# Mathematical models for the spread of infectious diseases

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## MATHEMATICAL MODELS FOR THE SPREAD OF INFECTIOUS DISEASES

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# MATHEMATICAL MODELS FOR THE SPREAD OF INFECTIOUS DISEASES

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## Introduction

Fecal-oral transmitted diseases such as typhoid fever, infectious hepatitis and cholera belong to a class of diseases which may be called Man-Environment-Man diseases (MEM). These disesases possess a mechanism of spread mainly due to an interaction between the human population and a polluted environment.

In order to know how to control these diseases we want a mathematical model to describe their spread. Epidemiologists are particularly interested in defining so called threshold parameters which discriminate situations in which the epidemic tends to extinction and situations in which it tends to some endemic state. Furthermore, the biological meaning of each model must be clear, in particular, the biological meaning of the parameters of the model.

In this report three basic models are presented. One model is based on a system of ordinary differential equations while the other two are based on parabolic systems of partial differential equations, i.e. reaction-diffusion systems.

## 1. A simple model based on a system of ODE

First of all we want to select a model which is as simple as possible but still contains the essentials.

Based on statistical analysis of data on typhoid fever and infectious hepatitis $A$ in the city of Bari, the following model is proposed

$$(1.1)\qquad \begin{cases} \dfrac{dz_1}{dt} = -a_{11} z_1 + a_{12} z_2 \\[2mm] \dfrac{dz_2}{dt} = -a_{22} z_2 + g(z_1). \end{cases}$$

Here $z_1$ denotes the (average) concentration of infectious agents in the environment, $z_2$ the infective human population, $\dfrac{1}{a_{11}}$ the mean life time of the agents in the environment, $\dfrac{1}{a_{22}}$ is the mean infectious period of the human infectives and $a_{12}$ the multiplication parameter of the infectious agent due to the human population. Finally, $g(z_1)$ describes the "force of infection" due to the agents on the human population. Consequently, we can influence $a_{12}$ in this model.

The simplest choice for $g(z_1)$ would be

$$g(z_1) = a_{21} z_1 .$$

In this case, the system (1.1) only admits the trivial state $z_1 = z_2 = 0$ as a stationary state. However, actual data show that we need a system which also possesses non-trivial stationary states.

We now make the following assumptions on the function $g : \mathbb{R}^+ \to \mathbb{R}^+$

(1.2.i) $g(0)=0$

(1.2.ii) $g$ is increasing

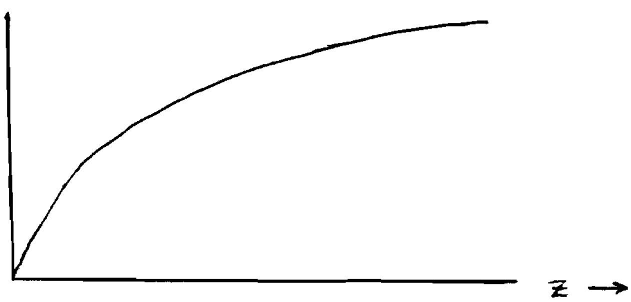
(1.2.iii) $g$ is twice continuously differentiable on $\mathbb{R}^+$ with

$$g_+'(0)=\lim_{z\downarrow 0} g'(z)>0$$

and $g''(z)<0$, $z\in \mathbb{R}^+-\{0\}$, i.e. $g$ is concave

(1.2.iv) $\lim_{z\to\infty} g'(z)=0.$

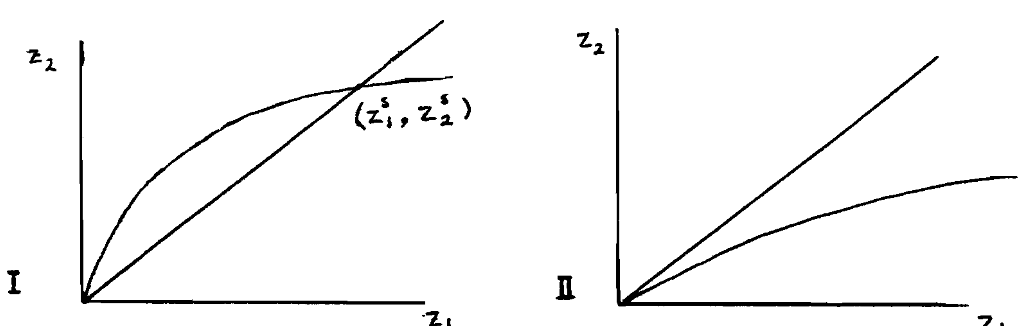
So a characteristic plot of $g$ is the following




A possible stationary state $(z_1^s, z_2^s)$ of the system (1.1) satisfies the following equations

$$(1.3)\qquad \begin{cases} -a_{11} z_1^s + a_{12} z_2^s = 0 \\ g(z_1^s) - a_{22} z_2^s = 0. \end{cases}$$

Hence the following situations occur




In case $I$, $g_+'(0)\,a_{12} > a_{11}\,a_{22}$ and in case II the opposite inequality is valid.

(1.4) **Theorem.**

Let $\theta=\dfrac{g_+'(0)\,a_{12}}{a_{11}\,a_{22}}$ and let $\mathbb{K}=\mathbb{R}^+\times\mathbb{R}^+$ be the positive cone in $\mathbb{R}^2$.

(a) If $0 < \theta < 1$, then the system (1.1) admits only the trivial equilibrium solution $z_1 = 0, z_2 = 0$ in the cone $I\!K$. This solution is globally asymptotically stable.

(b) If $\theta > 1$ then the system (1.1) admits two equilibrium solutions in $I\!K$. They are the origin (0,0) and the only non-trivial solution $(z_1^s, z_2^s) \in I\!K \setminus \{(0,0)\}$. In this case (0,0) is unstable, while $(z_1^s, z_2^s)$ is globally asymptotically stable.

A typical example of a function $g$ satisfying the conditions (1.2.i-iv) is given by

$$g(z) = \frac{\alpha z}{1 + \beta z}.$$

**Exercise.**
Theorem 1.4 strongly depends on the strict concavity of $g$ assumed in (1.2.iii). Another interesting choice for $g$ is

$$g(z) = \frac{\alpha z^2}{1 + \beta z^2}. \tag{1.5}$$

This function does not satisfy all stated conditions, in fact $g_+'(0) = 0$. Show that for this function $g$ there are two non-trivial equilibrium solutions, one which is globally asymptotically stable and one which is a stable point. Show that also the origin is globally asymptotically stable and determine the separatrix. Which situation is described by this model?

**Remark 1.**
The whole class of MEM-infectious diseases may be modelled by so called positive feedback systems

$$\begin{cases} \dfrac{dz_1}{dt} = f_1(z_1, z_2) \\ \dfrac{dz_2}{dt} = f_2(z_1, z_2) \end{cases} \tag{1.6}$$

where

$$\frac{\partial f_1}{\partial z_2} \geq 0 \quad \text{and} \quad \frac{\partial f_2}{\partial z_1} \geq 0.$$

**Remark 2.**
Data analysis shows seasonal dependence caused by the eating habits of the examined population. Therefore a periodic parameter $p(t)$ with one year as its period, has to be taken into account. The following model is suggested

$$
(1.7) \qquad \begin{cases} \dfrac{dz_1}{dt} = -a_{11} z_1 + a_{12} z_2 \\[2mm] \dfrac{dz_2}{dt} = -a_{22} z_2 + N \beta p(t) \dfrac{z_1}{1+z_1} \end{cases}
$$

where $N$ is the total population, $\beta$ the probability that a person is receptive. Here the problem is whether there exists a globally asymptotically stable periodic solution. For further details on this, see Section 6.

## 2. Some general considerations on systems of ODE

Consider the following first order ordinary differential equation

$$
(2.1) \qquad \frac{dz}{dt} = f(z), \quad z \in I\!R^m, \quad m \geq 1
$$

where $f$ is a function from $I\!R^m$ into $I\!R^m$.
Suppose we know that for a subset $D \subset I\!R^m$

$$
\forall_{z_0 \in D} \, \exists!_{z(t)} \text{ solution of (2.1) with } z(0) = z_0 .
$$

Then we also know (cf. [ ]) that there exists a family of non-linear operators $V(t) : D \to D$, $t \in I\!R^+$, such that $t \mapsto V(t) z_0$ is a solution of (2.1) with initial value $z_0$. The family $(V(t))_{t \geq 0}$ satisfies the following conditions

(2.2.i) $\quad V(0) = I$.

(2.2.ii) $\quad V(t+s) = V(t)\, V(s)$.

(2.2.iii) For all $t \in I\!R^+$,

$$
z_0 \mapsto V(t) z_0, \quad z_0 \in D,
$$

is a continuous mapping.

(2.2.iv) For all $z_0 \in D$

$$
t \mapsto V(t) z_0, \quad t \in I\!R^+
$$

is a continuous mapping.

**(2.3) Definition.**

i. The mapping $\Phi : t \mapsto V(t) z_0$ is called a *trajectory*.

ii. The image $\Phi(\mathbb{R}^+)$ of a trajectory $\Phi$ is called the *orbit.*

iii. A point $z^* \in D$ is called an equilibrium point if

$$\forall_{t \in \mathbb{R}^+} \; V(t) z^* = z^* .$$

iv. An equilibrium point $z^* \in D$ is called stable if

$$\forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{z_0 \in D} : \| z_0 - z^* \| < \delta \;\Rightarrow\; \| V(t) z_0 - z^* \| < \varepsilon .$$

v. An equilibrium point $z^* \in D$ is called asymptotically stable if

(a) $z^*$ is stable

(b) $z^*$ is a local attractor, i.e.

$$\exists_{\eta > 0} \forall_{z_0 \in D} : \| z - z_0 \| < \eta \;\Rightarrow\; \lim_{t \to \infty} \| V(t) z_0 - z^* \| = 0 .$$

vi. $z^*$ is called *globally asymptotically stable* in $D$ if

(a) $z^*$ is stable

(b) $z^*$ is a global attractor, i.e.

$$\forall_{z_0 \in D} \lim_{t \to \infty} \| V(t) z_0 - z^* \| = 0 .$$

Besides we introduce the following notation

(2.4) **Notation.**

(i) $\mathbb{K} = \{ \xi \in \mathbb{R}^m \mid \xi_i \geq 0, \quad i = 1, \ldots, m \} ,$

(ii) $\xi \geq \eta \; : \Leftrightarrow \; \xi - \eta \in \mathbb{K} ,$

(iii) $\xi > \eta \; : \Leftrightarrow \; \xi \geq \eta \;$ and $\; \xi \neq \eta ,$

(iv) $\xi \gg \eta \; : \Leftrightarrow \; \xi_i > \eta_i , \quad i = 1, \ldots, m ,$

(v) $[\xi, \eta] \; = \; \{ \zeta \in \mathbb{K} \mid \xi \leq \zeta \leq \eta \} .$

(2.5) **Definition.**
A function $f : \mathbb{K} \to \mathbb{R}^m$ is said to be quasi-monotonely increasing if

$$\forall_{\xi, \eta \in \mathbb{K}, \xi \geq \eta} \; : \; \xi_j = \eta_j \;\Rightarrow\; f_j(\xi) \geq f_j(\eta) .$$

In this connection we have the well-known theorem.

(2.6) **Theorem.**
Let $f : \mathbb{K} \to \mathbb{R}^m$ be quasi-monotonely increasing. Let $\underline{z}$ and $\overline{z}$ be functions from $\mathbb{R}^+$ into

$\mathbb{R}^m$. If $\underline{z}(0) \le \overline{z}(0)$ and

$$\frac{d}{dt}\underline{z} \le f(\underline{z}),\ \frac{d}{dt}\overline{z} \le f(\overline{z}),\ \text{then } \underline{z}(t) \le \overline{z}(t),\quad t \in \mathbb{R}^+.$$

On the function $f = (f_1, f_2, \ldots, f_m)$ we impose the following conditions throughout

C1 $f(0) = 0$

C2 $f$ is quasi-monotonely increasing

C3 $\forall_{\xi \in \mathbb{K}} \exists_{\xi_0 \in \mathbb{K}, \xi_0 \gg 0 \text{ and } \xi \le \xi_0}$:

$$f_j(\xi_0) < 0,\quad j = 1, \ldots, m.$$

Theorem (2.6) has the following consequences

(2.7) **Corollary.**
For each $t \in \mathbb{R}^+$, $V(t)$ is monotonely increasing, i.e.

$$\forall_{\zeta_1, \zeta_2 \in \mathbb{K}, \zeta_1 \le \zeta_2} : V(t)\zeta_1 \le V(t)\zeta_2.$$

(2.8) **Corollary.**
For each $t \in \mathbb{R}^+$, $V(t)(\mathbb{K}) \subset \mathbb{K}$.

*Proof.*
$f(0) = 0$ implies $V(t)0 = 0$, $t \in \mathbb{R}^+$. So if $\zeta_0 \in \mathbb{K}$ then $V(t)\zeta_0 \ge 0$, taking $\underline{z} \equiv 0$ and $\overline{z} : t \mapsto V(t)\zeta_0$. □

(2.9) **Definition.**
A region $\Sigma \subset \mathbb{R}^m$ is called *invariant* under the action of the family $(V(t))_{t \in \mathbb{R}^+}$ if $V(t)(\Sigma) \subset \Sigma$ for all $t \in \mathbb{R}^+$.

(2.10) **Lemma.**
Let $\Sigma$ be a region in $\mathbb{R}^m$ with boundary $\partial\Sigma$. If

$$\forall_{\zeta \in \partial\Sigma} : f(\zeta) \cdot n(\zeta) < 0$$

then $\Sigma$ is invariant for $(V(t))_{t \in \mathbb{R}^+}$.
Here $n(\zeta)$ denotes the outward pointed normal at $\partial\Sigma$, the dot denotes the Euclidean inner product in $\mathbb{R}^m$.

(2.11) **Definition.**
Let $\Sigma \subset \mathbb{R}^m$ be a bounded rectangle with $\zeta = 0$ in its interior. Then $\Sigma$ is called a *contracting rectangle* if

$$\forall_{\tau \in (0,1]} \forall_{\zeta \in \partial(\tau\Sigma)} : f(\zeta) \cdot n(\zeta) < 0$$

i.e. for each $\tau \in (0,1]$ the rectangle $\tau\Sigma$ is invariant.

(2.12) **Theorem.**
Consider the ordinary differential evolution equation

$$\frac{dz}{dt} = f(z).$$

If there exists a contracting rectangle $\Sigma$ with $\zeta = 0$ in its interior, then the trivial equilibrium solution $z(t) = 0$ is globally asymptotically stable.

*Proof.*
The function $L(\zeta) = \inf \{\tau \in (0,1] \mid \zeta \in \tau\Sigma\}$ is a Lyapunov function. Namely, in this case $\nabla L(\zeta)$ is proportional to $n(\zeta)$ ($\nabla L(\zeta) = \lambda(\zeta)\, n(\zeta)$, $\lambda(\zeta) > 0$). So

$$\frac{d}{dt} L(z(t)) = \nabla L(z(t)) \cdot f(z(t)) < 0$$

since $\Sigma$ is a bounded rectangle. $\square$

**Remark.**
We have proved global asymptotic stability by means of contracting rectangles. In the next section we extend this technique to the case of parabolic partial differential equations.

**Example.**
We consider the ordinary differential equation

$$\frac{dz}{dt} = B\, z$$

where $B$ is a quasi-monotone $m \times m$-matrix, i.e. $B = (b_{ij})$ with $b_{ij} \geq 0$, $i \neq j$. The corresponding propagation semigroup $(V(t))_{t \in \mathbb{R}^+}$ is given by $V(t) = \exp(tB)$. The following results can be proved

- $e^{tB}(\mathbb{K}) \subseteq \mathbb{K}$
- If $B$ is irreducible (i.e. $B \sim \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}$) then for all $\zeta \in \mathbb{K}$, $\zeta \neq 0$,

$$e^{tB}\, \zeta \gg 0.$$

Let $\mu := \max \{\mathrm{Re}\,\lambda \mid \lambda \in \sigma(B)\}$

- $\mu \in \sigma(B)$ and $\exists_{\xi \in I\!K} : B\xi = \mu\xi$.
- If $B$ is irreducible, then $\exists_{\xi >> 0} : B\xi = \mu\xi$.

The last two statements are called the Perron-Frobenius theorem.

## 3. Some general consideration on parabolic systems of PDE

In this section we try to extend certain techniques applied for systems of ODE, in particular the contracting rectangle technique.

Let $\Omega$ a bounded region in some $I\!R^n$. By $X_1$ we denote the Banach space of all continuous functions on $\overline{\Omega}$ endowed with the usual norm

$$\|u_1\|_\infty = \sup_{x \in \Omega} |u_1(x)|\ ,\quad u_1 \in X_1 .$$

Let $X$ denote the $m$-fold product of $X_1$,

$$X = X_1 \times X_1 \times \cdots \times X_1 \quad (m \text{ times})$$

with norm

$$\|u\| = \|(u_1, \ldots, u_m)\| = \|u_1\|_\infty + \cdots + \|u_m\|_\infty .$$

By $X_+$ we denote the positive cone in $X$

$$X_+ = \{u \in X \mid u_j(x) \geq 0 \text{ for } x \in \overline{\Omega} \text{ and } j = 1, \ldots, m\}.$$

In this section we study the following system of parabolic differential equations

$$\text{(3.1)} \qquad \begin{cases} \dfrac{\partial u_1}{\partial t}(x,t) = d_1 \Delta u_1 + f_1(u_1, \ldots, u_m) \\ \vdots \\ \dfrac{\partial u_m}{\partial t}(x,t) = d_m \Delta u_m + f_m(u_1, \ldots, u_m)\ , \quad (x,t) \in \Omega \times I\!R^+ \end{cases}$$

with boundary conditions

$$\frac{\partial u_j}{\partial \nu} + \alpha_j(x) u_j = 0\ ,\quad (x,t) \in \partial\Omega \times I\!R^+ .$$

Here $f = (f_1, \ldots, f_m) : I\!R_+^m \to I\!R^m$ is supposed to satisfy the conditions $C_1, C_2$ and $C_3$.

If we set $u = (u_1, \ldots, u_m) \in X$ and $D = \mathrm{diag}(d_1, \ldots, d_m)$, then we can write

$$\text{(3.1')} \qquad \frac{\partial u}{\partial t} = D \Delta u + f(u)$$

with

$$(B\,u)_j\,(x,t) = \frac{\partial u_j}{\partial \nu} + \alpha_j(x)\,u_j.$$

From literature we can conclude the existence of solutions of (3.1).

**(3.2) Theorem.**
For each "sufficiently" smooth $f$ and each initial condition $u^0 \in X$ there exists a unique solution $u$ with

$$u \in C^{2,1}(\Omega \times (0,\infty), \mathbb{R}^m) \cap C^{1,0}(\bar{\Omega} \times (0,\infty), \mathbb{R}^m)$$

such that $u(\cdot, 0) = u^0$.

As in the ODE-case by $V(t)\,u^0$ we denote the unique solution of (3.1) with initial condition $u^0 \in X$. For $V(t)$ we have similar results as stated in (2.2). ($(V(t))_{t \in \mathbb{R}^+}$ is called an evolution semigroup.) Indeed,

(3.3.i) $V(0) = I$.

(3.3.ii) $V(t+s) = V(t)\,V(s), \quad t,s \geq 0.$

(3.3.iii) $V(t)0 = 0, \quad t \geq 0.$

(3.3.iv) For all $t \geq 0$ the map

$$X \ni u^0 \;\mapsto\; V(t)\,u^0 \in X$$

is continuous uniformly on $[t_1, t_2]$.

(3.3.v) For all $u^0 \in X$ the map

$$\mathbb{R}^+ \ni t \to V(t)\,u^0 \in X$$

is continuous.

**(3.4) Lemma.**
For functions $u$ and $v$ on $\bar{\Omega} \times \mathbb{R}^+$ suppose that

$$\frac{\partial u}{\partial t}(x,t) \leq (D\,\Delta u)(x,t) + f(u(x,t))$$

$$\frac{\partial v}{\partial t}(x,t) \geq (D\,\Delta v)(x,t) + f(v(x,t))$$

for $(x,t) \in \Omega \times \mathbb{R}^+$, that

$$(B\,u)(x,t) \leq (B\,v)(x,t)$$

for $(x,t) \in \partial\Omega \times \mathbb{R}^+$ and that

$$u(x,0) \leq v(x,0)$$

for $x \in \overline{\Omega}$. Then

$$u(x,t) \leq v(x,t)$$

for all $(x,t) \in \overline{\Omega} \times \mathbb{R}^+$.

A consequence of the previous lemma is the monotonicity of the operators $V(t)$, $t \in \mathbb{R}^+$.

**(3.5) Corollary.**
Let $u^0, v^0 \in X_+$ with $u^0 \leq v^0$. Then $V(t)u^0 \leq V(t)v^0$ for all $t \geq 0$.

In correspondence with Definition (2.3) we have

**(3.6) Definition.**
Let $u^*$ be a stationary solution of equations (3.1) i.e.

$$D\Delta u^* - f(u^*) = 0 \quad \text{in} \quad \Omega$$

$$Bu^* = 0 \quad \text{in} \quad \partial\Omega.$$

Then $u^*$ is said to be *stable* if $\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{u^0 \in X}$

$$\|u^0 - u^*\| < \delta \Rightarrow \|V(t)u^0 - u^*\| < \varepsilon, \quad \forall t \in \mathbb{R}^+.$$

Besides, $u^*$ is said to be *globally asymptotically stable* in a domain $D \subset X$ if

(a) $u^*$ is stable

(b) $\forall_{u^0 \in D}$ : $\lim_{t \to \infty} \|V(t)u^0 - u^*\| = 0.$

Let us first, as an example of the system (3.1), investigate a linear PDE-system. Therefore we take $f(u) = Fu$ with $F$ a quasi-monotone $m \times m$-matrix:

(3.7) $$\frac{\partial u}{\partial t} = D\Delta u + Fu \quad \text{in} \quad \Omega$$

$$Bu = 0 \quad \text{in} \quad \partial\Omega$$

and with initial condition $u(0) = u^0 \in X$.
In this particular case the evolution operators $T(t)$, $t \in \mathbb{R}^+$, are linear. Now in addition we suppose that in the boundary operator all $\alpha_j$ are equal, i.e.

(3.8) $$\alpha_j(x) = \alpha(x) \geq 0, \quad j = 1, \ldots, m.$$

**(3.9) Lemma.**
Let $\phi$ be an eigenfunction of the operator $\Delta$ in $\Omega$, viz.

$$(3.10)\qquad \begin{cases} \Delta\phi + \lambda\phi = 0 & \text{in } \Omega \\ \dfrac{\partial\phi}{\partial\nu} + \alpha(x)\phi = 0 & \text{in } \partial\Omega. \end{cases}$$

Then $u_\xi(x,t) = \phi(x)\exp(-\lambda D + F)\xi$ is a solution of (3.7) with initial condition $u^0(x) = \phi(x)\xi$, $\xi \in \mathbb{R}^m$.

In the sequel by $\lambda_\alpha$ we denote the smallest eigenvalue of the eigenvalue equation (3.10). We observe that $\lambda_\alpha > 0$ whenever $\alpha$ is not identically equal to zero. Also, if $\alpha \equiv 0$ then $\lambda_\alpha = 0$. For $\lambda_\alpha$ there exists an eigenfunction $\phi_\alpha \in X_1$ with $\phi_\alpha(x) \geq 0$, $x \in \overline{\Omega}$ (cf. [ ]).

(3.11) **Theorem.**

i. If for all $\lambda \in \sigma(-\lambda_\alpha D + F)$, $\operatorname{Re}\lambda < 0$, then $u \equiv 0$ is globally asymptotically stable in $X_+$ for the system (3.7).
ii. If $\mu = \max\{\operatorname{Re}\lambda \mid \lambda \in \sigma(-\lambda_\alpha D + F)\} > 0$ and $F$ is irreducible, then $u \equiv 0$ is unstable. In fact,

$$\forall_{\xi \in \mathbb{K}, \xi \neq 0} \liminf_{t \to \infty} \|u_\xi(\cdot, t)\| e^{-\mu t} > 0$$

with $u_\xi(x,t) = \phi_\alpha(x)\exp(-\lambda_\alpha D + F)\xi$.

Next we look at the non-linear case

$$(3.1')\qquad \frac{\partial u}{\partial t} = D\Delta u + f(u)$$

$$B u \big|_{\partial\Omega} = 0.$$

We introduce the constants $\alpha_{\min}$ and $\alpha_{\max}$ by

$$(3.12)\qquad \alpha_{\min} = \min_{x \in \partial\Omega} \min\{\alpha_1(x), \ldots, \alpha_m(x)\}$$

$$\alpha_{\max} = \max_{x \in \partial\Omega} \max\{\alpha_1(x), \ldots, \alpha_m(x)\}.$$

By $\lambda_{\min}$ we denote the first eigenvalue of the equation

$$\Delta\phi + \lambda\phi = 0$$

$$\frac{\partial\phi}{\partial\nu} + \alpha_{\min}\phi = 0.$$

Similarly, $\lambda_{\max}$ is defined. By $\phi_{\min}$ and $\phi_{\max}$ we denote the corresponding positive eigenfunctions.

Now assume that the Jacobian $F = J_f(0^+)$ is quasi-monotonely increasing, irreducible and such that

(3.13) $\forall_{\xi \in I\!K} : f(\xi) \le F\,\xi.$

Under this assumption the following stability result is valid.

(3.14) **Theorem.**
If for all $\lambda \in \sigma(-\lambda_{min} D + F)$, $\mathrm{Re}\,\lambda < 0$, then the trivial equilibrium solution $u \equiv 0$ is globally asymptotically stable for the system (3.1).

The proof of this theorem is based on Lemma (3.4) and Theorem (3.11).

Let $F = J_f(0^+)$ be quasi-monotonely increasing, irreducible and such that $\forall_{\xi \in I\!K} : f(\xi) \le F\,\xi$.
Then for all $\varepsilon > 0$, $\varepsilon < \min\{f_{ij} \mid f_{ij} > 0,\ i \ne j\}$ there is $\delta(\varepsilon) > 0$ such that

$$\forall_{\xi \in I\!K,\ |\xi| < \delta(\varepsilon)} : f(\xi) \ge F_\varepsilon \xi$$

where the matrix $F_\varepsilon$ is quasi-monotonely increasing and irreducible with elements defined by

$$f_{ij}^{\varepsilon} = f_{ij} - \varepsilon \quad \text{if } f_{ij} > 0,\ i \ne j$$

$$f_{ij}^{\varepsilon} = 0 \quad \text{if } f_{ij} = 0,\ i \ne j$$

$$f_{ij}^{\varepsilon} = f_{ij} \quad \text{if } i = j.$$

In connection herewith we mention

(3.16) **Lemma.**
Suppose there is a quasi-monotonely increasing, irreducible matrix $\tilde{F}$ such that

$$\exists_{\delta > 0}\ \forall_{\xi \in I\!K,\ |\xi| \le \delta} : f(\xi) \ge \tilde{F}\,\xi.$$

Then for all $u^0 \in X_+$, $u^0 \ne 0$,

$$V(t)\,u^0 \gg 0, \quad t > 0.$$

The following result is on the instability of the trivial stationary state.

(3.17) **Theorem.**
Assume $F = J_f(0+)$ is a quasi-monotonely increasing and irreducible matrix such that

$$\forall_{\xi \in I\!K} : f(\xi) \le F\,\xi$$

and assume

$$\mu = \max\{\mathrm{Re}\,\lambda \mid \lambda \in \sigma(-\lambda_{max} D + F)\} > 0.$$

Then the trivial stationary state of (3.1) is not stable.

Finally we devote some attention to the existence and stability of non-trivial steady states. We approach their stability by means of a kind of contracting rectangle technique. So let us consider the equation

$$(3.18) \qquad \begin{cases} D\,\Delta\Phi + f(\Phi) = 0 \quad \text{in} \quad \Omega \\ B\,\Phi = 0 \quad \text{in} \quad \partial\Omega. \end{cases}$$

(3.19) **Definition.**
A function $\underline{\Phi}$ on $\overline{\Omega}$ is a subsolution of the system (3.18) if

$$D\,\Delta\underline{\Phi} + f(\underline{\Phi}) \geq 0 \quad \text{in} \quad \Omega$$

ànd

$$B\,\overline{\Phi} \geq 0 \quad \text{in} \quad \partial\Omega.$$

Correspondingly, $\overline{\Phi}$ on $\Omega$ is a supersolution of (3.18) if

$$D\,\Delta\overline{\Phi} + f(\overline{\Phi}) \geq 0 \quad \text{in} \quad \Omega$$

and

$$B\,\overline{\Phi} \geq 0 \quad \text{in} \quad \partial\Omega.$$

(3.20) **Lemma.**

i. Let $\underline{\Phi}$ be a subsolution of the system (3.18). Then $V(t)\,\underline{\Phi}$ increases on $\mathbb{R}^+$. Further, if

$$N_+(\underline{\Phi}) = \{\Psi \mid \Psi \text{ satisfies (3.18) and } \underline{\Phi} \leq \Psi\}$$

is not empty, then $\Phi_- = \min N_+(\underline{\Phi})$ exists and

$$\lim_{t\to\infty} \|V(t)\,\underline{\Phi} - \Phi_-\| = 0.$$

ii. Let $\overline{\Phi}$ be a supersolution of the system (3.18). Then $V(t)\,\overline{\Phi}$ decreases on $\mathbb{R}^+$. Further, if

$$N_-(\overline{\Phi}) = \{\Psi \mid \Psi \text{ satisfies (3.18) and } \overline{\Phi} \geq \Psi\}$$

is not empty, then $\Phi^+ = \max N_-(\overline{\Phi})$ exists and

$$\lim_{t\to\infty} \|V(t)\,\overline{\Phi} - \Phi_+\| = 0.$$

(3.21) **Theorem.**
Suppose there exist a real quasi-monotonely increasing $m \times m$ matrix $C$ such that $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(C)$ and

$$\exists_{\delta>0}\ \forall_{\xi\in I\!K,\ |\xi|\geq\delta}\ :\ f(\xi)\leq C\,\xi.$$

Then the collection of steady state solutions of (3.18) admits a maximum $\Phi_+$ for which

$$\lim_{t\to\infty}\ \mathrm{dist}\,(V(t)\,u^0\,,\,[0,\Phi_+])=0\,,\quad u^0\in X_+.$$

**(3.22) Theorem.**
Under the hypothesis of Theorems (3.17) and (3.21) the collection of steady state solutions admits a minimum $\Phi_-$ with $\Phi_- \gg 0$ for which for all $u^0 \in X^+$, $u^0 \neq 0$,

$$\lim_{t\to\infty}\ \mathrm{dist}\,(V(t)\,u^0\,,\,[\Phi_-,\infty))=0.$$

In combination with Theorem (3.21) we get for all $u^0 \in X_+ \setminus \{0\}$

$$\lim_{t\to\infty}\ \mathrm{dist}\,(V(t)\,u^0\,,\,[\Phi_-,\Phi_+])=0.$$

We introduce some extra conditions on functions $f$ from $I\!R^m$ into $I\!R^+$.

C4 For all $R > 0$ there exists a quasi-monotonely increasing and irreducible matrix $C_R$ such that

$$\forall_{\xi,\eta\in I\!K,\,0\leq\eta\leq\xi,\,|\xi|\leq R}\ :\ f(\xi)-f(\eta)\geq C_R(\xi-\eta).$$

C5 $f$ is sublinear, viz.

$$\forall_{\tau\in(0,1)}\ \forall_{\xi\gg0}\ :\ \tau f(\xi)\leq f(\tau\xi).$$

**Remark.** $f(\xi)-f(\eta)=\int_0^1 J_f(t\,\xi+(1-t)\eta)\,(\xi-\eta)\,dt$

We arrive at the main result of this subsection.

**(3.23) Theorem.**
Let $f$ satisfy the conditions C1 - C5 and assume, in addition, the conditions stated in Theorem 3.22. Then the parabolic system (3.1) admits a unique equilibrium solution $\Phi \gg 0$ which is globally asymptotically stable in $X_+ \setminus \{0\}$.

## 4. A model based on a reaction-diffusion system

Returning to our actual problem, modelling the spread of infectious diseases, we now take into account the spatial spread, also. So we introduce the spatial densities $u_1$ and $u_2$ of the interacting populations. With the notation of Section 1 we have

$$z_i(t) = \int_\Omega u_i(x,t)\,dx \quad (i = 1,2).$$

Our model is based on the following parabolic system

$$\text{(4.1)} \qquad \begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \Delta u_1 - a_{11} u_1 + a_{12} u_2 \\ \dfrac{\partial u_2}{\partial t} = d_2 \Delta u_2 - a_{22} u_2 + g(u_1) \quad \text{in } \Omega \end{cases}$$

with boundary conditions

$$\frac{\partial u_i}{\partial \nu} + \alpha_j(x) u_i = 0 \quad (j = 1,2) \ \text{ in } \partial\Omega.$$

We suppose that the function $g$ has the following properties

(4.2)

i. $0 < \zeta' < \zeta'' \ \Rightarrow \ 0 < g(\zeta') < g(\zeta'')$,

ii. $g(0) = 0$,

iii. $g$ is twice differentiable on $\mathbb{R}^+$ with $g''(\zeta) < 0$

iv. $\limsup_{\zeta \to \infty} \dfrac{g(\zeta)}{\zeta} < \dfrac{a_{11} a_{22}}{a_{12}}$.

In accordance with (3.1) we take

$$f(\xi) = \begin{bmatrix} -a_{11}\xi_1 + a_{12}\xi_2 \\ g(\xi_1) - a_{22}\xi_2 \end{bmatrix}, \ \xi \in \mathbb{R}^2.$$

Since $g(0) = 0$, we have $f(0) = 0$, and since $g$ is monotone on $\mathbb{R}_+$, $f$ is quasi-monotone on $\mathbb{R}^2_+$. So the condition C3 can be readily checked.

Let $B = J_f(0+)$, the Jacobian of $f$ at the origin,

$$B = \begin{bmatrix} -a_{11} & a_{12} \\ g'(0) & -a_{22} \end{bmatrix}.$$

Since $a_{12}, g'(0) > 0$, $B$ is quasi-monotonely increasing and, further, $B$ is irreducible. By (iii) $g$ is strictly concave, i.e.

$$\forall_{\zeta_1,\zeta_2 \in \mathbb{R}_+,\, \zeta_1 \neq \zeta_2} \ \forall_{\tau \in (0,1)} :$$

$$g(\tau\zeta_1 + (1-\tau)\zeta_2) > \tau g(\zeta_1) + (1-\tau)\, g(\zeta_2).$$

With $\zeta_2 = 0$ this becomes

$$\forall_{\zeta \in \mathbb{R}_+ \setminus \{0\}} \forall_{\tau \in (0,1)} : g(\tau \zeta) > \tau g(\zeta).$$

Thus we conclude that $f$ is sublinear (= C5)

$$\forall_{\xi \gg 0} \forall_{\tau \in (0,1)} : f(\xi) < \frac{1}{\tau} f(\tau \xi).$$

Letting $\tau \downarrow 0$ we see that

$$\forall_{\xi \in \mathbb{R}_+^2} \quad f(\xi) \leq B \xi.$$

Condition C4 follows from the equality

$$f(\xi) - f(\eta) = \int_0^1 J_f(t\xi + (1-t)\eta)(\xi - \eta)\, dt$$

where

$$J_f(\xi) = \begin{bmatrix} -a_{11} & a_{12} \\ g'(\xi) & -a_{22} \end{bmatrix},$$

and the fact that $g'$ is decreasing.

We put

$$\alpha_{\min} = \min_{x \in \partial\Omega} \min \{\alpha_1(x), \alpha_2(x)\}$$

$$\alpha_{\max} = \max_{x \in \partial\Omega} \max \{\alpha_1(x), \alpha_2(x)\}.$$

By $\lambda_{\min}$ and $\lambda_{\max}$ we denote the first eigenvalues belonging to the eigenvalue equations

$$\Delta\phi + \lambda\phi = 0$$

with respectively

$$\frac{\partial\phi}{\partial\nu} + \alpha_{\min}\phi = 0 \quad \text{and} \quad \frac{\partial\phi}{\partial\nu} + \alpha_{\max}\phi = 0.$$

The corresponding eigenfunctions $\phi_{\min}$ and $\phi_{\max}$ can be taken strictly positive.

On the basis of the theory presented in Section 3 the following statements can be proved.

**(4.3) Theorem.**
**If**

$$\theta_{\min} = \frac{a_{12}\, g'(0)}{(a_{11} + d_1 \lambda_{\min})(a_{22} + d_2 \lambda_{\min})} < 1,$$

then the trivial solution of 4.1 is globally asymptotically stable in $X_+$.

(4.4) **Theorem.**
If

$$\theta_{max} = \frac{a_{12}\, g'(0)}{(a_{11}+d_2\,\lambda_{max})\,(a_{22}+d_2\,\lambda_{max})} > 1\,,$$

then the trivial solution of (4.1) is instable. Furthermore,

$$\exists_{K>0}\ \forall_{u^0\in X^+\setminus\{0\}}\ :\ \lim_{t\to\infty} \|V(t)\,u^0\| > K.$$

(4.5) **Proposition.**
Let $\theta_{max} > 1$. Then $z^* \in \mathbb{R}^2_+ \setminus \{0\}$ satisfying

$$-a_{11}\, z_1^* + a_{12}\, z_2^* = 0$$

$$-a_{22}\, z_2^* + g(z_1^*) = 0$$

generates the supersolution $\overline{\Phi}(x) = z^*\,,\quad x \in \overline{\Omega}$.

(4.6) **Proposition.**
Let $\theta_{max} > 1$. By $z^*_{max}$ we denote the unique non-trivial equilibrium solution of the system of ordinary differential equations

$$\frac{dz}{dt} = -\lambda_{max}\, D\, z + f(z)\,,\quad t > 0.$$

Then for any $\varepsilon > 0$ with $\|\varepsilon\,\phi_{max}\| \leq 1$ the function

$$\underline{\Phi}(x) = \varepsilon\,\phi_{max}(x)\, z^*_{max}\,,\quad x \in \overline{\Omega}$$

is a subsolution of (4.1).

Now given a decreasing sequence $(\varepsilon_k)$ with $\varepsilon_k \downarrow 0$ we obtain a decreasing sequence of subsolutions $\underline{\Phi}_k = \varepsilon_k\,\phi_{max}\, z^*_{max}$. Herewith by Lemma 3. we can construct a decreasing sequence of equilibrium solutions $\Phi_k$ of (4.1) satisfying $\Phi_k \gg 0$. On the basis of the following lemma there exists an equilibrium solution $\Phi_- \gg 0$ with

$$\lim_{k\to\infty} \|\Phi_k - \Phi_-\| = 0.$$

(4.7) **Lemma.**
Let $(\Theta_k)$ denote a decreasing sequence of equilibrium solutions of the system (3.1) (in particular (4.1)) such that $\Theta_k \gg 0$ (in $\overline{\Omega}$). Then there exists an equilibrium solution $\Theta \gg 0$ (in $\overline{\Omega}$) of (3.1) sucht hat $\lim_{k\to\infty} \|\Theta_k - \Theta\| = 0$.

In addition we have $0 \ll \Phi_- \leq \Phi_+$.

Finally we present the following result.

**(4.8) Theorem.**
Let $\Theta_{\max} > 1$. Then the system (4.1) admits a unique equilibrium solution $\Phi \gg 0$ which is globally asymptotically stable in $X_+ \setminus \{0\}$.

## 5. Some general considerations on periodic systems

Consider the non-autonomous system

$$(5.1) \qquad \frac{\partial u_j}{\partial t} = d_j \Delta u_j + f_j(t; u_1, \ldots, u_m), \quad (x,t) \in \Omega \times \mathbb{R}, \quad j = 1, \ldots, m$$

with boundary conditions

$$\frac{\partial u_j}{\partial \nu} + \alpha_j(x) u_j = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}, \quad j = 1, \ldots, m.$$

So in comparison with (3.1) the nonlinear term $f = (f_1, \ldots, f_m)$ also depends on $t$.
We assume that for each $t \in \mathbb{R}$ the function $f(t; \cdot)$ satisfies the conditions C1, C2 and C3.
In addition, periodicity is assumed, viz.

$$(5.2) \qquad \exists_{T>0} \forall_{t \in \mathbb{R}} : f(t; \cdot) = f(t+T; \cdot).$$

With these conditions on $f$ it can be proved that for each $s \in \mathbb{R}$ and for each $u^0 \in X$ there exists a solution $u(x,t)$, $t \geq s$, such that $u(x,s) = u^0(x)$. Thus there are operators $U(t,s)$, $s \leq t$, on $X$ with the following properties.

i. For each $u^0 \in X$, $u(x,t) = (U(t,s) u^0)(x)$ is a solution of (5.1) with $u(x,s) = u^0(x)$.

ii. For each $s \in \mathbb{R}$, $U(s,s) = I$.

iii. For each $s, \tau, t \in \mathbb{R}$ with $s \leq \tau \leq t$

$$U(t,\tau)\, U(\tau,s) = U(t,s).$$

iv. For each $s,t \in \mathbb{R}$, $s \leq t$, $U(t,s) 0 = 0$.

v. For each $s,t \in \mathbb{R}$, $s \leq t$, $U(t+T, s+T) = U(t,s)$.

vi. For each $s \in \mathbb{R}$

$$u^0 \mapsto U(t,s) u^0, \quad u^0 \in X$$

is equicontinuous for $t \in [s, \Sigma]$

vii. For each $u^0 \in X$ and each $s \in \mathbb{R}$

$$t \mapsto U(t,s) u^0, \quad t \geq s,$$

is continuous.

viii. For each $u^0, v^0 \in X_+$ with $u^0 \leq v^0$

$$U(t,s)\,u^0 \leq U(t,s)\,v^0, \quad t \geq s.$$

By $U := U(T,0)$ we denote the *monodromy operator*. The following statements are equivalent

a. $u(x,t)$ is a periodic solution of the system (5.1).

b. $\forall_{s \in I\!R}$ : $u(s,x)$ is a fixed point of the operator $U$.

For non-autonomous systems there are the following notions of stability and asymptotic stability.

**(5.3) Definition.**
Let $\Phi(x,t)$ be a solution of (5.1). Then $\Phi$ is said to be *stable* if $\forall_{\varepsilon>0}\, \forall_{s \in I\!R}\, \exists_{\delta>0}\, \forall_{u^0 \in X}$

$$\|u^0 - \Phi(\cdot\,,s)\| < \delta \;\Rightarrow\; \|U(t,s)\,u^0 - \Phi(\cdot\,,t)\| < \varepsilon.$$

The stability is said to be *uniform* if $\delta$ does not depend on the choice of $s$.

**(5.4) Definition.**
Let $\Phi(x,t)$ be a solution of (5.1). We say that $\Phi$ is globally asymptotically stable in $D \subset X$ if $\Phi$ is stable and globally attractive in $D$, i.e.

$$\forall_{s \in I\!R}\, \forall_{u^0 \in D} \; : \; \lim_{t \to \infty} \|U(t,s)\,u^0 - \Phi(\cdot\,,t)\| = 0.$$

Further, $\Phi$ is said to be uniformly globally asymptotically stable if $\Phi$ is uniformly stable ànd

$$\forall_{s \in I\!R}\, \forall_{\varepsilon>0}\, \exists_{\Sigma=\Sigma(\varepsilon)>0}\, \forall_{u^0 \in D}$$

$$\|U(t,s)\,u^0 - \Phi(\cdot\,,t)\| < \varepsilon, \quad t > \Sigma + s.$$

**(5.5) Lemma.**
Let $\Phi(x,t)$ be a periodic solution of (5.1). Then $\Phi$ is stable (asymptotically stable) if and only if $\Phi(\cdot\,,0)$ is a stable (asymptotically stable) fixed point of the monodromy operator $U$ related to the system.

The previous lemma requires the following definition.

**(5.6) Definition.**
Let $\phi \in X$ be a fixed point of the operator $U$. Then $\phi$ is stable if

$$\forall_{\varepsilon>0}\, \exists_{\delta>0}\, \forall_{u^0\in X} \,:\, \|u^0-\phi\|<\delta \;\Rightarrow\; \|U^n u^0-\phi\|<\varepsilon$$

and $\phi$ is globally asymptotically stable in $D \subset X$ if $\phi$ is stable and globally attractive in $D$,

$$\forall_{u^0\in D} \,:\, \lim_{n\to\infty} \|U^n u^0 = \phi\| = 0.$$

## 6. A model with seasonality dependence

In this section we analyze the following system of ordinary differential equations which generalizes the system (1.1)

$$(6.1) \qquad \begin{cases} \dfrac{dz_1}{dt} = -a_{11}\, z_1 + a_{12}\, z_2 \\ \dfrac{dz_2}{dt} = -a_{22}\, z_2 + p(t)\, h(z_1). \end{cases}$$

Here $a_{11}, a_{12}$ and $a_{22}$ are positive constants and $p$ is a positive Hölder continuous $T$-periodic function. We set

$$p_{\max} = \max_{t\in[0,T]} p(t)\,,\quad p_{\min} = \min_{t\in[0,T]} p(t).$$

The function $h$ is supposed to satisfy the conditions 4.2 i.-iii. and in addition

iv' $$\limsup_{z\to\infty} \frac{h(z)}{z} = \frac{a_{11}\, a_{22}}{a_{12}\, p_{\max}}.$$

Define $f=(f_1,f_2)$ by

$$f_1\,(t\,;z_1,z_2) = -a_{11}\, z_1 + a_{12}\, z_2$$

$$f_2\,(t\,;z_1,z_2) = -a_{22}\, z_2 + p(t)\, h(z_1).$$

Then for each $t\in I\!R$, $f(t\,;\cdot)$ satisfies the conditions $C_1, C_2$ and $C_3$.
The ODE-system (6.1) can be written as

$$(6.1') \qquad \frac{dz}{dt} = f(t\,;z).$$

By $V(t,s)$, $t\geq s$, we denote the corresponding translation operators and by $V$ the monodromy operator $V(T,0)$. The operator $V$ is monotone

$$\forall_{\xi,\eta\in I\!K} \,:\, \xi\leq\eta \;\Rightarrow\; V\xi\leq V\eta.$$

In particular, $V$ is positive,

$$\forall_{\xi\in I\!K} \,:\, V\xi\in I\!K.$$

Let $B_{\max}$ be the matrix

$$B_{\max} = \begin{bmatrix} -a_{11} & a_{12} \\ p_{\max} h'(0) & -a_{22} \end{bmatrix}.$$

The conditions on $h$ ensure that

$$\forall_{\xi \in I\!K} \forall_{t \in I\!R} : f(t;\xi) \leq B_{\max} \xi.$$

So if the eigenvalues of $B_{\max}$ are strictly negative, the trivial solution of (6.1) is stable. Indeed,

**(6.2) Theorem.**
Suppose

$$\theta^{\max} = \frac{a_{12} p_{\max} h'(0)}{a_{11} a_{22}} < 1.$$

Then the trivial solution of (6.1) is globally asymptotically stable in the positive cone $I\!K$.

Next we discuss non-trivial periodic solutions.

**(6.3) Theorem.**
Suppose the monodromy operator $V$ corresponding to the system (6.1') satisfies

i. $\forall_{\xi \in I\!K} : V\xi \gg 0$ (strong positivity)

ii. $\forall_{\xi \gg 0} \forall_{\tau \in (0,1)} \exists_{\alpha > 0} : V[\tau\xi] \geq (1+\alpha)\tau V[\xi]$ (strong concavity).

Then each non-trivial $T$-periodic solution $\xi^*(t)$, $t \geq 0$ of (6.1') with $\xi^*(t) \in I\!K$ is asymptotically stable.

In literature the following results on the existence of periodic solutions can be found.

**(6.4) Theorem.**
Let $C(t)$ be a $T$-periodic matrix function with continuous entries such that

$$\exists_{\delta > 0} \forall_{t \in I\!R} \forall_{\xi \in I\!K, |\xi| \geq \delta} : f(t,\xi) \leq C(t)\xi.$$

Let $Z$ denote the monodromy matrix associated to the system

$$\frac{dz}{dt} = C(t)z$$

and suppose all eigenvalues of $Z$ satisfy $|\lambda| < 1$. Then the system (6.1') has a $T$-periodic solution $\xi$ with $\xi(t) \in I\!K$, $t \geq 0$.

**(6.5) Theorem.**
Under the assumptions of Theorem (6.4), let there also exists a $T$-periodic matrix valued

function $B(t)$ with continuous entries and such that $B(t)$ is irreducible for all $t$. Suppose

$$\exists_{\delta>0}\ \forall_{t\in I\!R}\ \forall_{\xi\in I\!K,\,|\xi|\le\delta}\ :\ f(t\,;\xi)\ge B(t)\,\xi.$$

Then (6.1') admits a *non-trivial* $T$-periodic solution.

The previous results are at the basis of the following theorem concerning the system (6.1).

(6.6) **Theorem.**
Let

$$\theta^{\min} = \frac{a_{12}\,p_{\min}\,h'(0)}{a_{11}\,a_{22}} > 1.$$

Then (6.1) has a $T$-periodic non-trivial strictly positive solution which is globally asymptotically stable in $I\!K\setminus\{0\}$.

Finally we consider the non-autonomous version of (4.1).

(6.7)
$$\begin{aligned}\frac{\partial u_1}{\partial t} &= d_1\,\Delta u_1 - a_{11}\,u_1 + a_{12}\,u_2\\ \frac{\partial u_2}{\partial t} &= d_2\,\Delta u_2 - a_{22}\,u_2 + p(t)\,h(u_1)\quad\text{in}\quad \Omega\times I\!R\end{aligned}$$

with

$$\frac{\partial u_i}{\partial \nu} + \alpha_i(x)\,u_i = 0\quad\text{in}\quad \partial\Omega\times I\!R.$$

Let $\alpha_{\min}$, $\alpha_{\max}$, $\lambda_{\min}$ and $\lambda_{\max}$ be as in Section 4.

(6.8) **Theorem.**
Suppose

$$\theta_1 = \frac{a_{12}\,p_{\max}\,h'(0)}{(a_{11}+d_1\,\lambda_{\min})\,(a_{22}+d_2\,\lambda_{\min})} < 1.$$

Then the trivial solution of (6.6) is globally asymptotically stable in $X_+$.
Suppose

$$\theta_2 = \frac{a_{12}\,p_{\min}\,h'(0)}{(a_{11}+d_1\,\lambda_{\max})\,(a_{22}+d_2\,\lambda_{\max})} > 1.$$

Then the trivial solution is instable.

With the aid of so called $T$-subsolutions and $T$-supersolutions the existence of a non-trivial $T$-periodic solution can be proved.

(6.9) **Theorem.**
If $\theta_2 > 1$, then (6.7) admits a unique non-trivial $T$-periodic solution which globally asymptotically stable in $X_+ \setminus \{0\}$.

**Appendix**

There has been proposed a third mathematical model for describing the spread of an infectious disease. It is based on the following parabolic system of partial differential equations

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \Delta u_1 - a_{11} u_1 \\ \dfrac{\partial u_2}{\partial t} = \qquad\qquad - a_{22} u_2 + g(u_1) \end{cases}$$

with boundary conditions

$$\begin{cases} \dfrac{\partial u_1}{\partial \nu} = \int\limits_{\Omega} k(x,x')\, u_2(x',t)\, dx', \quad x \in \partial\Omega \\ \dfrac{\partial u_2}{\partial \nu} = 0. \end{cases}$$

So in this model only random dispersal of the infectious agent is supposed to take place. The kernel $k(x,x')$ in the boundary conditions describes the transfer of infectious agent at $x' \in \Omega$ to $x \in \partial\Omega$.

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