

On subset selection from Logistic populations

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On subset selection from
Logistic populations

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On subset selection from Logistic populations

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Summary

Some distributional results are derived for subset selection from Logistic populations, differing only in their location parameter. The probability of correct selection is determined. Exact and numerical results concerning the expected subset size are presented.

Keywords and Phrases: subset selection, Logistic distribution, probability of correct selection, expected subset size.

1. Introduction

An important class of problems is concerned with selection of the best of $k (\geq 2)$ populations $\pi_1, \pi_2, \dots, \pi_k$. These populations may be among other things treatments or production processes. Given are k random variables X_1, X_2, \dots, X_k , which may be sample means, associated with these populations. We assume that the distributions of these random variables differ only in their location parameter. The problem considered in this paper is to select a non-empty subset, as small as possible, such that the probability of selecting the best population in the subset is at least equal to a specified value $P^* (k^{-1} < P^* < 1)$. This so called subset selection procedure has been introduced by Gupta (1965).

In selection problems populations with large (small) values of a certain parameter are usually considered good. We define the population with the largest of the unknown values of the k location parameters to be the best. If there are more than one contenders of the best, we suppose that one of these is appropriately tagged.

In Van der Laan (1989) some results are given concerning Bechhofer's Indifference Zone selection procedure for Logistic populations. In Han (1987) and Gupta and Han (1987) the relevant distribution theory has been solved using Edgeworth expansions. Lorentzen and McDonald (1981) considered the problem of selecting the best Logistic population using sample medians. In this paper we shall study the distribution theory for subset selection procedures, starting from Logistic populations. Using the Logistic distribution it is possible to solve analytically certain distribution problems. An interesting point is the striking resemblance between the Normal and Logistic distribution for a suitable choice of the parameters. An illustration of this resemblance can be given using results from Van der Laan (1989). Using the subset selection rule with the model assumption of Normality, whereas the distribution used is in fact Logistic, each with variance one, the actual lower bound of the probability of correct selection has been given in the next table for some values of k and P^* .

| k | P^* | | |
|-----|-------|-------|-------|
| | .90 | .95 | .99 |
| 2 | .9063 | .9510 | .9872 |
| 5 | .9000 | .9444 | .9835 |
| 10 | .8897 | .9368 | .9801 |

If X_i ($i = 1, 2, \dots, k$) are the sample means based on $n \geq 2$ independently and Logistically distributed observations, the deviations will tend to become smaller for increasing n . For $n = 10$ and a simulation with 5000 runs (cf. Van der Laan and Van Putten (1989)) this tendency can be illustrated by presenting some results in the following table.

| k | P^* | | |
|-----|-------|------|------|
| | .90 | .95 | .99 |
| 2 | .899 | .952 | .989 |
| 5 | .892 | .952 | .989 |
| 10 | .889 | .963 | .987 |

Finally, some results concerning the expected subset size and its maximum value in a specific subspace of the parameter space are provided.

2. Subset selection from Logistic populations

Let X_1, X_2, \dots, X_k be $k (\geq 2)$ independent random variables with probability densities $f(x - \theta_1), f(x - \theta_2), \dots, f(x - \theta_k)$, respectively. These k random variables characterize populations $\pi_1, \pi_2, \dots, \pi_k$. We are interested in choosing the population with the largest value of θ (the best population). The ranked location parameters $\theta_1, \theta_2, \dots, \theta_k$ are denoted by $\theta_{[1]} \leq \theta_{[2]} \leq \dots \leq \theta_{[k]}$. If there are more than one contenders of the best, we suppose that one of these is appropriately tagged. We choose a non-empty subset such that the probability is at least P^* (with $k^{-1} < P^* < 1$) that the selected subset contains the population with the largest value of θ by following the next selection rule:

Select π_i iff $x_i \geq x_{\max} - d$,

where x_i is the observed value of X_i ($i = 1, 2, \dots, k$) and x_{\max} is the observed value of $X_{\max} = \max_{1 \leq i \leq k} X_i$. A correct selection CS means selection of any subset which includes the best one.

The probability of CS is equal to

$$P(CS) = P(X_{(k)} \geq X_{\max} - d),$$

where $X_{(k)}$ is the unknown random variable which is associated with $\theta_{[k]}$. Now we can write (cf. Gupta (1965))

$$P(CS) = \int_{-\infty}^{\infty} f(t) \prod_{i=1}^{k-1} F(t + d + \theta_{[k]} - \theta_{[i]}) dt,$$

where $F(\cdot)$ and $f(\cdot)$ are the distribution function and the density, respectively, of $X_i - \theta_i$ ($i = 1, 2, \dots, k$) and

$$\inf P(CS) = \int_{-\infty}^{\infty} f(t) F^{k-1}(t+d) dt ,$$

which is attained for $\theta_{[1]} = \theta_{[k]}$. The smallest value of d has to be chosen for which

$$\int_{-\infty}^{\infty} f(t) F^{k-1}(t+d) dt = P^* ,$$

to be sure that $P(CS) \geq P^*$ for all configurations of $\theta_1, \theta_2, \dots, \theta_k$.

Assuming X_1, X_2, \dots, X_k are Logistically distributed with known scale parameter λ we get

$$f(x - \theta_i) = \lambda e^{-\lambda(x-\theta_i)} \{1 + e^{-\lambda(x-\theta_i)}\}^{-2}$$

and

$$F(x - \theta_i) = \{1 + e^{-\lambda(x-\theta_i)}\}^{-1}$$

for $-\infty < x < \infty$ and $i = 1, 2, \dots, k$. The probability requirement can now be written as follows

$$\begin{aligned} P^* &= \int_{-\infty}^{\infty} \lambda e^{-\lambda x} \{1 + e^{-\lambda x}\}^{-2} \{1 + e^{-\lambda(x+d)}\}^{-(k-1)} dx \\ &= a^{k-1} \int_0^{\infty} (s+1)^{-2} (s+a)^{-k+1} ds \end{aligned}$$

with $a = e^{\lambda d}$. From Van der Laan (1989) it can easily be seen that this equality reduces to

$$P^* = 1 - (k-1) a^{k-1} (a-1)^{-k} S_a(k-1) , \quad (1)$$

where for $c > 0$ and integer m the following definition holds: $S_c(m) = \ln c - \sum_{i=1}^m \frac{1}{i} (1 - \frac{1}{c})^i$ and

$$\sum_{i=1}^m \frac{1}{i} (1 - \frac{1}{c})^i = 0 \text{ voor } m \leq 0.$$

In table 1 values of $\sqrt{3} \pi^{-1} \lambda d$ are given for which (1) holds. The value of d can be deduced in a simple way.

3. Expected size of the subset

Consistent with the probability requirement, we would like the size of the selected subset to be as small as possible. This size S is a random variable with possible outcomes $1, 2, \dots, k$. A criterion for the efficiency of the selection procedure is the expected value $E(S)$ of S . We have

$$\begin{aligned}
 E(S) &= \sum_{i=1}^k P(\text{selecting the population with } \mu_{[i]}) \\
 &= \sum_{i=1}^k P(X_{(i)} \geq X_{\max} - d) \\
 &= \sum_{i=1}^k \int_{-\infty}^{\infty} f(t) \prod_{\substack{j=1 \\ j \neq i}}^k F(t + d + \theta_{[i]} - \theta_{[j]}) dt .
 \end{aligned}$$

From Gupta (1965) we have

$$\begin{aligned}
 \max_{\Omega} E(S) &= k \int_{-\infty}^{\infty} F^{k-1}(x+d) f(x) dx \\
 &= kP^* ,
 \end{aligned}$$

where Ω is the parameter space consisting of all configurations of θ 's. In the subset $\Omega(\delta)$, defined by $\theta_{[i]} \leq \theta_{[k]} - \delta$ for $i = 1, 2, \dots, k-1$ and $\delta > 0$, $E(S)$ takes on its maximum value M when $\theta_{[i]} = \theta_{[k]} - \delta$ for all $i (\leq k-1)$ and hence

$$\begin{aligned}
 M &= \max_{\Omega(S)} E(S) \\
 &= \int_{-\infty}^{\infty} F^{k-1}(x+d+\delta) dF(x) + (k-1) \int_{-\infty}^{\infty} F^{k-2}(x+d) F(x+d-\delta) dF(x) .
 \end{aligned}$$

Now we can prove the following theorem.

Theorem 1. For Logistic populations we have for all $k \geq 2$

$$M = \begin{cases} \frac{1}{2} (k+1) - \frac{k-1}{a^2} \left(\frac{a^2}{a^2-1} \right)^k S_{a^2}(k-1) - \frac{1}{2} (k-1)(k-2) \left\{ \frac{1}{a} - (k-1) \frac{a^{k-2}}{(a-1)^k} S_a(k-1) \right\} \\ \text{for } d=\delta \\ \frac{ab}{ab-1} - \frac{k-1}{ab} \left(\frac{ab}{ab-1} \right)^k S_{ab}(k-2) + \frac{k-1}{a-b} \left[a - (k-2) \left(\frac{a}{a-1} \right)^{k-1} S_a(k-2) - \right. \\ \left. - \frac{ab}{a-b} \left\{ \left(\frac{a}{a-1} \right)^{k-2} S_a(k-3) - \left(\frac{b}{b-1} \right)^{k-2} S_b(k-3) \right\} \right] \\ \text{for } d \neq \delta \end{cases}$$

where $b = e^{\lambda \delta}$.

Proof: We write

$$M = I_1 + (k - 1) I_2 ,$$

where

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} F^{k-1}(x+d+\delta) dF(x) \\ &= \lambda(e^{\lambda(d+\delta)})^{k-1} \int_{-\infty}^{\infty} (e^{\lambda(d+\delta)} + e^{-\lambda x})^{-k+1} (1 + e^{-\lambda x})^{-2} e^{-\lambda x} dx \\ &= (ab)^{k-1} \int_0^{\infty} (t+ab)^{-k+1} (t+1)^{-2} dt \\ &= \left[\frac{ab}{ab-1} \right]^k \left\{ \frac{ab-1}{ab} - (k-1) \frac{\ln(ab)}{ab} + \frac{1}{ab} \sum_{i=1}^{k-2} \frac{k-1-i}{i} \left(1 - \frac{1}{ab}\right)^i \right\} \\ &= \frac{ab}{ab-1} - \frac{k-1}{ab} \left[\frac{ab}{ab-1} \right]^k S_{ab}(k-2) \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} F^{k-2}(x+d) F(x+d-\delta) dF(x) \\ &= \frac{a^{k-1}}{b} J \end{aligned}$$

with

$$J = \int_0^{\infty} (t+a)^{-k+2} \left(t + \frac{a}{b}\right)^{-1} (t+1)^{-2} dt .$$

First we consider the case $d = \delta$ (or $a = b$), then

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} F^{k-1}(x+2d) dF(x) \\ &= \frac{a^2}{a^2-1} - \frac{k-1}{a^2} \left[\frac{a^2}{a^2-1} \right]^k S_{a^2}(k-2) \\ &= 1 - \frac{k-1}{a^2} \left[\frac{a^2}{a^2-1} \right]^k S_{a^2}(k-1) \end{aligned}$$

and

$$\begin{aligned}
 I_2 &= a^{k-2} \int_0^{\infty} (t+1)^{-3} (t+a)^{-k+2} dt \\
 &= \frac{1}{2} - \frac{1}{2} (k-2) a^{k-2} \int_0^{\infty} (t+1)^{-2} (t+a)^{-k+1} dt \\
 &= \frac{1}{2} - \frac{1}{2} (k-2) \left\{ \frac{1}{a-1} - (k-1) \frac{a^{k-2}}{(a-1)^k} S_a(k-2) \right\} .
 \end{aligned}$$

After some elementary calculations the result of theorem 1 follows.

Now we consider the case $d \neq \delta$ (thus $a \neq b$). For $k \geq 4$ we get

$$\begin{aligned}
 J &= \frac{b}{a-b} \int_0^{\infty} (t+a)^{-k+2} (t+1)^{-1} \left\{ (t+1)^{-1} - \left(t + \frac{a}{b}\right)^{-1} \right\} dt \\
 &= \frac{b}{a-b} \left\{ a^{-k+3} (a-1)^{-1} - (k-2) (a-1)^{-k+1} S_a(k-3) \right\} - \\
 &\quad - \left(\frac{b}{a-b}\right)^2 \left\{ \int_0^{\infty} (t+a)^{-k+2} (t+1)^{-1} dt - \int_0^{\infty} (t+a)^{-k+2} \left(t + \frac{a}{b}\right)^{-1} dt \right\} \\
 &= \frac{b}{a-b} \left\{ a^{-k+3} (a-1)^{-1} - (k-2) (a-1)^{-k+1} S_a(k-3) \right\} - \\
 &\quad - \left(\frac{b}{a-b}\right)^2 \left\{ (a-1)^{-k+2} S_a(k-3) - b^{k-2} a^{-k+2} (b-1)^{-k+2} S_b(k-3) \right\} .
 \end{aligned}$$

From this it follows

$$\begin{aligned}
 M &= \frac{ab}{ab-1} - \frac{k-1}{ab} \left(\frac{ab}{ab-1}\right)^k S_{ab}(k-2) + \frac{k-1}{a-b} \left[\frac{a^2}{a-1} - (k-2) \left(\frac{a}{a-1}\right)^{k-1} S_a(k-3) - \right. \\
 &\quad \left. - \frac{ab}{a-b} \left\{ \left(\frac{a}{a-1}\right)^{k-2} S_a(k-3) - \left(\frac{b}{b-1}\right)^{k-2} S_b(k-3) \right\} \right]
 \end{aligned}$$

and the result of theorem 1 follows.

For $k = 2$ we get

$$I_1 = \left(\frac{ab}{ab-1}\right)^2 \left\{ \frac{ab-1}{ab} - \frac{\ln(ab)}{ab} \right\}$$

and

$$\begin{aligned}
 I_2 &= \frac{a}{b} \int_0^a (t + \frac{a}{b})^{-1} (t + 1)^{-2} dt \\
 &= \frac{a}{a-b} \int_0^\infty (t + 1)^{-1} \{ (t + 1)^{-1} - (t + \frac{a}{b})^{-1} \} dt \\
 &= \frac{a}{a-b} \{ 1 - \frac{b}{a-b} \ln(\frac{a}{b}) \} ,
 \end{aligned}$$

thus

$$M = \frac{ab}{ab-1} - \frac{ab}{(ab-1)^2} \ln(ab) + \frac{a}{a-b} \{ 1 - \frac{b}{a-b} \ln(\frac{a}{b}) \}$$

and this is the general expression for $d \neq \delta$ with $k = 2$. For $k = 3$ we get

$$\begin{aligned}
 I_2 &= \frac{a^2}{b} \int_0^\infty (t + 1)^{-2} (t + a)^{-1} (t + \frac{a}{b})^{-1} dt \\
 &= \frac{a^2}{a-b} [\int_0^\infty (t + 1)^{-2} (t + a)^{-1} dt - \int_0^\infty (t + 1)^{-1} (t + a)^{-1} (t + \frac{a}{b})^{-1} dt] \\
 &= \frac{a^2}{a-b} [\frac{1}{a-1} - \frac{1}{(a-1)^2} \ln a - \frac{b}{a-b} \{ \int_0^\infty (t + 1)^{-1} (t + a)^{-1} dt - \int_0^\infty (t + a)^{-1} (t + \frac{a}{b})^{-1} dt \}] \\
 &= \frac{a^2}{a-b} [\frac{1}{a-1} - \frac{\ln a}{(a-1)^2} - \frac{b}{a-b} \{ \frac{\ln a}{a-1} - \frac{b}{a} \frac{\ln b}{b-1} \}]
 \end{aligned}$$

and I_1 can be determined as for $k = 4$, thus

$$M = \frac{ab}{ab-1} - \frac{2}{ab} (\frac{ab}{ab-1})^3 S_{ab}(1) + \frac{2a^2}{a-b} [\frac{1}{a-1} - \frac{\ln a}{(a-1)^2} - \frac{b}{a-b} \{ \frac{\ln a}{a-1} - \frac{b}{a} \frac{\ln b}{b-1} \}]$$

and it can easily be seen that this result is equal to the general expression of theorem 1 for $k = 3$.

For different values of d , δ and k the value of M has been computed. In table 2 the results can be found.

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Table 1 Values of $\sqrt{3} \pi^{-1} \lambda d$ fulfilling the probability requirement for some values of k and P^* .

| k | P^* | | | |
|-----|--------|--------|--------|--------|
| | .75 | .90 | .95 | .99 |
| 2 | .9000 | 1.7581 | 2.3109 | 3.4578 |
| 3 | 1.3636 | 2.1906 | 2.7311 | 3.8635 |
| 4 | 1.6151 | 2.4319 | 2.9682 | 4.0955 |
| 5 | 1.7878 | 2.5995 | 3.1338 | 4.2585 |
| 6 | 1.9193 | 2.7280 | 3.2610 | 4.3842 |
| 7 | 2.0255 | 2.8322 | 3.3643 | 4.4865 |
| 8 | 2.1146 | 2.9198 | 3.4514 | 4.5728 |
| 9 | 2.1912 | 2.9954 | 3.5265 | 4.6474 |
| 10 | 2.2586 | 3.0619 | 3.5926 | 4.7131 |
| 25 | 2.8112 | 3.6104 | 4.1394 | 5.2577 |
| 50 | 3.2084 | 4.0063 | 4.5348 | 5.6524 |

Table 2 $M = \max_{\Omega(\delta)} E(S)$ for some values of d , δ and k .

| | | k | | | | | | | |
|-----|----------|--------|--------|--------|--------|---------|---------|---------|---------|
| | | 2 | 3 | 5 | 10 | 25 | 50 | 75 | 100 |
| d | δ | | | | | | | | |
| 0.1 | 0 | 1.0604 | 1.0922 | 1.1256 | 1.1576 | 1.1809 | 1.1896 | 1.1926 | 1.1942 |
| | 0.1 | 1.0602 | 1.0920 | 1.1254 | 1.1575 | 1.1809 | 1.1896 | 1.1926 | 1.1942 |
| | 0.2 | 1.0596 | 1.0914 | 1.1249 | 1.1572 | 1.1808 | 1.1896 | 1.1926 | 1.1942 |
| | 1 | 1.0439 | 1.0712 | 1.1043 | 1.1421 | 1.1744 | 1.1870 | 1.1913 | 1.1933 |
| | 2 | 1.0186 | 1.0325 | 1.0532 | 1.0854 | 1.1296 | 1.1580 | 1.1708 | 1.1780 |
| | 5 | 1.0003 | 1.0006 | 1.0011 | 1.0022 | 1.0049 | 1.0087 | 1.0119 | 1.0148 |
| 0.5 | 0 | 1.2942 | 1.4806 | 1.7108 | 1.9831 | 2.2337 | 2.3442 | 2.3855 | 2.4072 |
| | 0.1 | 1.2933 | 1.4795 | 1.7097 | 1.9825 | 2.2334 | 2.3442 | 2.3855 | 2.4072 |
| | 0.2 | 1.2907 | 1.4761 | 1.7065 | 1.9803 | 2.2326 | 2.3439 | 2.3854 | 2.4071 |
| | 1 | 1.2181 | 1.3731 | 1.5855 | 1.8719 | 2.1734 | 2.3160 | 2.3691 | 2.3964 |
| | 2 | 1.0959 | 1.1738 | 1.2975 | 1.5082 | 1.8309 | 2.0632 | 2.1777 | 2.2461 |
| | 5 | 1.0016 | 1.0032 | 1.0060 | 1.0124 | 1.0288 | 1.0514 | 1.0711 | 1.0888 |
| 1 | 0 | 1.5453 | 1.9546 | 2.5511 | 3.4388 | 4.5370 | 5.1668 | 5.4375 | 5.5891 |
| | 0.1 | 1.5440 | 1.9527 | 2.5490 | 3.4369 | 4.5361 | 5.1664 | 5.4373 | 5.5890 |
| | 0.2 | 1.5399 | 1.9470 | 2.5422 | 3.4310 | 4.5328 | 5.1649 | 5.4364 | 5.5884 |
| | 1 | 1.4255 | 1.7681 | 2.3004 | 3.1605 | 4.3291 | 5.0432 | 5.3559 | 5.5311 |
| | 2 | 1.2079 | 1.3902 | 1.7017 | 2.2894 | 3.3212 | 4.1695 | 4.6290 | 4.9208 |
| | 5 | 1.0042 | 1.0083 | 1.0161 | 1.0342 | 1.0814 | 1.1481 | 1.2063 | 1.2592 |
| 2 | 0 | 1.8511 | 2.6230 | 3.9973 | 6.7879 | 12.3802 | 17.7885 | 21.1225 | 23.4347 |
| | 0.1 | 1.8501 | 2.6214 | 3.9949 | 6.7847 | 12.3769 | 17.7859 | 21.1205 | 23.4331 |
| | 0.2 | 1.8471 | 2.6164 | 3.9875 | 6.7747 | 12.3662 | 17.7771 | 21.1134 | 23.4274 |
| | 1 | 1.7532 | 2.4511 | 3.7186 | 6.3618 | 11.8352 | 17.2687 | 20.6639 | 23.0336 |
| | 2 | 1.4956 | 1.9686 | 2.8578 | 4.8221 | 9.3068 | 14.2786 | 17.6475 | 20.1223 |
| | 5 | 1.0194 | 1.0386 | 1.0762 | 1.1676 | 1.4238 | 1.8086 | 2.1571 | 2.4791 |

Table 2 (continued).

| | | <i>k</i> | | | | | | | |
|----------|----------|----------|--------|--------|--------|---------|---------|---------|---------|
| | | 2 | 3 | 5 | 10 | 25 | 50 | 75 | 100 |
| <i>d</i> | δ | | | | | | | | |
| 3 | 0 | 1.9611 | 2.8959 | 4.7010 | 8.9202 | 19.8791 | 34.6453 | 46.6745 | 56.8543 |
| | 0.1 | 1.9608 | 2.8953 | 4.6999 | 8.9183 | 19.8762 | 34.6417 | 46.6708 | 56.8506 |
| | 0.2 | 1.9597 | 2.8933 | 4.6966 | 8.9126 | 19.8670 | 34.6304 | 46.6589 | 56.8385 |
| | 1 | 1.9211 | 2.8215 | 4.5686 | 8.6798 | 19.4534 | 34.0790 | 46.0458 | 56.1973 |
| | 2 | 1.7717 | 2.5322 | 4.0222 | 7.5881 | 17.2137 | 30.7241 | 42.0496 | 51.8073 |
| | 5 | 1.0744 | 1.1487 | 1.2949 | 1.6590 | 2.7255 | 4.4283 | 6.0523 | 7.6088 |
| 4 | 0 | 1.9911 | 2.9755 | 4.9266 | 9.7162 | 23.4916 | 44.9606 | 64.9996 | 83.8868 |
| | 0.1 | 1.9910 | 2.9754 | 4.9263 | 9.7155 | 23.4904 | 44.9588 | 64.9973 | 83.8842 |
| | 0.2 | 1.9907 | 2.9748 | 4.9253 | 9.7136 | 23.4866 | 44.9530 | 64.9901 | 83.8760 |
| | 1 | 1.9796 | 2.9535 | 4.8857 | 9.6354 | 23.3226 | 44.6912 | 64.6580 | 83.4895 |
| | 2 | 1.9254 | 2.8469 | 4.6792 | 9.2007 | 22.3229 | 42.9788 | 62.3985 | 80.7882 |
| | 5 | 1.2273 | 1.4545 | 1.8941 | 3.0079 | 6.3243 | 11.7725 | 17.1280 | 22.3967 |
| 5 | 0 | 1.9981 | 2.9948 | 4.9839 | 9.9354 | 24.6340 | 48.7000 | 72.3083 | 95.5151 |
| | 0.1 | 1.9981 | 2.9947 | 4.9839 | 9.9352 | 24.6337 | 48.6995 | 72.3076 | 95.5143 |
| | 0.2 | 1.9980 | 2.9946 | 4.9836 | 9.9348 | 24.6326 | 48.6977 | 72.3052 | 95.5115 |
| | 1 | 1.9954 | 2.9895 | 4.9739 | 9.9147 | 24.5869 | 48.6179 | 72.1972 | 95.3788 |
| | 2 | 1.9805 | 2.9601 | 4.9165 | 9.7909 | 24.2862 | 48.0673 | 71.4314 | 94.4217 |
| | 5 | 1.5000 | 1.9999 | 2.9993 | 5.4959 | 12.9687 | 25.3681 | 37.7001 | 49.9663 |
| 6 | 0 | 1.9996 | 2.9990 | 4.9969 | 9.9874 | 24.9266 | 49.7326 | 74.4365 | 99.0480 |
| | 0.1 | 1.9996 | 2.9990 | 4.9969 | 9.9874 | 24.9266 | 49.7325 | 74.4364 | 99.0480 |
| | 0.2 | 1.9996 | 2.9989 | 4.9969 | 9.9873 | 24.9264 | 49.7322 | 74.4360 | 99.0474 |
| | 1 | 1.9990 | 2.9978 | 4.9949 | 9.9832 | 24.9167 | 49.7147 | 74.4116 | 99.0169 |
| | 2 | 1.9956 | 2.9909 | 4.9827 | 9.9567 | 24.8510 | 49.5914 | 74.2369 | 98.7949 |
| | 5 | 1.7727 | 2.5453 | 4.6335 | 9.1747 | 22.7937 | 45.4765 | 68.1404 | 90.7857 |

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List of COSOR-memoranda - 1990

| Number | Month | Author | Title |
|---------|----------|--|---|
| M 90-01 | January | I.J.B.F. Adan J. Wessels W.H.M. Zijm | Analysis of the asymmetric shortest queue problem Part 1: Theoretical analysis |
| M 90-02 | January | D.A. Overdijk | Meetkundige aspecten van de productie van kroonwielen |
| M 90-03 | February | I.J.B.F. Adan J. Wessels W.H.M. Zijm | Analysis of the asymmetric shortest queue problem Part II: Numerical analysis |
| M 90-04 | March | P. van der Laan L.R. Verdooren | Statistical selection procedures for selecting the best variety |
| M 90-05 | March | W.H.M. Zijm E.H.L.B. Nelissen | Scheduling a flexible machining centre |
| M 90-06 | March | G. Schuller W.H.M. Zijm | The design of mechanizations: reliability, efficiency and flexibility |
| M 90-07 | March | W.H.M. Zijm | Capacity analysis of automatic transport systems in an assembly factory |
| M 90-08 | March | G.J. v. Houtum W.H.M. Zijm | Computational procedures for stochastic multi-echelon production systems |

| Number | Month | Author | Title |
|---------|-------|--|---|
| M 90-09 | March | P.J.M. van Laarhoven W.H.M. Zijm | Production preparation and numerical control in PCB assembly |
| M 90-10 | March | F.A.W. Wester J. Wijngaard W.H.M. Zijm | A hierarchical planning system versus a schedule oriented planning system |
| M 90-11 | April | A. Dekkers | Local Area Networks |
| M 90-12 | April | P. v.d. Laan | On subset selection from Logistic populations |