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Abstract

Two criteria are given such that an infinite matrix generates a C_0 semigroup in l^2.

§1. Introduction

Let $(a_{jk})_{j,k\in \mathbb{N}_0}$ be an infinite matrix of complex numbers with j,k the row index and column index respectively. We suppose that

$$\{a_{\cdot k}\} \in l^2, \ \forall k \in \mathbb{N}_0 ; \{a_{j\cdot}\} \in l^2, \ \forall j \in \mathbb{N}_0.$$

Corresponding to the matrix (a_{jk}) we can define an operator $A_{\max} \equiv \mathrm{Op}(a_{jk})$ in l^2 as follows:

$$A_{\max} u = \{\sum_{k=0}^{\infty} a_{jk} u_k\}_{j\in \mathbb{N}_0}$$

with

$$D(A_{\max}) = \{u = (u_k) \in l^2 \mid \sum_{k=0}^{\infty} a_{jk} u_k \text{ converges for all } j \in \mathbb{N}_0 \text{ and } \{\sum_{k=0}^{\infty} a_{jk} u_k\}_{j\in \mathbb{N}_0} \in l^2\}.$$

If instead of the matrix (a_{jk}) we use its complex conjugate $(\overline{a}_{kj})$, then we obtain another operator on l^2, denoted by $A^+_{\max}$. Under the above conditions it is easy to see that $l^2_c \subset D(A_{\max})$ and $l^2_c \subset D(A^+_{\max})$, where $l^2_c = \{u = (u_k) \mid u_k = 0 \text{ if } k \geq K \text{ depending on } u\}$, the subspace of finite sequences. We fix the notations $A_{\max} \restriction l^2_c = A_{\min}$ and $A^+_{\max} \restriction l^2_c = A^+_{\min}$. Thus both the operators $A_{\max}$ and $A^+_{\max}$ are densely defined. Actually they are also closed and obey the following relations:

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$$(A_{\max})^* \subset (A_{\min})^* = A^+_{\max}\ ; \quad (A^+_{\max})^* \subset (A^+_{\min})^* = A_{\max}.$$

The above facts are simple to prove and can be found in standard text books, e.g. in [6].

In the present note we give two criteria which determine classes of matrices (a_{jk}) whose corresponding maximal operators $A_{\max}$ or restrictions thereof are infinitesimal generators of C_0 semigroups on l^2. Nontrivial criteria of such kind seem missing in the literature. We shall obtain our criteria by applying the perturbation theorem of Rellich-Kato-Gustafson-Chernoff and the auxiliary operator trick of De Graaf in §2 and §3 respectively. These abstract results read:

Theorem I. (Rellich-Kato-Gustafson-Chernoff) Suppose that $A : D(A) \subset X \to X$ be an infinitesimal generator of a C_0 semigroup on a Banach space $(X, \|\cdot\|)$. Let P be another operator in X such that $D(P) \supset D(A)$ and

$$\|P\,u\| \le M\,\|u\| + \alpha\|A\,u\|\,,\ \ \forall u \in D(A) \tag{9}$$

where M and α are nonnegative constants with $\alpha < 1$. Then the operator $A + P$ as defined on $D(A)$ generates a C_0 semigroup on X.

If instead of (9) above a weaker condition

$$\|P\,u\| \le M\,\|u\| + \|A\,u\|\,,\ \ \forall u \in D(A) \tag{10}$$

is satisfied, then the operator $A + P$ as defined in $D(A)$ is closable and its closure $\overline{A+P}$ generates a C_0 semigroup on X. □

Theorem II. (De Graaf) Let A be a closable densely defined operator in a Hilbert space $(H, (\cdot\,,\cdot))$. Assume that there exists a strictly positive self-adjoint operator Q such that $D(Q) \subset D(A)$, $R(Q) = H$ and

$$\left.\begin{array}{l} \mathrm{Re}(u,\,Au) \le \omega(u,u) \\ \mathrm{Re}(Qu,\,Au) \le \omega(Qu,u) \end{array}\right\} \ \forall u \in D(Q). \tag{11}$$

Then $D(Q)$ is a core for the operator A and $\bar{A}$ generates a quasi-contractive C_0 semigroup on X (i.e. a C_0 semigroup $\{T(t) \mid t \ge 0\}$ such that $\{e^{-\omega t}\,T(t) \mid t \ge 0\}$ is contractive for some $\omega \ge 0$). □

Theorem II was given in [3]. Theorem I is quite standard and can be found in standard books on functional analysis or monographs on operator semigroups, cf. e.g. [1], [2], [4] and [5].

For the sake of simplicity in formulation from now on we always assume that all the matrices are tridiagonal. Extensions to cases involving more general matrices are immediate.

§2. Diagonal Dominant Generators

Theorem 1. Suppose that (a_{jk}) is a tridiagonal matrix with the diagonal elements $a_{kk} \geq d > 0$ for all $k \in I\!N_0$ and $(A_{\max} - Q) \upharpoonright l_c^2$ quasi-accretive. Here $Q = \mathrm{Op}[\mathrm{diag}(a_{kk})]$. Assume that there exist constants $K \in I\!N_0$ and $c_l \geq 0$ and $c_u \geq 0$ with $c_l + c_u \leq 1$ such that

$$| a_{k+1,k} | \, a_{kk}^{-1} \leq c_l , \quad | a_{k-1,k} | \, a_{kk}^{-1} \leq c_u , \quad \forall k \geq K. \tag{31}$$

Then the closure of $A_{\max} \upharpoonright l_c^2$ generates a c_0 semigroup of operators on l^2.

Proof. From the condition $a_{kk} \geq d > 0$ it follows easily that the operator Q generates a c_0 semigroup on l^2 with l_c^2 as a core of Q.

For $u \in D(Q)$ we have the estimate

$$(\sum_{k=0}^{\infty} | a_{k,k-1} u_{k-1} + a_{k,k+1} u_{k+1} |^2)^{1/2}$$

$$\leq (\sum_{k=0}^{\infty} | a_{k,k-1} u_{k-1} |^2)^{1/2} + (\sum_{k=0}^{\infty} | a_{k,k+1} u_{k+1} |^2)^{1/2}$$

$$\leq M_k \| u \| + (c_l + c_u) \; \| Q u \|$$

where

$$M_k = 2 \max_{0 \leq k \leq K} \{ | a_{k,k-1} | , \; | a_{k,k+1} | \}.$$

From this estimate we have $D(Q) \subset D(A_{\max})$ and for $u \in D(Q)$

$$\| (A - Q) u \| \leq 2 M_K \| u \| + (c_l + c_u) \| Q u \| \tag{32}$$

and

$$\| A u \| \leq 2 M_K \| u \| + (1 + c_l + c_u) \| Q u \|. \tag{33}$$

By virtue of (32) with $c_l + c_u \leq 1$ Theorem I ensures that the closure of $A_{\max} \upharpoonright D(Q)$ generates a c_0 semigroup on l^2. Moreover (33) implies that l_c^2 is a core for $A_{\max} \upharpoonright D(Q)$, and hence also for its closure. □

Corollary 2. If in the above theorem instead of condition (31) we assume the stronger condition that

$$\limsup \mid a_{k+1,k} \mid a_{kk}^{-1} + \limsup \mid a_{k,k-1} \mid a_{kk}^{-1} < 1 \tag{34}$$

then $A_{\max} \restriction D(Q)$ generates a c_0 semigroup in l^2 with l_c^2 as a core for $A_{\max} \restriction D(Q)$. □

We remark that if $(a_{jk}) = \operatorname{diag}(a_{kk}) + (a_{jk}^{(1)}) + (a_{jk}^{(2)})$ with $(a_{jk}^{(1)})$ a bounded matrix on l^2 and $(a_{jk}^{(2)})$ skew symmetric, then $(A_{\max} - Q) \restriction l_c^2$ is quasi-accretive.

Example 3. With (a_{jk}) given by $(\mu, \nu < 1)$

$$\operatorname{diag}(1,2,3,\ldots) + \begin{bmatrix} 0 & -1^\nu & & & \\ 1^\nu & 0 & -2^\nu & & \\ & 2^\nu & 0 & -3^\nu & \\ & & 3^\nu & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

$$+ i \begin{bmatrix} 0 & 1^\mu & & & \\ 1^\mu & 0 & 2^\mu & & \\ & 2^\mu & 0 & 3^\mu & \\ & & 3^\mu & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix} + \begin{bmatrix} 0 & 1-\frac{1}{1} & & & \\ 2+\frac{1}{1} & 0 & 1-\frac{1}{2} & & \\ 0 & 2+\frac{1}{2} & 0 & 1-\frac{1}{3} & \\ & & 2+\frac{1}{3} & \ddots & \ddots \\ & & & \ddots & \end{bmatrix}$$

the operator $A_{\max} \restriction D(Q)$ generates a c_0 semigroup in l^2 with l_c^2 as a core of it. Here Q is the maximal operator corresponding to $\operatorname{diag}(1,2,3,\ldots)$. □

§3. Skew-adjoint Like Generators

Theorem 4. For an infinite matrix (a_{jk}) assume there exists a diagonal matrix $\operatorname{diag}(q_0, q_1, \ldots)$ with $0 < q_k \to \infty$ such that its corresponding maximal operator Q and the operators $A_{\max}$ and $A_{\max}^+$ satisfy the following conditions:

$$\operatorname{Op}(a_{jk} q_k^{-1}) \text{ is a Hilbert–Schmidt operator on } l^2 \tag{35}$$

$$\operatorname{Op}(a_{jk} + \bar{a}_{kj}) \restriction D(Q) \text{ is quasi–accretive} \tag{36}$$

$$\mathrm{Op}(q_j^{1/2}\, a_{jk}\, q_k^{-1/2} + q_j^{-1/2}\, \overline{a}_{kj}\, q_k^{1/2}) \restriction D(Q^{1/2}) \text{ is quasi-accretive.} \tag{37}$$

Then $A_{\max} \restriction D(Q)$ is closable and its closure generates a c_0 semigroup on l^2.

Proof. Condition (35) implies in particular that $D(Q) \subset D(A_{\max})$. Theorem II will yield the wanted conclusion if we can check the conditions that

$$\mathrm{Re}(u\,,A_{\max}\,u) \geq \beta(u,u)\,,\ \ \forall u \in D(Q) \tag{38}$$

$$\mathrm{Re}(Q\,u\,,A_{\max}\,u) \geq \beta(Q\,u,u)\,,\ \ \forall u \in D(Q) \tag{39}$$

where β is a constant.

Let $u \in D(Q)$. We have

$$(u\,,A_{\max}\,u) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_j\, \overline{a}_{jk}\, \overline{u}_k$$

and

$$\begin{aligned}
\overline{(u\,,A_{\max}\,u)} &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \overline{u}_j\, a_{jk}\, u_k \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \overline{(q_j\, u_j)}\, (q_j^{-1}\, a_{jk}\, q_k^{-1})\, (q_k\, u_k) \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \overline{(q_j\, u_j)}\, (q_j^{-1}\, a_{jk}\, q_k^{-1})\, (q_k\, u_k) \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \overline{u}_j\, a_{jk}\, u_k \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_j\, a_{kj}\, \overline{u}_k .
\end{aligned}$$

Note that the double summations above are interchangable because of the fact that $(q_j^{-1}\, a_{jk}\, a_k^{-1})$ is a Hilbert-Schmidt matrix in l^2. Consequently

$$\begin{aligned}
\mathrm{Re}(u\,,A_{\max}\,u) &= \tfrac{1}{2}\,[(u\,,A_{\max}\,u) + \overline{(u\,,A_{\max}\,u)}] \\
&= \tfrac{1}{2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_j\, (\overline{a}_{jk} + a_{kj})\, \overline{u}_k
\end{aligned}$$

$$= \frac{1}{2}\,(u\,,\mathrm{Op}\,(a_{jk} + \bar{a}_{kj})\,u)\,,\ \ u \in D(Q).$$

This together with the condition (36) implies (38).

Similarly, for $\mathrm{v} \in D(Q^{1/2})$ we have

$$(Q^{1/2}\,\mathrm{v}\,,A_{\max}\,Q^{-1/2}\,\mathrm{v}) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q_j^{1/2}\,\mathrm{v}_j\,\bar{a}_{jk}\,q_k^{-1/2}\,\bar{\mathrm{v}}_k.$$

And

$$\overline{(Q^{1/2}\,\mathrm{v}\,,A_{\max}\,Q^{-1/2}\,\mathrm{v})} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (q_j^{1/2}\,\bar{\mathrm{v}}_j)\,(a_{jk}\,q_k^{-1})\,q_k^{1/2}\,\mathrm{v}_k$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} (q_j^{1/2}\,\bar{\mathrm{v}}_j)\,(a_{jk}\,q_k^{-1})\,q_k^{1/2}\,\mathrm{v}_k$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q_j^{-1/2}\,\mathrm{v}_j\,a_{kj}\,q_k^{1/2}\,\bar{\mathrm{v}}_k.$$

The interchange of summations above is allowable since $\mathrm{Op}\,(a_{jk}\,q_k^{-1})$ is Hilbert-Schmidt. Therefore for $u = Q^{-1/2}\,\mathrm{v} \in D(Q)$

$$\mathrm{Re}\,(Q\,u\,,A_{\max}\,u) = \mathrm{Re}(Q^{1/2}\,\mathrm{v}\,,A_{\max}\,Q^{-1/2}\,\mathrm{v})$$

$$= \frac{1}{2}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \mathrm{v}_j\,(q_j^{1/2}\,\bar{a}_{jk}\,q_k^{-1/2} + q_k^{-1/2}\,a_{kj}\,q_k^{1/2})\,\bar{\mathrm{v}}_k$$

$$= \frac{1}{2}\,(\mathrm{v}\,,\mathrm{Op}\,(q_j^{1/2}\,a_{jk}\,q_k^{-1/2} + q_j^{-1/2}\,\bar{a}_{kj}\,q_k^{1/2})\,\mathrm{v}).$$

Condition (39) then follows from (37) and the above relation. □

We have a few remarks on the conditions in the previous theorem. Condition (35) is satisfied if we take

$$q_k = \alpha_k^{-1}\max\{\,|\,a_{k-1,k}\,|\,,\ |\,a_{kk}\,|\,,\ \ |\,a_{k+1,k}\,|\,\}$$

with $\{\alpha_k\}$ in l^2. Condition (36) is verified if (a_{jk}) is a sum of a skew-symmetric matrix and a bounded matrix. In case (a_{jk}) s skew-symmetric, then the entry at $(k, k-1)$ of the matrix $(c_{jk}) = (q_j^{1/2}\,a_{jk}\,q_k^{-1/2} + q_j^{-1/2}\,\bar{a}_{kj}\,q_k^{1/2})$ is

$$a_{k,k-1}\left[\left(\frac{q_k}{q_{k-1}}\right)^{1/2} - \left(\frac{q_k}{q_{k-1}}\right)^{-1/2}\right].$$

Thus if $q_k/q_{k-1} \to 1$ the operator in (37) might be bounded and hence quasi-accretive in spite of $\{|a_{k,k-1}|\}$ being unbounded.

Example 5. If

$$(a_{jk}) = \begin{pmatrix} 0 & 1^\nu & & & \\ -1^\nu & 0 & 2^\nu & & \\ & -2^\nu & 0 & 3^\nu & \\ & & -3^\nu & \ddots & \ddots \\ & & & \ddots & \end{pmatrix} \qquad (\nu \le 1)$$

and $Q = \mathrm{Op}[\mathrm{diag}(1^\mu, 2^\mu, 3^\mu, \ldots)]$ $(\mu > \nu + \frac{1}{2})$, then the conditions in the above theorem are satisfied. So $\mathrm{Op}(a_{jk}) \restriction D(Q)$ essentially generates a c_0 semigroup in l^2. □

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