c,x,y be any mutually adjacent vertices of $(\Omega,E)$, with $c \neq f(x,y)$. Then

$$f(f(c,x),y) = f(c,f(x,y)) .$$

Proof. Since $c \neq f(x,y)$, there is a vertex d adjacent to x and to y, and non-adjacent to c. Referring to figure F we call $f(c,x) = x'$, $f(x,y) = s$, $f(c,s) = s'$. We have to show that $f(x',y) = s'$. Now any vertex u is adjacent to one or three of x,y,s. It follows that u is adjacent to one or three of $x',y,s'$; this is easily checked for the vertices u of N, K, L, M, {c}, {d}. By lemma 4.2 the proof is completed.

Remark 4.4. The present lemma 4.3 will serve to show that addition (to be defined) is associative. Analogously, the following relation for any quadrangle c,x,d,z in $(\Omega,E)$ may be proved

$$f(f(c,x),f(d,z)) = f(f(c,z),f(d,x)) .$$

Lemma 4.5. $(\Omega,E)$ is a regular graph of valency k.

Proof. The subgraphs on K, on L, on M are isomorphic. Hence the valencies of any non-adjacent c and d are equal, viz. $k = 2|K|$. This implies that the valency of every vertex equals k, since no $x \in K$ is adjacent to all of N, L, M.

Lemma 4.6. The subgraph on K of $(\Omega,E)$ satisfies hypothesis 4.1; the subgraph on $K \cup N$ satisfies the triangle property.

Proof. Any $x,y \in K$ have $f(x,y) \in K$, and so do any $x,y \in N$. Any $x \in K$, $y \in N$ have $f(x,y) \in N$. No $x \in K$ is adjacent to all of $K \setminus \{x\}$, since otherwise $x$ would be adjacent to all of $L$ and of $M$, contrary to lemma 4.5. We shall see later that there may exist $x \in N$ adjacent to all of $(K \cup N) \setminus \{x\}$.

Lemma 4.7. The graph $(\Omega \cup \{\omega\},E)$, where $\omega$ is an additional isolated vertex, belongs to a regular two-graph $(\Omega \cup \{\omega\},\Delta)$ with the parameter $k$.

Proof. Let $\Delta$ be the set of the triples of $\Omega \cup \{\omega\}$ which carry an odd number of edges from $(\Omega \cup \{\omega\},E)$. Since $(\Omega,E)$ is regular, $\omega$ and any $c \in \Omega$ are on $k$ triples of $\Delta$. Any non-adjacent $c,d \in \Omega$ are on $|K| + |M| = k$ triples of $\Delta$. Referring to lemma 4.2 we observe that, as a consequence of the regularity of $(\Omega,E)$, the sets $Q$, $R$, $S$ have equal cardinality. Hence any adjacent $u,v \in \Omega$ are on $1 + 1 + (k - 2) = k$ triples of $\Delta$, corresponding to $\omega$, $f(u,v)$, and the vertices of $P \cup S$.

Lemma 4.8. The regular two-graph $(\Omega \cup \{\omega\},\Delta)$ contains a graph whose adjacency matrix $A$, when written on the sets $\{\omega\} \cup K$, $\{c\} \cup L$, $\{d\} \cup M$, $N$, takes the following form:

$$A - I = \begin{bmatrix} B & -B & -B & C \\ -B & B & -B & C \\ -B & -B & B & C \\ C^T & C^T & C^T & D \end{bmatrix} .$$

Proof. Referring to figure F, we switch the graph $(\Omega \cup \{\omega\},E)$ with respect to the vertices of $K$. The graph thus obtained has three isomorphic subgraphs on $\{\omega\} \cup K$, on $\{c\} \cup L$, on $\{d\} \cup M$. The mutual adjacencies of these subgraphs are complementary to the adjacencies within each subgraph. Any $n \in N$ is adjacent to all or none of $x \in K$, $x' \in L$, $x'' \in M$. Thus, we arrive at the adjacency matrix of the lemma.

Lemma 4.9. For the eigenvalues $\lambda$, $\mu$, and the order $n$ of $A - I$ one of the following holds, for some integer $m \geq 1$:

case I : $\lambda = 2^{m-1}$, $\mu = -2^m$, $n = 2^{2m-1} + 2^{m-1}$ ;

case II : $\lambda = 2^m$, $\mu = -2^m$, $n = 2^{2m}$ ;

case III: $\lambda = 2^{m+1}$, $\mu = -2^m$, $n = 2^{2m+1} - 2^m$ .

Proof. From theorem 2.5 and lemma's 4.5, 4.7 we know that the eigenvalues of $A - I$ are even integers, $\lambda \geq 0$ and $\mu$, say. Hypothesis 4.1 implies $\lambda \neq 0$, since $(\Omega,E)$ is not the complete graph. From tr $A = 0$ we have $\mu \leq -2$, and $\mu = -2$ if and only if $(\Omega,E)$ is the void graph. Substitution in

$$(A - I - \lambda I)(A - I - \mu I) = 0$$

of the matrix $A - I$ of lemma 4.8 yields

$$3B^2 - (\lambda + \mu)B + \lambda\mu I + CC^T = 0,\ -B^2 + (\lambda + \mu)B + CC^T = 0 ,$$

$$D^2 - (\lambda + \mu)D + \lambda\mu I + 3C^TC = 0,\ BC + (\lambda + \mu)C = CD .$$

It follows that

$$(2B - \lambda I)(2B - \mu I) = 0,\ 4CC^T + 2(\lambda + \mu)B + \lambda\mu I = 0 .$$

Hence the eigenvalues of $B$ are $\{\frac{1}{2}\lambda, \frac{1}{2}\mu\}$, and those of $CC^T$ are

$$\{-\tfrac{1}{4}\lambda(\lambda + 2\mu),\ -\tfrac{1}{4}\mu(2\lambda + \mu)\} .$$

This implies:

$$\tfrac{1}{2}\lambda \text{ even};\ \tfrac{1}{2}\mu \text{ even};\ 0 < \tfrac{1}{2}\lambda \leq -\mu \leq 2\lambda \leq -4\mu .$$

Now, by use of lemma 4.6, we repeat the process for the regular two-graph on $K \cup \{\omega\}$, with the matrix $B$. Then the eigenvalues are halved again. By iterating $m - 1$ steps, say, we finally end up with the void graph, that is, with $\mu = -2$ and $0 < \frac{1}{2}\lambda \leq 2 \leq 2\lambda \leq 8$, whence $\lambda = 1,2,3,4$. However, the case $\lambda = 3$, $\mu = -2$ cannot occur, since no matrix with the eigenvalues $\lambda = 6$, $\mu = -4$ exists. Indeed, if it would exist then $n = 1 - (1 + \lambda)(1 + \mu) = 22$, and the integer multiplicities $\mu_1$, $\mu_2$ would satisfy

$$\mu_1 + \mu_2 = 22 ,\quad 6\mu_1 - 4\mu_2 = -22 ,$$

which is impossible. So we are left with ultimate matrices of the following three types:

case I : $\lambda = 1$, $\mu = -2$, $n = 3$ ;

case II : $\lambda = 2$, $\mu = -2$, $n = 4$ ;

case III : $\lambda = 4$, $\mu = -2$, $n = 6$ ,

that is, with the void graphs on 2, 3, 5 vertices, respectively, extended by the isolated vertex $\omega$. Going up $m - 1$ steps, we arrive at the statement of the lemma.

<u>Lemma 4.10</u>. For $m \geq 2$, in the three cases we have the following eigenvalues:

| matrix | case I | case II | case III |
|---|---|---|---|
| $A - I$ | $2^{m-1}, -2^m$ | $2^m, -2^m$ | $2^{m+1}, -2^m$ |
| $B$ | $2^{m-2}, -2^{m-1}$ | $2^{m-1}, -2^{m-1}$ | $2^m, -2^{m-1}$ |
| $D$ | $2^{m-1}, -2^{m-2}$ | $2^{m-1}, -2^{m-1}$ | $2^{m-1}, -2^m$ |
| $\begin{bmatrix} B & C \\ C^T & D \end{bmatrix}$ | $2^{m-1}, -2^{m-1}$ | $0, 2^m, -2^m$ | $2^m, -2^m$ |

All matrices belong to regular two-graphs, except for the last matrix in case II.

<u>Proof</u>. We refer to the proof of lemma 4.9 for the equations for B, C, D. In case II these imply

$$CC^T = 2^{2m-2}I = C^TC,\ D^2 = B^2 = 2^{2m-2}I,\ \begin{bmatrix} C^T & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} B & C \\ C^T & D \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} =$$

$$= \begin{bmatrix} D & D \\ D & D \end{bmatrix} 2^{2m-2} ,$$

hence $\begin{bmatrix} B & C \\ C^T & D \end{bmatrix}$ has the eigenvalues $0$, $2^m$, $-2^m$, with the multiplicities $2^{2m-2}$, $2^{2m-3} - 2^{m-2}$, $2^{2m-3} + 2^{m-2}$, respectively. In case I, from the eigenvalues of B those of $CC^T$, of $C^TC$, and of D are calculated. Then we observe that

$$B^2 + CC^T = 2^{2m-2}I,\ D^2 + C^TC = 2^{2m-2}I,\ BC + CD = 0$$

hold, by eliminating B, D, $CC^TC$, respectively. This yields the eigenvalues of $\begin{bmatrix} B & C \\ C^T & D \end{bmatrix}$. Case III is treated likewise.

Definition 4.11. A symplectic set $S_m$ in $(\Omega,E)$ is a set of 2m vertices $\{c_1,\ldots,c_m, d_1,\ldots,d_m\}$ all of which are adjacent except for the pairs $\{c_i,d_i\}$, $i = 1,\ldots,m$. A maximal clique is a clique (= complete subgraph) whose vertices are not all adjacent to any further vertex of $\Omega$.

Lemma 4.12. $(\Omega,E)$ contains symplectic sets $S_m$, and maximal cliques of length $2^m - 1$. Such a maximal clique, together with the isolated vertex $\omega$, may be viewed as a vector space $V(m,2)$. Any vertex of $(\Omega,E)$ is adjacent to $2^{m-1} - 1$, and non-adjacent to $2^{m-1}$ vertices of any maximal clique not containing that vertex.

Proof. Take any non-adjacent $c_1,d_1 \in \Omega$; consider its set K according to figure F; take any non-adjacent $c_2,d_2 \in K$; and iterate this process. At the final step we have for $c_m$ and $d_m$ a choice between 2, 3, 5 non-adjacent vertices, in the cases I, II, III, respectively. This follows from the end of the proof of lemma 4.9. The resulting set $\{c_1,\ldots,c_m,d_1,\ldots,d_m\}$ is symplectic. The set $\{c_1,\ldots,c_m\}$ forms a clique. Define $c_i + c_j$ to be the unique vertex $f(c_i,c_j)$, for $i \neq j = 1,\ldots,m$. It is adjacent to any $c_k$, and it does not coincide with $c_k$ because of the corresponding $d_k$. For any $k \neq i,j$, define $(c_i + c_j) + c_k$ to be the unique $f((c_i + c_j),c_k)$. The associative law holds because of lemma 4.3. The vertex $c_i + c_j + c_k$ does not coincide with a vertex already obtained, because of the corresponding vertex from the set $\{d_1,\ldots,d_m\}$. Thus proceeding, we arrive at $2^m - 1$ distinct vertices $\Sigma_{i=1}^m \gamma_i c_i$, with $\gamma_i \in GF(2)$ not all zero. These constitute a projective space $PG(m - 1, 2)$ with the lines

$$\{\Sigma_{i=1}^m \gamma_i c_i,\ \Sigma_{i=1}^m \gamma_i' c_i,\ \Sigma_{i=1}^m (\gamma_i + \gamma_i')c_i\}\ .$$

Indeed, lemma 4.3 expresses the axiom of Pasch, cf. [3], p. 24. Adjoining the isolated vertex $\omega$ as the origin, we arrive at the vector space $V(m,2)$. Any $d_1$, non-adjacent to $c_1$, is adjacent to the vertices $\Sigma_{i=2}^{m} \gamma_i c_i$ and non-adjacent to the vertices $c_1 + \Sigma_{i=2}^{m} \gamma_i c_i$.

<u>Lemma 4.13</u>. In case II the graph $(\Omega,E)$ is the symplectic graph $S(2m,2)$.

<u>Proof</u>. Let $S_m = \{c_1,\ldots,c_m,d_1,\ldots,d_m\}$ be a symplectic set, constructed as in lemma 4.12. With respect to the non-adjacent $c_1$ and $d_1$ we define the set K, L, M, N as in figure F. From lemma's 4.6 and 4.10 we know that the subgraph on $K \cup N$ satisfies the triangle property, but fails to be strongly regular. Hence there exists a vertex $e_1$ belonging to N (not to K), which is adjacent to all other vertices of $K \cup N$. There is just one such $e_1$, since it is obtained from $S_m$ by reversing the construction of lemma 4.12 as follows: starting with $c_m$ and $d_m$, going up to $c_2$ and $d_2$, and leaving the pairwise non-adjacent $c_1,d_1,e_1$. As a consequence, to any non-adjacent $c_1,d_1 \in \Omega$ there exists a unique $e_1$ such that every vertex of $\Omega \setminus \{c_1,d_1,e_1\}$ is adjacent to one or three of $\{c_1,d_1,e_1\}$. Now let

$$C = \{\Sigma_{i=1}^{m} \gamma_i c_i \mid \gamma_i \in GF(2)\},\ D = \{\Sigma_{i=1}^{m} \delta_i d_i \mid \delta_i \in GF(2)\}$$

be the maximal cliques obtained from the vertices of the symplectic set $S_m$. Considering each of these maximal cliques as a vertex space $V(m,2) \setminus \{0\}$, we shall prove that $\Omega \cup \{\omega\}$ is the direct product $V \times V$. Any $c \in C$ and $d \in D$ determine a unique third vertex: for non-adjacent c and d this is the vertex e referred to above; for adjacent c and d this is the vertex $f(c,d)$ according to the triangle property. In either case, any further vertex of $\Omega$ is adjacent to one or three of c, d, and its third vertex. We claim that each $g \in \Omega \setminus (C \cup D)$ is the third vertex of a unique pair $c \in C$, $d \in D$. Indeed, suppose g is the third vertex of $c' \in C$, $d' \in D$ and of $c'' \in C$, $d'' \in D$. Any $x \in C$ different from $c'$ and $c''$ is adjacent to $c'$ and to $c''$. Therefore, x is adjacent to 0 or 2 of $\{g,d'\}$, of $\{g,d''\}$, whence of $\{d',d''\}$. It follows that each of the $2^m - 3$ vertices $x \in C$ is adjacent to $f(d',d'')$. However, this is in contradiction to lemma 4.12. This proves our claim. It follows that each of the $2^{2m}$ vertices of $\Omega \cup \{\omega\}$ is a unique element of the direct product $V \times V$, hence is a vector of the vector space $V(2m,2)$. For any two distinct vertices their unique third vertex acts as their sum. The form $B(x,y)$ is defined by its values 1 for any distinct non-adjacent $x,y \in \Omega$, and 0 otherwise. Bilinearity follows from

$$B(x,y+z) + B(x,y) + B(x,z) = 0 .$$

Thus, $V(2m,2)$ carries a symplectic geometry, and $(\Omega,E)$ is the symplectic graph.

Lemma 4.14. The graph $(\Omega,E)$ is the orthogonal graph $O^+(2m,2)$ in case I, and the orthogonal graph $O^-(2m+2, 2)$ in case III.

Proof. Referring to lemma 4.8 we take

$$A - I = \begin{bmatrix} B & -B & -B & C \\ -B & B & -B & C \\ -B & -B & B & C \\ C^T & C^T & C^T & D \end{bmatrix}$$

as the matrix, of order $2^{2m-1} + 2^{m-1}$, belonging to a graph of case I. By lemma's 4.6 and 4.10, the subgraph on $K \cup N$ satisfies the triangle property and contains no vertex adjacent to all other vertices. Lemma 4.13 implies that $\begin{bmatrix} B & C \\ C^T & D \end{bmatrix}$, of order $2^{2m-2}$, represents symplectic geometry $V(2m-2, 2)$ with the bilinear form $B(x,y)$, which equals 0 for adjacent, and 1 for non-adjacent vertices $x,y \in \Omega$. We shall first prove that the submatrix $B$, which is a case I matrix of the order $2^{2m-3} + 2^{m-2}$, represents $O^+(2m-2, 2)$ extended by an isolated vertex. To that end we consider the dissection of $V(2m-2, 2)$ into the disjoint parts $K \cup \{\omega\}$ and $N$. If $B(x,y) = 0$ then $x \in K$, $y \in K$ imply $x + y \in K$, and $x \in N$, $y \in N$ imply $x + y \in K$, whereas $x \in K$, $y \in N$ imply $x + y \in N$. Hence the characteristic function $Q$ of $N$ satisfies

$$Q(x+y) + Q(x) + Q(y) = B(x,y) .$$

We also have to verify this formula for $B(x,y) = 1$, so for mutually non-adjacent $x$, $y$, $x + y$. For $x \in K$, $y \in K$ we cannot have $x + y \in K$ since we are in case I. Now consider $x \in N$, $y \in N$, and suppose $x + y \in K$. Choose any $z \in K$ non-adjacent to $x + y$, then $z$ is adjacent to $x$ and non-adjacent to $y$, say, and $z + (x + y) \in N$. However, $x + z \in N$ is adjacent to $y \in N$, whence $y + (x + z) \in K$, a contradiction. Hence $x \in N$, $y \in N$ imply $x + y \in N$. In addition, $x \in K$, $y \in N$ imply $x + y \in K$. This completes the verification of the formula; $Q$ is a quadratic form, and the matrix $B$ corresponds to the orthogonal graph $O^+(2m - 2, 2)$. We still have to show that the matrix $A - I$ of order $2^{2m-1} + 2^{m-1}$ is imbeddable in a case II matrix of order $2^{2m}$, as the matrix $B$ was imbedded in $\begin{bmatrix} B & C \\ C^T & D \end{bmatrix}$. This becomes clear from the following block matrices, which are the same up to permutation of rows and columns:

$$\left[\begin{array}{cccc|cccc}
B & -B & -B & C & -C & C & C & -B \\
-B & B & -B & C & C & -C & C & -B \\
-B & -B & B & C & C & C & -C & -B \\
C^T & C^T & C^T & D & -D & -D & -D & -C^T \\
\hline
-C^T & C^T & C^T & -D & D & -D & -D & C^T \\
C^T & -C^T & C^T & -D & -D & D & -D & C^T \\
C^T & C^T & -C^T & -D & -D & -D & D & C^T \\
-B & -B & -B & -C & C & C & C & B
\end{array}\right],$$

$$\left[\begin{array}{cc|cc|cc|cc}
B & -C & -B & C & -B & C & -B & C \\
-C^T & D & C^T & -D & C^T & -D & C^T & -D \\
\hline
-B & C & B & -C & -B & C & -B & C \\
C^T & -D & -C^T & D & C^T & -D & C^T & -D \\
\hline
-B & C & -B & C & B & -C & -B & C \\
C^T & -D & C^T & -D & -C^T & D & C^T & -D \\
\hline
-B & C & -B & C & -B & C & B & -C \\
C^T & -D & C^T & -D & C^T & -D & -C^T & D
\end{array}\right].$$

These matrices, of order $2^{2m}$, represent symplectic geometry of dimension 2m. This is seen from the second matrix, when switched with respect to the diagonal blocks D. Now the imbedding of A - I is evident, and the lemma is proved for the case I. The case III is treated by use of (cf. the proof of theorem 3.5)

$$Q^{-}(x + e) = Q^{+}(x) + 1 , \quad \text{for } e = (1,1,0,\ldots,0) .$$

Summarizing 4.1 through 4.14 we have proved the following theorem.

Theorem 4.15. The only graphs satisfying the triangle property, in which no vertex is adjacent to all other vertices, are the void, the symplectic, and the orthogonal graphs.

## 5. A problem by Hamelink

Hypothesis 5.1.

i) $\Gamma$ is a spanning subset of symplectic space $V(2m,2)$, $m > 1$;

ii) $\Gamma$ is not the join of 2 disjoint non-void orthogonal subsets;

iii) $\forall_{u,v\in\Gamma}$ $((u + v \in \Gamma) \iff (B(u,v) = 1))$.

Hamelink (private communication, see also [4]) proposed the question which sets $\Gamma$ satisfy hypothesis 5.1. We shall answer this question by applying Shult's theorem 4.15. B. Fischer (private communication) has observed that this question is answered also by results of McLaughlin [5].

Definition 5.2. $(\Gamma,E)$ is a graph whose vertex set $\Gamma$ satisfies hypothesis 5.1, any 2 vertices being adjacent whenever they are orthogonal.

Theorem 5.3. The only graphs $(\Gamma,E)$ satisfying definition 5.2 are the complements of the triangular graphs on $2m + 1$, and on $2m$ symbols, and the graphs of theorem 3.11.

Proof. $(\Gamma,E)$ is not a complete graph, by ii). Let $a,b \in \Gamma$ be any non-adjacent vertices. From the set $\{a,b,a + b\} \subset \Gamma$ there are 1 or 3 vertices which are adjacent to any further $x \in \Gamma$. Therefore, $\Gamma \setminus \{a,b,a + b\}$ is partitioned into the following 4 disjoint subsets: $\Omega$ (the vertices $\neq a + b$ non-adjacent to $a$ and to $b$), $\Omega_a$ (the vertices $\neq b$ non-adjacent to $a$ and to $a + b$), $\Omega_b$ (the vertices $\neq a$ non-adjacent to $b$ and to $a + b$), $\Delta$ (the vertices adjacent to $a$, to $b$, to $a + b$). If $x$ runs through $\Omega$, then $x + a$ runs through $\Omega_a$, and $x + b$ runs through $\Omega_b$. The subgraphs on these 3 sets are isomorphic, since for all $x,y \in \Omega$ we have

$$B(x,y) = B(x + a, y + a) = B(x + b, y + b) .$$

No vertex $t \in \Delta$ is adjacent to all vertices of $\Omega$. Indeed, let $\Delta'$ be the set of all such vertices $t$. Any $z \in \Delta \setminus \Delta'$ is non-adjacent to some $x \in \Omega$, and to $x + z \in \Omega$, so $z$ is the sum of 2 elements of $\Omega$. Hence any $t \in \Delta'$ is adjacent to all elements of $\Gamma \setminus \Delta'$, and ii) implies $\Delta' = \emptyset$. No vertex $x \in \Omega$ is adjacent to all vertices of $\Omega \setminus \{x\}$. Indeed, if so, then x would be adjacent to all vertices of $\Delta$, and the vector $x + a + b$ would be orthogonal to all elements of $\Gamma$, contrary to i). These observations imply that $\Delta \neq \emptyset$, and that $V(2m,2)$ is spanned by $\Omega \cup \{a\} \cup \{b\}$.

The subgraph on $\Omega$ satisfies the triangle property. Indeed, for any adjacent $x,y \in \Omega$ the vertex $((x + a) + y) + b$ belongs to $\Omega$, and is adjacent to x and to y. By

$$B(x + y + a + b, u) + B(x,u) + B(y,u) = B(a + b, u)$$

any further $u \in \Omega$ is adjacent to 1 or 3 from $\{x,y,x + y + a + b\}$. Now we are in a position to apply theorem 4.15 on the subgraph on $\Omega$. Since $\Omega$ has to span a subspace of dimension $2m - 2$, we distinguish the following cases:

Case I. The subgraph on $\Omega$ is the void graph on $2m - 1$, or on $2m - 2$ vertices. Then the vertices of the subgraph on $\Delta$ correspond to the unordered pairs from $\Omega$, two vertices of $\Delta$ being adjacent whenever the corresponding pairs have no element of $\Omega$ in common. This graph is called the complement of the triangular graph on $|\Omega|$ symbols, cf. [6]. It follows that $(\Gamma,E)$ is the complement of the triangular graph on $2m + 1$, or on $2m$ symbols.

Case II. The subgraph on $\Omega$ is the symplectic graph $S(2m - 2, 2)$. This case is impossible. Indeed, the vector space $V(2m,2)$ contains the set

$$\Omega_{a+b} := \{x + a + b \mid x \in \Omega\} ,$$

which is disjoint to $\Gamma$. If $\Omega$ would carry the symplectic graph on $2^{2m-2} - 1$ vertices, then $\Omega$, $\Omega_a$, $\Omega_b$, $\Omega_{a+b}$, together with a, b, $a + b$, and 0, would exhaust $V(2m,2)$, leaving $\Delta = \emptyset$, which is impossible.

Case III. The subgraph on $\Omega$ is the orthogonal graph $O^{\varepsilon}(2m - 2, 2)$. Let $c,d \in \Omega$ be non-adjacent, then $c + d \in \Delta$. We partition $\Omega \setminus \{c,d\}$ into the 4 disjoint sets K, L, M, N, as we did in figure F. For any $k \in K$ the element

$$a + b + c + d + k = ((b + c) + k) + (a + d)$$

belongs to $\Delta$. For any $n \in N$ the elements

$$c + n,\ d + n,\ a + b + c + d + n$$

belong to $\Delta$. Indeed, any $n \in N$ is non-adjacent to some $x \in K$, and

$$a + b + c + d + n = (a + b + c + d + x) + (x + n) .$$

In view of lemma's 4.9 and 4.10 this amounts to a total of

$$1 + |K| + 3|N| = 1 + 2^{2m-5} + \varepsilon 2^{m-3} - 1 + 3(2^{2m-5} - \varepsilon 2^{m-3}) =$$

$$= 2^{2m-3} - \varepsilon 2^{m-2}$$

distinct elements of $\Delta$. These elements exhaust $\Delta$, since the disjoint sets

$$\Omega \cup \{a + b\},\ \Omega_a \cup \{b\},\ \Omega_b \cup \{a\},\ \Delta;\ \Delta_{a+b},\ \Delta_b,\ \Delta_a,\ \Omega_{a+b} \cup \{0\}$$

exhaust $V(2m,2)$, where $\Delta_p := \{z + p \mid z \in \Delta\}$. Hence $\Gamma$, and $V \setminus (\Gamma \cup \{0\})$, have

$$2^{2m-1} + \varepsilon 2^{m-1}, \text{ and } 2^{2m-1} - \varepsilon 2^{m-1} - 1 ,$$

elements, respectively. It follows that the subgraph on $\Omega \cup \Delta$ satisfies the triangle property, whence is the symplectic graph $S(2m - 2, 2)$. Indeed, consider

$$a + b + c + d + x + z = 0 \ ; \quad c,d \in \Omega .$$

If any $c + d \in \Delta$ is adjacent to any $x \in \Omega$, then $x \in K$ or $x \in N$, and $z \in \Delta$ serves as the third vertex. If any $c + d \in \Delta$ is adjacent to any $z \in \Delta$, then $x \in \Omega$ serves as the third vertex. This explains the structure of the subgraphs on $\Omega$, on $\Delta$, on $\Gamma$, and on $V \setminus (\Gamma \cup \{0\})$. Now the theorem is proved by reference to theorems 3.11 and 4.14.

References

1. E. Artin, Geometric algebra, Interscience, New York, 1957.

2. F. Buekenhout and E. Shult, On the foundations of polar geometry, to be published.

3. P. Dembowski, Finite geometries, Springer, Berlin, 1968.

4. R.C. Hamelink, Lie algebras of characteristic 2, Trans. Amer. Math. Soc. 144 (1969), 217-233.

5. J. McLaughlin, Some subgroups of $SL_n(F_2)$, Illinois J. Math. 13 (1969), 108-115.

6. J.J. Seidel, Strongly regular graphs of $L_2$-type and of triangular type, Proc. Kon. Ned. Akad. Wet. A. 70 (= Indag. Math. 29), (1967), 188-196.

7. J.J. Seidel, Strongly regular graphs with (-1,1,0) adjacency matrix having eigenvalue 3, Lin. Alg. and Appl. 1 (1968), 281-298.

8. E. Shult, The graph-extension theorem, Proc. Amer. Math. Soc. 33 (1972), 278-284.

9. E. Shult, Characterizations of certain classes of graphs, J. Combin. Theory (B) 13 (1972), 142-167.

10. D.E. Taylor, Some topics in the theory of finite groups, Ph.D. thesis, Univ. Oxford, 1971.

11. D.G. Higman, Remark on Shult's graph extension theorem, Proc. Confer. on finite groups, University of Florida, to be published.

12. P.W.H. Lemmens and J.J. Seidel, Equiangular lines, J. of Algebra 24 (1973), in press.