

## Matrix generators of $C_0$ semigroups in $\mathbb{R}^2$

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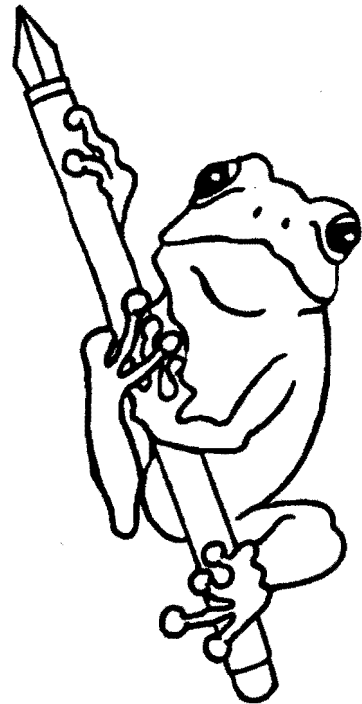
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# MATRIX GENERATORS OF $C_0$ SEMIGROUPS IN $l^2$

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## Abstract

Two criteria are given such that an infinite matrix generates a  $C_0$  semigroup in  $l^2$ .

## §1. Introduction

Let  $(a_{jk})_{j,k \in \mathbb{N}_0}$  be an infinite matrix of complex numbers with  $j, k$  the row index and column index respectively. We suppose that

$$\{a_{\cdot k}\} \in l^2, \forall k \in \mathbb{N}_0; \{a_{j \cdot}\} \in l^2, \forall j \in \mathbb{N}_0.$$

Corresponding to the matrix  $(a_{jk})$  we can define an operator  $A_{\max} \equiv \text{Op}(a_{jk})$  in  $l^2$  as follows:

$$A_{\max} u = \left\{ \sum_{k=0}^{\infty} a_{jk} u_k \right\}_{j \in \mathbb{N}_0}$$

with

$$D(A_{\max}) = \left\{ u = (u_k) \in l^2 \mid \sum_{k=0}^{\infty} a_{jk} u_k \text{ converges for all } j \in \mathbb{N}_0 \text{ and } \left\{ \sum_{k=0}^{\infty} a_{jk} u_k \right\}_{j \in \mathbb{N}_0} \in l^2 \right\}.$$

If instead of the matrix  $(a_{jk})$  we use its complex conjugate  $(\bar{a}_{kj})$ , then we obtain another operator on  $l^2$ , denoted by  $A_{\max}^+$ . Under the above conditions it is easy to see that  $l_c^2 \subset D(A_{\max})$  and  $l_c^2 \subset D(A_{\max}^+)$ , where  $l_c^2 = \{u = (u_k) \mid u_k = 0 \text{ if } k \geq K \text{ depending on } u\}$ , the subspace of finite sequences. We fix the notations  $A_{\max} \upharpoonright l_c^2 = A_{\min}$  and  $A_{\max}^+ \upharpoonright l_c^2 = A_{\min}^+$ . Thus both the operators  $A_{\max}$  and  $A_{\max}^+$  are densely defined. Actually they are also closed and obey the following relations:

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$$(A_{\max})^* \subset (A_{\min})^* = A_{\max}^+; \quad (A_{\max}^+)^* \subset (A_{\min}^+)^* = A_{\max}.$$

The above facts are simple to prove and can be found in standard text books, e.g. in [6].

In the present note we give two criteria which determine classes of matrices  $(a_{jk})$  whose corresponding maximal operators  $A_{\max}$  or restrictions thereof are infinitesimal generators of  $C_0$  semigroups on  $l^2$ . Nontrivial criteria of such kind seem missing in the literature. We shall obtain our criteria by applying the perturbation theorem of Rellich-Kato-Gustafson-Chernoff and the auxiliary operator trick of De Graaf in §2 and §3 respectively. These abstract results read:

**Theorem I.** (Rellich-Kato-Gustafson-Chernoff) Suppose that  $A : D(A) \subset X \rightarrow X$  be an infinitesimal generator of a  $C_0$  semigroup on a Banach space  $(X, \|\cdot\|)$ . Let  $P$  be another operator in  $X$  such that  $D(P) \supset D(A)$  and

$$\|P u\| \leq M \|u\| + \alpha \|A u\|, \quad \forall u \in D(A) \tag{9}$$

where  $M$  and  $\alpha$  are nonnegative constants with  $\alpha < 1$ . Then the operator  $A + P$  as defined on  $D(A)$  generates a  $C_0$  semigroup on  $X$ .

If instead of (9) above a weaker condition

$$\|P u\| \leq M \|u\| + \|A u\|, \quad \forall u \in D(A) \tag{10}$$

is satisfied, then the operator  $A + P$  as defined in  $D(A)$  is closable and its closure  $\overline{A + P}$  generates a  $C_0$  semigroup on  $X$ . □

**Theorem II.** (De Graaf) Let  $A$  be a closable densely defined operator in a Hilbert space  $(H, (\cdot, \cdot))$ . Assume that there exists a strictly positive self-adjoint operator  $Q$  such that  $D(Q) \subset D(A)$ ,  $R(Q) = H$  and

$$\left. \begin{array}{l} \operatorname{Re}(u, Au) \leq \omega(u, u) \\ \operatorname{Re}(Qu, Au) \leq \omega(Qu, u) \end{array} \right\} \quad \forall u \in D(Q). \tag{11}$$

Then  $D(Q)$  is a core for the operator  $A$  and  $\overline{A}$  generates a quasi-contractive  $C_0$  semigroup on  $X$  (i.e. a  $C_0$  semigroup  $\{T(t) \mid t \geq 0\}$  such that  $\{e^{-\omega t} T(t) \mid t \geq 0\}$  is contractive for some  $\omega \geq 0$ ). □

Theorem II was given in [3]. Theorem I is quite standard and can be found in standard books on functional analysis or monographs on operator semigroups, cf. e.g. [1], [2], [4] and [5].

For the sake of simplicity in formulation from now on we always assume that all the matrices are

tridiagonal. Extensions to cases involving more general matrices are immediate.

## §2. Diagonal Dominant Generators

**Theorem 1.** Suppose that  $(a_{jk})$  is a tridiagonal matrix with the diagonal elements  $a_{kk} \geq d > 0$  for all  $k \in \mathbb{N}_0$  and  $(A_{\max} - Q) \upharpoonright l_c^2$  quasi-accretive. Here  $Q = \text{Op}[\text{diag}(a_{kk})]$ . Assume that there exist constants  $K \in \mathbb{N}_0$  and  $c_l \geq 0$  and  $c_u \geq 0$  with  $c_l + c_u \leq 1$  such that

$$|a_{k+1,k}| a_{kk}^{-1} \leq c_l, \quad |a_{k-1,k}| a_{kk}^{-1} \leq c_u, \quad \forall k \geq K. \quad (31)$$

Then the closure of  $A_{\max} \upharpoonright l_c^2$  generates a  $c_0$  semigroup of operators on  $l^2$ .

*Proof.* From the condition  $a_{kk} \geq d > 0$  it follows easily that the operator  $Q$  generates a  $c_0$  semigroup on  $l^2$  with  $l_c^2$  as a core of  $Q$ .

For  $u \in D(Q)$  we have the estimate

$$\begin{aligned} & \left( \sum_{k=0}^{\infty} |a_{k,k-1} u_{k-1} + a_{k,k+1} u_{k+1}|^2 \right)^{1/2} \\ & \leq \left( \sum_{k=0}^{\infty} |a_{k,k-1} u_{k-1}|^2 \right)^{1/2} + \left( \sum_{k=0}^{\infty} |a_{k,k+1} u_{k+1}|^2 \right)^{1/2} \\ & \leq M_K \|u\| + (c_l + c_u) \|Qu\| \end{aligned}$$

where

$$M_K = 2 \max_{0 \leq k \leq K} \{ |a_{k,k-1}|, |a_{k,k+1}| \}.$$

From this estimate we have  $D(Q) \subset D(A_{\max})$  and for  $u \in D(Q)$

$$\|(A - Q)u\| \leq 2M_K \|u\| + (c_l + c_u) \|Qu\| \quad (32)$$

and

$$\|Au\| \leq 2M_K \|u\| + (1 + c_l + c_u) \|Qu\|. \quad (33)$$

By virtue of (32) with  $c_l + c_u \leq 1$  Theorem I ensures that the closure of  $A_{\max} \upharpoonright_{D(Q)}$  generates a  $c_0$  semigroup on  $l^2$ . Moreover (33) implies that  $l_c^2$  is a core for  $A_{\max} \upharpoonright_{D(Q)}$ , and hence also for its closure.  $\square$

**Corollary 2.** If in the above theorem instead of condition (31) we assume the stronger condition

that

$$\limsup |a_{k+1,k}| a_{kk}^{-1} + \limsup |a_{k,k-1}| a_{kk}^{-1} < 1 \quad (34)$$

then  $A_{\max} \upharpoonright_D(Q)$  generates a  $c_0$  semigroup in  $l^2$  with  $l_c^2$  as a core for  $A_{\max} \upharpoonright_D(Q)$ .  $\square$

We remark that if  $(a_{jk}) = \text{diag}(a_{kk}) + (a_{jk}^{(1)}) + (a_{jk}^{(2)})$  with  $(a_{jk}^{(1)})$  a bounded matrix on  $l^2$  and  $(a_{jk}^{(2)})$  skew symmetric, then  $(A_{\max} - Q) \upharpoonright_{l_c^2}$  is quasi-accretive.

**Example 3.** With  $(a_{jk})$  given by  $(\mu, \nu < 1)$

$$\begin{aligned} & \text{diag}(1, 2, 3, \dots) + \begin{bmatrix} 0 & -1^\nu & & & \\ 1^\nu & 0 & -2^\nu & & \\ & 2^\nu & 0 & -3^\nu & \\ & & 3^\nu & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \\ & + i \begin{bmatrix} 0 & 1^\mu & & & \\ 1^\mu & 0 & 2^\mu & & \\ & 2^\mu & 0 & 3^\mu & \\ & & 3^\mu & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix} + \begin{bmatrix} 0 & 1 - \frac{1}{1} & & & \\ 2 + \frac{1}{1} & 0 & 1 - \frac{1}{2} & & \\ 0 & 2 + \frac{1}{2} & 0 & 1 - \frac{1}{3} & \\ & & 2 + \frac{1}{3} & \ddots & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \end{aligned}$$

the operator  $A_{\max} \upharpoonright_D(Q)$  generates a  $c_0$  semigroup in  $l^2$  with  $l_c^2$  as a core of it. Here  $Q$  is the maximal operator corresponding to  $\text{diag}(1, 2, 3, \dots)$ .  $\square$

### §3. Skew-adjoint Like Generators

**Theorem 4.** For an infinite matrix  $(a_{jk})$  assume there exists a diagonal matrix  $\text{diag}(q_0, q_1, \dots)$  with  $0 < q_k \rightarrow \infty$  such that its corresponding maximal operator  $Q$  and the operators  $A_{\max}$  and  $A_{\max}^+$  satisfy the following conditions:

$$\text{Op}(a_{jk} q_k^{-1}) \text{ is a Hilbert-Schmidt operator on } l^2 \quad (35)$$

$$\text{Op}(a_{jk} + \bar{a}_{kj}) \upharpoonright_D(Q) \text{ is quasi-accretive} \quad (36)$$

$$\text{Op}(q_j^{1/2} a_{jk} q_k^{-1/2} + q_j^{-1/2} \bar{a}_{kj} q_k^{1/2}) \upharpoonright_{D(Q)^{1/2}} \text{ is quasi-accretive.} \quad (37)$$

Then  $A_{\max} \upharpoonright_{D(Q)}$  is closable and its closure generates a  $c_0$  semigroup on  $l^2$ .

*Proof.* Condition (35) implies in particular that  $D(Q) \subset D(A_{\max})$ . Theorem II will yield the wanted conclusion if we can check the conditions that

$$\text{Re}(u, A_{\max} u) \geq \beta(u, u), \quad \forall u \in D(Q) \quad (38)$$

$$\text{Re}(Qu, A_{\max} u) \geq \beta(Qu, u), \quad \forall u \in D(Q) \quad (39)$$

where  $\beta$  is a constant.

Let  $u \in D(Q)$ . We have

$$(u, A_{\max} u) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_j \bar{a}_{jk} \bar{u}_k$$

and

$$\begin{aligned} \overline{(u, A_{\max} u)} &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \bar{u}_j a_{jk} u_k \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \overline{(q_j u_j)} (q_j^{-1} a_{jk} q_k^{-1}) (q_k u_k) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \overline{(q_j u_j)} (q_j^{-1} a_{jk} q_k^{-1}) (q_k u_k) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \bar{u}_j a_{jk} u_k \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_j a_{kj} \bar{u}_k. \end{aligned}$$

Note that the double summations above are interchangeable because of the fact that  $(q_j^{-1} a_{jk} q_k^{-1})$  is a Hilbert-Schmidt matrix in  $l^2$ . Consequently

$$\begin{aligned} \text{Re}(u, A_{\max} u) &= \frac{1}{2} [(u, A_{\max} u) + \overline{(u, A_{\max} u)}] \\ &= \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_j (\bar{a}_{jk} + a_{kj}) \bar{u}_k \end{aligned}$$

$$= \frac{1}{2} (u, \text{Op}(a_{jk} + \bar{a}_{kj}) u), \quad u \in D(Q).$$

This together with the condition (36) implies (38).

Similarly, for  $v \in D(Q^{1/2})$  we have

$$(Q^{1/2} v, A_{\max} Q^{-1/2} v) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q_j^{1/2} v_j \bar{a}_{jk} q_k^{-1/2} \bar{v}_k.$$

And

$$\begin{aligned} \overline{(Q^{1/2} v, A_{\max} Q^{-1/2} v)} &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (q_j^{1/2} \bar{v}_j) (a_{jk} q_k^{-1}) q_k^{1/2} v_k \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (q_j^{1/2} \bar{v}_j) (a_{jk} q_k^{-1}) q_k^{1/2} v_k \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q_j^{-1/2} v_j a_{kj} q_k^{1/2} \bar{v}_k. \end{aligned}$$

The interchange of summations above is allowable since  $\text{Op}(a_{jk} q_k^{-1})$  is Hilbert-Schmidt. Therefore for  $u = Q^{-1/2} v \in D(Q)$

$$\begin{aligned} \text{Re}(Q u, A_{\max} u) &= \text{Re}(Q^{1/2} v, A_{\max} Q^{-1/2} v) \\ &= \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} v_j (q_j^{1/2} \bar{a}_{jk} q_k^{-1/2} + q_k^{-1/2} a_{kj} q_k^{1/2}) \bar{v}_k \\ &= \frac{1}{2} (v, \text{Op}(q_j^{1/2} a_{jk} q_k^{-1/2} + q_j^{-1/2} \bar{a}_{kj} q_k^{1/2}) v). \end{aligned}$$

Condition (39) then follows from (37) and the above relation. □

We have a few remarks on the conditions in the previous theorem. Condition (35) is satisfied if we take

$$q_k = \alpha_k^{-1} \max \{ |a_{k-1,k}|, |a_{kk}|, |a_{k+1,k}| \}$$

with  $\{\alpha_k\}$  in  $l^2$ . Condition (36) is verified if  $(a_{jk})$  is a sum of a skew-symmetric matrix and a bounded matrix. In case  $(a_{jk})$  is skew-symmetric, then the entry at  $(k, k-1)$  of the matrix  $(c_{jk}) = (q_j^{1/2} a_{jk} q_k^{-1/2} + q_j^{-1/2} \bar{a}_{kj} q_k^{1/2})$  is



$$a_{k,k-1} \left[ \left( \frac{q_k}{q_{k-1}} \right)^{1/2} - \left( \frac{q_k}{q_{k-1}} \right)^{-1/2} \right].$$

Thus if  $q_k / q_{k-1} \rightarrow 1$  the operator in (37) might be bounded and hence quasi-accretive in spite of  $\{ | a_{k,k-1} | \}$  being unbounded.

**Example 5.** If

$$(a_{jk}) = \begin{bmatrix} 0 & 1^\nu & & & \\ -1^\nu & 0 & 2^\nu & & \\ & -2^\nu & 0 & 3^\nu & \\ & & -3^\nu & \ddots & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \quad (\nu \leq 1)$$

and  $Q = \text{Op}[\text{diag}(1^\mu, 2^\mu, 3^\mu, \dots)]$  ( $\mu > \nu + \frac{1}{2}$ ), then the conditions in the above theorem are satisfied. So  $\text{Op}(a_{jk}) \upharpoonright_{D(Q)}$  essentially generates a  $c_0$  semigroup in  $l^2$ . □

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